Eulerian Volume Transport Induced by Spatially Damped Internal Equatorial Kelvin Waves

JAN ERIK H. WEBER

Department of Geosciences, University of Oslo, Norway

(Manuscript received 26 May 2014, in final form 7 May 2015)

ABSTRACT

The Eulerian volume transport in internal equatorial Kelvin waves subject to viscous attenuation is investigated theoretically by integrating the horizontal momentum equations in the vertical. In terms of small perturbations, the time-averaged horizontal transports are determined to second order in wave steepness. The total Lagrangian volume transport in this problem consists of a Stokes transport plus an Eulerian transport. It is known that the Stokes transport, that is, the vertically integrated Stokes drift, in inviscid internal equatorial Kelvin waves vanishes identically in the rigid-lid approximation for arbitrary vertical variation of the Brunt–Väisälä frequency. The present study considers spatial wave damping due to viscosity. The Stokes transport still becomes zero, but now the radiation stresses due to decaying waves become source terms for the Eulerian mean transport. Calculations of the wave-induced Eulerian transport yield a jetlike symmetric mean flow along the equator from west to east for each baroclinic component, with compensating westward flows on both sides. The flow system scales as the internal equatorial Rossby radius in the north–south direction. The total eastward part of the Eulerian volume flux centered about the equator is estimated to about 0.2 Sv (1 Sv = 10^6 m^3 s^-1) for the first baroclinic mode.

1. Introduction

The combination of a pronounced thermocline and the changing sign of the Coriolis parameter have made the equatorial ocean a guide for various trapped internal waves. These waves appear to be generated by temporal/spatial variation of the trade wind system (Philander 1981). Also, barotropic and/or baroclinic instability of the upper-ocean current systems and eddy formation are important in this connection [see the review of oceanic intraseasonal variability by Kessler (2005, and references therein)]. The eastward propagating Kelvin wave is special in that the meridional wave velocity vanishes identically. For nonzero meridional velocity, there exists an infinite number of equatorial waves with trapping scales of the same order as that for the Kelvin waves, that is, the Rossby radius of deformation (Matsuno 1966; Blandford 1966; Munk and Moore 1968).

The main focus of the present study is the mean drift induced by internal equatorial Kelvin waves. Such waves possess mean momentum and hence induce a Stokes drift. As first shown by Longuet-Higgins (1953), the net particle velocity (the Lagrangian mean velocity) in the wave motion can be written as the sum of a Stokes drift (Stokes 1847) and an Eulerian mean current. A more general theory of wave-mean flow interaction has been developed by Andrews and McIntyre (1978). Earlier treatments of the net transport in barotropic inviscid long waves in a rotating ocean have demonstrated that the Eulerian mean flow is decoupled from the wave motion (Moore 1970) and cannot be uniquely determined in a channel with no meridional boundaries. By taking the effect of friction into account, this discrepancy is remedied. This is in fact most easily seen by performing direct calculations of the wave-induced drift using a Lagrangian description of motion [see, e.g., Weber (1983) for temporally attenuated surface waves]. A related Lagrangian analysis of the net equatorial circulation due to damped Yanai waves has been reported by Ascani et al. (2010).

Generally, the temporal frictional damping of the waves induces a virtual wave stress at the surface (Longuet-Higgins 1969) that drives the mean Eulerian flow. For spatially damped waves, the Eulerian mean current is driven by the radiation stress in the waves.

Corresponding author address: Jan Erik H. Weber, Department of Geosciences, University of Oslo, P.O. Box 1022 Blindern, 0315 Oslo, Norway.
E-mail: j.e.weber@geo.uio.no

DOI: 10.1175/JPO-D-14-0102.1

© 2015 American Meteorological Society
This theme has been thoroughly discussed for barotropic flows by Longuet-Higgins and Stewart (1960). It has also been studied for interfacial coastal Kelvin waves in a reduced gravity context (Støylen and Weber 2010), and for continuous stratification by Weber and Ghaﬀari (2014).

The Stokes drift in internal equatorial Kelvin waves has been analyzed for inviscid flow by Weber et al. (2014). It is a simple task to include the effect of friction in the wave problem, and this is demonstrated in the appendix. The rest of this paper is organized as follows: in section 2 we present the basic assumptions and the governing equations, and in section 3 (with reference to the appendix) we consider the spatially damped wave motion. Section 4 presents the solutions for the Stokes drift and the Eulerian mean flow, and section 5 contains a discussion and some concluding remarks.

