

Eikonal Calculations for Energy Transfer in the Deep-Ocean Internal Wave Field near Mixing Hotspots

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ABSTRACT

In the proximity of mixing hotspots in the deep ocean, the observed internal wave spectra are usually distorted from the Garrett–Munk (GM) spectrum and are characterized by the high energy level $E$ as well as a shear–strain ratio $R_v$ quite different from that of the GM spectrum. On the basis of the eikonal theoretical model, Ijichi and Hibiya (IH) recently proposed the revised finescale parameterization of turbulent dissipation rates in the distorted internal wave field, although the vertical velocity associated with background internal waves and the strict WKB scale separation, for example, were not taken into account. To see the effects of such simplifying assumptions on the revised parameterization, this study carries out a series of eikonal calculations for energy transfer through various internal wave spectra distorted from the GM. Although the background vertical velocity and the strict WKB scale separation somewhat affect the calculated energy transfer rates, their parameter dependence is confirmed as expected; the calculated energy transfer rates $\varepsilon$ follow the basic scaling $\varepsilon \propto E^2N^2f$ with the local buoyancy frequency $N$ and the local inertial frequency $f$ and exhibit strong $R_v$ dependence quite similar to that predicted by IH.

1. Introduction

Turbulent mixing in the deep ocean, which plays an important role in controlling the oceanic overturning circulation, is largely sustained by breaking of internal waves. The internal wave energy available for deep-ocean mixing is originally supplied at large vertical scales of $O(1)$ km by tide–topography interactions. It is then transferred across the internal wave spectrum down to small vertical scales of $O(10)$ m by nonlinear wave–wave interactions, where wave breaking eventually occurs (Hibiya et al. 1996, 1998). Thus, specifying the spectral shape of intermediate-scale internal waves is key to accurately quantifying the spectral energy transfer toward small dissipation scales and the resulting deep-ocean mixing.

In the ocean interior away from the boundary regions, the internal wave field appears to maintain the Garrett–Munk (GM) spectral shape (Garrett and Munk 1975; Munk 1981) with its energy level $E$ increasing or decreasing. Henyey et al. (1986, hereinafter HWF) used an eikonal (ray tracing) approach to predict that the rate of energy transfer $\varepsilon_{\text{HWF}}$ through such an internal wave spectrum toward dissipation scales satisfies $\varepsilon_{\text{HWF}} \propto E^2N^2f$, where $N$ is the local buoyancy frequency and $f$ is the local inertial frequency. Note that $E$ can be estimated in terms of either finescale $(10–100$ m