2. Basic assumptions and governing equations

We consider an (eddy) viscous ocean of constant depth \( H \), and we choose a Cartesian coordinate system \((x, y, z)\) such that the origin is situated at the undisturbed surface, the \( x \) axis is directed eastward along the equator, and the \( z \) axis is directed vertically upward. The respective unit vectors are \((\mathbf{i}, \mathbf{j}, \mathbf{k})\). The reference system rotates about the vertical axis with angular velocity \( \beta \), where \( \beta = 2.3 \times 10^{-11} \, \text{m}^{-1} \, \text{s}^{-1} \).

Furthermore, we use an Eulerian description of motion, which means that all dependent variables are functions of \( x, y, z \) and time \( t \). We take that the horizontal scale of the motion is so large compared to the depth that we can make the hydrostatic approximation in the vertical. Furthermore, we apply the Boussinesq approximation for the density \( \rho \). We also take that the density of an individual fluid particle is conserved. The governing equations for this problem then become

\[
\frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}_h = -\mathbf{f} \times \mathbf{v}_h - \frac{1}{\rho_r} \nabla p + \frac{\partial}{\partial z} \left( \frac{\tau_v}{\rho_r} \right),
\]

(1)

\[
\frac{\partial p}{\partial z} = -\rho g,
\]

(2)

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0, \quad \text{and}
\]

\[
\nabla \cdot \mathbf{v} = 0.
\]

(3)

Here \( \mathbf{v} = (u, v, w) \) is the velocity vector, \( p \) is the pressure, subscript \( h \) refers to the horizontal values, and \( \tau_v = \left[ \tau_h^{(x)}, \tau_h^{(y)} \right] \) is the turbulent stress in the horizontal direction. Furthermore, \( \rho_r \) is a constant reference density, and \( g \) is the acceleration due to gravity.

We take that there is no forcing from the atmosphere in this problem, that is, at the surface:

\[
\tau_h^{(x)} = \tau_h^{(y)} = p_{\infty} = 0.
\]

The surface is material and given by \( z = \eta(x, y, t) \). Integrating the governing equations from the bottom \( z = -H \), where \( w = 0 \), to the moving surface, we obtain equations for the horizontal volume transport \((U_L, V_L)\) in the fluid:

\[
\frac{\partial U_L}{\partial t} = f V_L = -\frac{1}{\rho_r} \frac{\partial}{\partial x} \int_{-H}^{\eta} p \, dz - \frac{\partial}{\partial x} \int_{-H}^{\eta} u^2 \, dz - \frac{\partial}{\partial y} \int_{-H}^{\eta} uv \, dz - \frac{\tau_B^{(x)}}{\rho_r},
\]

\[
\frac{\partial V_L}{\partial t} = f U_L = -\frac{1}{\rho_r} \frac{\partial}{\partial y} \int_{-H}^{\eta} p \, dz - \frac{\partial}{\partial x} \int_{-H}^{\eta} uv \, dz - \frac{\partial}{\partial x} \int_{-H}^{\eta} u^2 \, dz - \frac{\tau_B^{(y)}}{\rho_r},
\]

(5)

where

\[
U_L = \int_{-H}^{\eta} u \, dz, \quad V_L = \int_{-H}^{\eta} v \, dz.
\]

(6)

Furthermore, \([\tau_B^{(x)}, \tau_B^{(y)}]\) are the turbulent stresses at \( z = -H \). Since we integrate the particle velocity in the fluid between the bounding material surfaces, (6) represents the total horizontal mass transport per unit density (see, e.g., Phillips 1977). We denote them by the subscript \( L \), since they actually are the Lagrangian transports, which in principle contain contributions from the Eulerian mean flow as well as the wave motion.  

3. Linear internal equatorial Kelvin waves

We consider equatorially trapped internal periodic waves propagating along the \( x \) axis (the equator) with frequency \( \omega \). In the following analysis we separate the variables into mean quantities (marked by an overbar) and periodic wave components with zero mean (marked by a tilde), that is, for any quantity \( W = \overline{W} + \tilde{W} \), where

\[
\overline{W} = \frac{1}{T} \int_{0}^{T} \tilde{W} \, dt.
\]

(7)

Here \( T = 2\pi/\omega \) is the wave period. The waves result from small perturbations from a state of rest
characterized by a horizontally uniform stable stratification $\rho_0(z)$. We take that the velocity in the $y$ direction vanishes identically, characterizing the Kelvin wave. The variables may be separated into normal modes (Lighthill 1969), and we refer to Gill and Clarke (1974) for details. The approach for free, spatially damped waves follows very closely that for internal coastal Kelvin waves (see, e.g., Weber and Ghafrari 2014). For didactic reasons we give a short account in the appendix.

In summary, letting real parts represent the physical solution, we have from the appendix for the linear vertical isopycnal displacement and the horizontal wave velocity:

$$\xi_n = \sum_{n=1}^{\infty} \xi_n(x,y) \phi_n(z), \quad \hat{u} = \sum_{n=1}^{\infty} u_n(x,y) \phi'_n(z),$$

where $\phi_n$ is given by (A2), and

$$u_n = c_n A_n \exp(-\alpha_n x - y^2/\alpha_n^2) \left[ \cos(k_n x + l_n^2 y^2 - \omega t) + \frac{\alpha_n}{k_n} \sin(k_n x + l_n^2 y^2 - \omega t) \right].$$

Here $k_n$ is the wavenumber for mode $n$, while $\alpha_n$ is a small frictional damping coefficient ($\alpha_n/k_n \ll 1$) [see (A17)]. Furthermore, $\alpha_n$ is the Rossby radius and $l_n$ is a small frictional wavenumber defined by (A20).

We note that the lines of constant phase for spatially damped equatorial Kelvin waves are parabolas curving westward on both sides of the equator. This is the same effect as found in temporally damped equatorial waves (Martsenis and Weber 1981). In principle, the displacement amplitudes $A_1, A_2, A_3, \ldots$ must be determined from field observations or analytical/numerical models runs with appropriate forcing.

4. The mean drift

a. The Stokes drift

As first shown by Stokes (1847), irrotational periodic waves possess nonzero mean wave momentum, leading to a net drift of particles in the fluid. This mean drift is referred to as the Stokes drift and is basically related to the inviscid part of the wave field, here modified by a slow spatial decay of the wave amplitude due to friction. To second order in wave steepness the Stokes drift in the $x$ direction can be expressed by the Eulerian wave field (Longuet-Higgins 1953):

$$\pi_S = \left( \int \hat{u} \, dt \right) \hat{u}_x + \left( \int \hat{v} \, dt \right) \hat{u}_y + \left( \int \hat{w} \, dt \right) \hat{u}_z,$$

where a subscript denotes partial differentiation and the averaging process is defined by (7). In the present problem, $\hat{v} = 0$ and $\hat{w} = \xi_i$. Hence, from (8) and (9) for internal Kelvin waves, neglecting small terms of order $\alpha_n^2/k_n^2$,

$$\xi_i = A_n \exp(-\alpha_n x - y^2/\alpha_n^2) \cos(k_n x + l_n^2 y^2 - \omega t),$$

$$u_n = c_n A_n \exp(-\alpha_n x - y^2/\alpha_n^2) \left[ \cos(k_n x + l_n^2 y^2 - \omega t) + \frac{\alpha_n}{k_n} \sin(k_n x + l_n^2 y^2 - \omega t) \right].$$

Tang et al. (1988) have shown from observations that the first baroclinic Kelvin wave mode is a dominant feature in the eastern equatorial Pacific Ocean. It is well known that the vertical variation of the eigenfunctions and the eigenvalues depend on the profile of the Brunt–Väisälä frequency. In this paper we consider a slightly different $N(z)$ than in Weber et al. (2014) [see (A6)]. To approximate the observations by Hayes et al. (1985) and Tang et al. (1988), the maximum peak value has been slightly reduced, and a small constant $N$, representing the average deep water value, has been added. This addition forces the depth of the zero crossing for the first mode from near peak depth for $N$ to larger depths (W. S. Kessler 2015, personal communication). The numerical calculations of the first mode eigenfunctions $\phi_1$ and $d\phi_1/dZ$, where $Z = z/H_0$, are shown in Fig. 1. They are normalized such that $(d\phi_1/dZ)_{Z=0} = 1$.

It is seen that zero crossing for the horizontal velocity occurs at about 1500-m depth. This fits well with the results based on observations obtained by Hayes et al. (1985) and Tang et al. (1988). We find that the corresponding phase speed is $c_1 = 2.47 \text{ m s}^{-1}$, while for the second mode we obtain $c_2 = 1.79 \text{ m s}^{-1}$. This fits fairly well with the estimates of Kessler and McPhaden (1995) (2.73 and 1.74 m s$^{-1}$, respectively).

Interestingly enough, even with a zero crossing at near 1500-m depth, the Stokes drift velocity for the first mode is still basically confined to the upper layer, as shown in Fig. 2. Here we have displayed the normalized nondimensional Stokes drift for the first mode (solid line). For comparison we have also shown the normalized Stokes drift for the case of a constant $N$ (broken line).
We note that the Stokes drift basically varies within an upper layer of thickness $2H_0$, where $H_0$ is the peak depth of $N^2$ (here 150 m). This is very much the same as that obtained when the stability is virtually zero in the deep ocean (Weber et al. 2014).

b. The Eulerian mean transport

The Stokes transport is defined as

$$US = \int_{-H}^{0} uS \, dz.$$  \hfill (12)

By application of the boundary conditions of (A4), we find from (11), valid for weakly damped internal Kelvin waves, that $US = 0$, which is similar to Weber et al. (2014) for an inviscid fluid. Since the wave motion occurs in the $x$–$z$ plane, it is also obvious that both the Stokes drift and the Stokes transport in the $y$ direction are identically zero for equatorial Kelvin waves. Hence, from Longuet-Higgins (1953), we infer that for small frictional effects on the wave motion, the Lagrangian and Eulerian mean transports are equal.

A more direct way of obtaining a general result is by averaging the exact Lagrangian transports [(6)] in time:

$$UL = \int_{-H}^{0} \bar{u} \, dz = \int_{-H}^{0} \bar{\pi} \, dz + \int_{-H}^{0} \bar{\eta} \, dz = \int_{-H}^{0} \bar{\pi} \, dz + \bar{\eta}(z = 0),$$

$$VL = \int_{-H}^{0} \bar{v} \, dz = \int_{-H}^{0} \bar{\psi} \, dz + \int_{-H}^{0} \bar{\nu} \, dz = \int_{-H}^{0} \bar{\psi} \, dz + \bar{\nu}(z = 0).$$  \hfill (13)

The last expressions on the right-hand side are valid to second order in wave steepness. For baroclinic motion, we apply the rigid-lid assumption. Hence, $\bar{u}(z = 0) = \bar{\eta}(z = 0)$. The remaining terms are the Eulerian mean volume transports:

$$\mathcal{U}, \mathcal{V} = \int_{-H}^{0} (\bar{\pi}, \bar{\psi}) \, dz.$$  \hfill (14)

Hence, to second order in wave steepness, $(UL, VL) = (\mathcal{U}, \mathcal{V})$ for this problem.

Utilizing (13), and the fact that $\mathcal{V} = 0$, we find from (5) (valid to second order) that

$$\frac{\partial \mathcal{U}}{\partial t} - f \mathcal{V} = -\frac{\partial \mathcal{P}}{\partial x} - \frac{\partial \bar{Q}}{\partial x} - \frac{\bar{F}_B^{(x)}}{\rho_r},$$

$$\frac{\partial \mathcal{V}}{\partial t} + f \mathcal{U} = -\frac{\partial \mathcal{P}}{\partial y} - \frac{\bar{F}_B^{(y)}}{\rho_r}.$$  \hfill (15)

Here we have defined the averaged integrated pressure $\mathcal{P}$ as
\[
\bar{P} = \frac{1}{\rho_r} \int_{-H}^H p \, dz. \tag{16}
\]

Furthermore, we have a wave-forcing term \( \bar{Q} \), valid to second order in wave steepness, defined by
\[
\bar{Q} = \int_{-H}^0 \overline{\alpha} \, dz. \tag{17}
\]

In this context, \( \bar{Q} \) is the radiation stress component associated with the baroclinic wave field that drives the mean flow in the wave-propagation direction. By writing \( p = \bar{p} + \bar{p} \), where \( \bar{p} \sim \rho_r \overline{\alpha} \kappa \eta \eta' \) [see (A1)], we realize that the internal wave field does not contribute to (16), so for the baroclinic case, \( \bar{P} = \int_{-H}^0 \bar{p} \, dz \). Inserting from (8) and applying the orthogonality condition [(A5)], we obtain for the forcing function:
\[
\bar{Q} = \frac{1}{4H} \sum_{n=1}^\infty c_n^2 A_n^2 \exp(-2\alpha_n x - 2y^2/a_n^2). \tag{18}
\]

By integrating the continuity [(4)] in the vertical, we obtain the exact relation
\[
\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x} \int_{-H}^0 u \, dz - \frac{\partial}{\partial y} \int_{-H}^0 v \, dz
\]
\[
= -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}. \tag{19}
\]

We seek steady solutions to the mean drift problem. Utilizing (14), (19) then reduces to
\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \tag{20}
\]

Since the mean motion in the direction normal to the wave propagation is very small (to be verified posteriori), we neglect the effect of friction [\( \tau_B^{(y)} \)] in this direction. Hence, in the \( y \) direction, the integrated pressure gradient is in geostrophic balance with the Coriolis force. From the vorticity of the steady version of (15), utilizing (20), we obtain
\[
\beta V = \frac{\partial^2 \bar{Q}}{\partial x \partial y} + \frac{\partial \tau_B^{(x)}}{\partial y}. \tag{21}
\]

The mean stress at the lower boundary is generally a function of the mean Eulerian flow in the fluid. Since the motion here is turbulent, a quadratic function of the vertically averaged mean velocity is often assumed, that is, \( \tau_B^{(x)} = \rho_c c_D \bar{U} \bar{U} / H^2 \), where \( c_D \) is a nondimensional drag coefficient of order \( 10^{-5} \). However, for small velocities, we can simplify and introduce a linear friction coefficient \( R = c_D |U| H^2 \), where \( u_B \) is a typical velocity in this region [Nøst 1994]. With this simplification, we have that
\[
\tau_B^{(x)} = \rho_r R \bar{U}. \tag{22}
\]

We obtain from (21), utilizing (20) and (22):
\[
\frac{\partial^2 \bar{V}}{\partial y^2} + \frac{\beta}{R} \frac{\partial \bar{V}}{\partial x} = \frac{\partial^2 \bar{Q}}{R \partial x^2 \partial y}. \tag{23}
\]

For Kelvin waves this problem is symmetric about the equator, and the motion is trapped, so the boundary conditions become
\[
\bar{V} = 0, \quad y = 0, \quad \bar{V} \text{ finite}, \quad y \to \pm \infty. \tag{24}
\]

Since the forcing \( \bar{Q} \) in (23) is quadratic in the wave amplitude, that is, \( -\exp(-2\alpha_n x) \), we assume a separable solution for \( \bar{V} \) of the form
\[
\bar{V} = \sum_{n=1}^\infty V_n(y) \exp(-2\alpha_n x). \tag{25}
\]

Utilizing (25), (23) reduces to
\[
\frac{d^2 V}{dy^2} - \frac{2\beta}{R} V_n = -\frac{4\alpha_n^2 c_c^2 A_n^2}{H \rho_r} y \exp(-2y^2/a_n^2). \tag{26}
\]

For each mode we scale the north–south distance by the internal Rossby radius, that is, we choose an inverse length scale \( \delta_n = 2^{1/2} a_n \). The nondimensional meridional coordinate \( Y \) becomes
\[
Y = \delta_n y. \tag{27}
\]

Then (26) can be written
\[
V'' - \gamma_n^2 V_n = -B_n Y \exp(-Y^2), \tag{28}
\]
where a prime now denotes derivative with respect to \( Y \). Using the results of (A17) and (A20), we find for the coefficients in (28):
\[
\gamma_n^2 = \frac{K_n}{R}, \quad B_n = \frac{2^{1/2} a_n K_n^2}{4HR A_n^2}, \tag{29}
\]
where \( K_n \) is given by (A11).

The solution of (28) follows easily from the variation of parameters method. Using the fact that \( V_n = 0 \) at \( Y = 0 \), and \( V_n \) is finite when \( Y \to \pm 0 \), we find
\[ V_n = V_0 \gamma_n^2 \exp(\gamma_n^2/4) \{ \exp(\gamma_n Y) \left[ \exp\left( Y + \gamma_n/2 \right) \right] + \exp\left( -Y - \gamma_n/2 \right) \} \]

where

\[ V_0 = \frac{\pi^{1/2} K_n c_n \Gamma_n^2}{16H R^{1/2}}. \]  

(31)

In (30) the error function is defined as

\[ \text{erf}(\theta) = \frac{2}{\pi^{1/2}} \int_0^{\theta} \exp(-q^2) \, dq. \]  

(32)

We note the important factor \( \gamma_n = (K_n/R)^{1/2} \) in (30). It expresses the relative importance between forcing from the wave field (through the radiation stress) versus the restraint of boundary friction on the mean flow. In balanced flow, \( K_n/R \) should be of order unity. In Fig. 3 we have displayed \( V_n/V_0 \) for various values of \( \gamma_n \) in this range.

For each specific mode, which is seen to scale well in the meridional direction with the corresponding Rossby

\[ U_n = U_0 \gamma_n^3 \exp(\gamma_n^2/4) \left\{ \exp(\gamma_n Y) \left\{ -1 + \exp\left( Y + \gamma_n/2 \right) + \frac{2}{\pi^{1/2} \gamma_n} \exp\left[ -(Y + \gamma_n/2)^2 \right] \} \right\} \\
+ \exp(-\gamma_n Y) \left\{ -1 + \exp(\gamma_n/2) + \exp(-\gamma_n/2) - \frac{2}{\pi^{1/2} \gamma_n} \exp\left[ -\left( Y - \gamma_n/2 \right)^2 \right] \} \right\}. \]  

(35)

where

\[ U_0 = \frac{\pi^{1/2} c_n \Gamma_n^2}{16H}. \]  

(36)

In Fig. 4 we have depicted \( U_n/U_0 \) for various values of \( \gamma_n \).

We note from Fig. 4 that we have a pronounced jet-like flow along the equator from west to east for each mode limited laterally by the internal Rossby radius. Outside this region, we observe a compensating reversed flow. Calculations show that

\[ \int_{-\infty}^{\infty} U_n \, dy = 0, \]  

(37)

so the total zonal transport is zero. Again we note that increasing bottom friction (smaller \( \gamma_n \)) leads to smaller maximum currents.

In this analysis \( K_n \), given by (A11), is a small quantity related to the turbulent frictional effect on the wave motion. Also the friction coefficient \( R \) acting on the mean flow is small, but for the Eulerian mean flow

\[ \int_{-\infty}^{\infty} V_n \, dy = 0, \]  

so we can vary \( \gamma_n \) by changing the bottom friction coefficient \( R \) in (29). From the figure we note that increasing bottom friction (smaller \( \gamma_n \)) means weaker north–south transports. Calculations show that the north–south transport for each mode is exactly antisymmetric about the equator, so the total meridional transport is zero.

The transport along the equator is obtained from (20). Assuming

\[ U = \sum_{n=1}^{\infty} U_n(y) \exp\left( -2\alpha_n y \right), \]  

(33)

and using the scaling of (27), we obtain

\[ U_n(Y) = \frac{2^{1/2}}{2\alpha_n a_n} V_n(Y). \]  

(34)

After some algebra, we find

\[ U_n = U_0 \gamma_n^3 \exp(\gamma_n^2/4) \left\{ \exp(\gamma_n Y) \left\{ -1 + \exp(Y + \gamma_n/2) + \frac{2}{\pi^{1/2} \gamma_n} \exp\left[ -(Y + \gamma_n/2)^2 \right] \} \right\} \\
+ \exp(-\gamma_n Y) \left\{ -1 + \exp(\gamma_n/2) + \exp(-\gamma_n/2) - \frac{2}{\pi^{1/2} \gamma_n} \exp\left[ -\left( Y - \gamma_n/2 \right)^2 \right] \} \right\}. \]  

(35)

driven by the radiation stress, we suppose that \( K_n/R \) is of order unity for balanced flow. We note by comparing (31) and (36) that

\[ \frac{V_0}{U_0} = \frac{a_1 K_n}{2^{1/2} c_n}. \]  

(38)

For the first baroclinic mode we have that \( c_1 = 2.47 \text{ m s}^{-1} \) and \( a_1 = 463 \times 10^3 \text{ m} \). For an order of magnitude estimate we apply the maximum value of \( N^2 \) and take that \( v_T \sim 10^{-3} \text{ m}^2 \text{s}^{-1} \) in the pycnocline (10 times the usual interior ocean value). We then find that \( K_1 = v_T N^2/c_1^2 \sim 0.53 \times 10^{-7} \text{ s}^{-1} \). Hence in (38), \( V_0/U_0 \sim 7 \times 10^{-3} \ll 1 \).

This justifies our neglect of the turbulent boundary friction in the y direction in (15).

5. Discussion and concluding remarks

The Eulerian mean volume transport induced by internal viscous equatorial Kelvin waves has been derived without the need to specify the explicit form of the
Brunt–Väisälä frequency $N$. The flow along the equator is eastward. It is symmetric about the equator, with compensating westward flows on both sides. We also find a much weaker transport in the meridional direction that is antisymmetric about the equator. Both flows are trapped within the scale of the internal Rossby radius. However, to obtain the numerical magnitude of the transport, we need to know the various baroclinic phase speeds, which in turn depend on the vertical distribution of $N$.

We know from (11) that the total zonal Stokes transport in weakly damped internal equatorial Kelvin waves is zero. This is due to the alternating sign in the distribution of each Stokes drift component with depth. Interestingly, the result [(37)] shows that the total Kelvin-wave-induced Eulerian zonal transport is also zero, but this is caused by the meridional variation of the vertically integrated Eulerian mean current.

We have obtained Eulerian mean transports for each baroclinic component. However, the corresponding amplitudes are not easy to assess. Tang et al. (1988) have analyzed observations from the eastern equatorial Pacific Ocean. Data from 1027-m depth shows vertical displacements for the first baroclinic mode with amplitudes from 20 to 80 m. Hence, an amplitude of 45 m at the depth of zero crossing could be a realistic value. From Fig. 4 we note that typically the total eastward volume transport $F_1$ along the equator is

$$
F_1 \sim 2U_0 \frac{2}{\delta_1} = \frac{2\theta_0 a_{1}}{2^{1/2}} = \frac{\pi^{1/2} c_{1}^{3/2} A_{1}^{2}}{4H^{1/2}}. \tag{39}
$$

Using values for the equatorial region for the first baroclinic mode, $c_{1} = 2.47 \text{ m s}^{-1}$, $a_{1} = 463 \times 10^{3} \text{ m}$, $A_{1} = 45 \text{ m}$, and $H = 3500 \text{ m}$, we obtain from (39) that $F_1 \sim 0.2 \text{ Sv}$ (1 Sv $= 10^{6} \text{ m}^{3} \text{ s}^{-1}$).

In this simplified study we do not discuss what may happen at the eastern boundary, from which we assume that no wave energy is reflected. From a mass balance point of view there is no need to assume mass accumulation (and associated mean pressure gradients) here since both the total Stokes transport and the total Eulerian mean zonal transport are zero through any meridional section.

Our analysis yields a net mean eastward volume transport along the equator aiding the Equatorial Undercurrent. The depth-averaged velocity that can be derived from (39) is modest, but the present procedure masks the possible vertical variation of the wave-induced Eulerian mean current. Like the Stokes drift (see, e.g., Fig. 2), the Eulerian mean current could have a significant depth variation in the pycnocline region. However, this problem is not likely to be resolved without numerical modeling.

Acknowledgments. Financial support from the Research Council of Norway through Grant 233901 (Experiments on waves in oil and ice) is gratefully acknowledged.
APPENDIX

Damped Internal Equatorial Kelvin Waves

According to the adopted approach, we assume for the linear periodic wave quantities that

\[
\begin{align*}
\tilde{u} &= \sum_{n=1}^{\infty} u_n(x,y,t)\varphi_n(z), \\
\tilde{v} &= \rho r \sum_{n=1}^{\infty} p_n(x,y,t)\varphi'_n(z), \\
\tilde{\xi} &= \sum_{n=1}^{\infty} \xi_n(x,y,t)\varphi_n(z).
\end{align*}
\]

(A1)

where \(\varphi_n\) is the vertical displacement of the isopycnals. The eigenfunctions \(\varphi_n\) are solutions of

\[
\varphi''_n + \frac{N^2}{c_n} \varphi_n = 0,
\]

(A2)

where \(N\) is the Brunt–Väisälä frequency defined by

\[
N^2 = \frac{g}{\rho r} \frac{dp_0(z)}{dz}.
\]

(A3)

For the baroclinic modes we assume rigid lids at the surface and the lower boundary. Hence, the boundary conditions become

\[
\varphi_n = 0, \quad z = -H, 0.
\]

(A4)

It is easily shown that the eigenfunctions \(\varphi'_n\) constitute an orthogonal set for arbitrary \(N = N(z)\). Assuming that \(\varphi_n\) is dimensionless, we normalize by assuming

\[
\int_{-H}^{0} \varphi'^2_n dz = \frac{1}{2H}.
\]

(A5)

In Weber et al. (2014) an idealized model of the Pacific equatorial thermocline was applied. Here we use a more realistic model that has a small constant \(N^2\) in the deep water, and a slightly less maximum peak value. Hence, we approximate \(N^2(z)\) by

\[
N^2(z) = N^2_d + N^2_0 \exp[-b(z/H_0 + 1)^2].
\]

(A6)

From Hayes et al. (1985) we take that \(N^2_0 = 3.24 \times 10^{-4} \text{s}^{-2}\) and \(N^2_d = 3 \times 10^{-6} \text{s}^{-2}\). The depth \(H_0\), where \(N^2(z)\) attains a maximum, is approximately 150 m (Kessler 2005). Furthermore, \(b\) is a dimensionless coefficient. The value \(b = 5\) appears to yield a good fit with observations. Equations (A2) and (A4), with \(N^2(z)\) given by (A6), are easily solved by a simple shooting procedure. For the first and second mode we find \(c_1 = 2.47 \text{m s}^{-1}\) and \(c_2 = 1.79 \text{m s}^{-1}\), respectively, which fits fairly well with previously reported values (Wunsch and Gill 1976; Kessler and McPhaden 1995).

The results for the first mode are plotted in Fig. 1. We note that the zero crossing of \(d\varphi_1/dz\) here is close to 1500 m, which corresponds well with earlier analyses of equatorial Pacific Ocean data (see, e.g., Hayes et al. 1985; Tang et al. 1988). Utilizing the orthogonality property, the governing equations may be written

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial t} &= -c_n \frac{\partial \varphi_n}{\partial x} + \varphi'_n z, \\
\beta \tilde{u} &= -c_n \frac{\partial \varphi'_n}{\partial y}, \\
\frac{\partial \varphi'_n}{\partial t} &= -\frac{\partial \varphi_n}{\partial x}.
\end{align*}
\]

(A7)

where the baroclinic modes are given by \(n = 1, 2, 3\), etc. Furthermore, we have defined

\[
\varphi_n(x,y,t) = \frac{2H}{\rho r} \int_{-H}^{0} \frac{\partial \tilde{u}(x,y,t)}{\partial z} \varphi'_n dz.
\]

(A8)

where \(\tilde{u}(x,y,t)\) is the turbulent stress on the wave motion in the \(x\) direction. Since \(\tilde{v} = 0\), we have taken \(\tilde{u}(x,y,t) = 0\). In the present problem we model the friction on internal Kelvin waves by a turbulent diffusion coefficient of momentum \(\nu_T\) such that \(\tilde{u}_n = \rho_T \varphi_n \tilde{u}_n/\partial z\). With \(\tilde{u}\) given by the series expansions of (A1), this formulation cannot capture the effect of bottom stress, but works well for the eddy dissipation in the bulk of the fluid for the baroclinic modes. This makes sense since the horizontal wave motion basically occurs in the upper ocean (see, e.g., Fig. 1). In this analysis we assume that \(\nu_T N^2\) is independent of \(z\) (Fjeldstad 1964). More specifically, from Williams and Gibson (1974), \(\nu_T N^2 = C \varepsilon\), where \(\varepsilon\) is the rate of dissipation of turbulent kinetic energy. Utilizing (A1) and (A5), (A8) reduces to

\[
\varphi_n(x,y,t) = -\frac{\nu_T N^2}{c_n} \tilde{u}_n.
\]

(A9)

The governing equations for mode \(n\) then become
\[ \frac{\partial u_n}{\partial t} = -c_n^2 \frac{\partial^2 \xi_n}{\partial x^2} - K_n u_n, \]
\[ \beta u_n = -c_n^2 \frac{\partial \xi_n}{\partial y}, \]
\[ \frac{\partial \xi_n}{\partial t} = -\frac{\partial u_n}{\partial x}, \]
\tag{A10} \]

where
\[ K_n = \nu \frac{N^2}{c_n^2}. \]
\tag{A11} \]

The trapping condition can be written
\[ u_n(x, t), \xi_n \to 0, \quad y \to \pm \infty. \]
\tag{A12} \]

Formally, we consider spatially damped waves in the domain \( x \approx 0 \) and take that
\[ u_n(x, t), \xi_n = \{u_n(y, t), \xi_n(y)\} \exp[i(\kappa_n x - \omega t)]. \]
\tag{A13} \]

where \( \omega \) is a real positive frequency and \( \kappa_n \) is the complex wavenumber for mode \( n \). We define \( \kappa_n = \kappa_n + i\alpha_n \), where \( \kappa_n \) is a real wavenumber and \( \alpha_n \) is the spatial decay rate. In practice the \( x \) direction is limited. In the application to the equatorial ocean, the origin is placed at the west coast, where the wave generation is assumed to occur. We do not consider any reflection at the east coast in the form of equatorially trapped waves with a westward group velocity. Hence, our Kelvin waves either dissipate at the eastern boundary or propagate northward and southward as coastal Kelvin waves.

By eliminating \( u_n \) from (A10), we obtain
\[ \frac{\partial^2 \xi_n}{\partial t^2} - c_n^2 \frac{\partial^2 \xi_n}{\partial x^2} + K_n \frac{\partial \xi_n}{\partial t} = 0. \]
\tag{A14} \]

We assume solutions of the form
\[ \xi_n = A_n G_n(y) \exp[i(\kappa_n x - \omega t)]. \]
\tag{A15} \]

Inserting into (A14), we obtain the dispersion relation
\[ \omega^2 + iK_n\omega - c_n^2 \kappa_n^2 = 0. \]
\tag{A16} \]

Assuming that \( |\alpha_n|/\kappa_n \ll 1 \), the real and imaginary parts of (A16) yield
\[ \kappa_n = \pm \frac{\omega}{c_n}, \quad \alpha_n = \frac{k_n K_n}{2\omega}. \]
\tag{A17} \]

Finally, from the geostrophic balance in (A10) we obtain
\[ \frac{\partial \xi_n}{\partial y} + \frac{\beta y}{\omega} (k_n - i\alpha_n) \xi_n = 0. \]
\tag{A18} \]

Inserting from (A15), we find that
\[ G_n(y) = \exp\left(-\frac{\beta k_n}{2\omega} + \frac{i\beta \alpha_n}{2\omega}\right)^2. \]
\tag{A19} \]

We note that trapping at each side of the equator requires that \( \kappa_n > 0 \), that is, we must choose the positive sign in (A17), implying eastward wave propagation. The appropriate internal Rossby radius \( a_n \) for the equatorial problem and the friction-induced wavenumber \( l_n \) in the north–south direction become
\[ a_n = \left(\frac{2\omega}{\beta k_n}\right)^{1/2}, \quad l_n = \left(\frac{\beta \alpha_n}{2\omega}\right)^{1/2}. \]
\tag{A20} \]

REFERENCES


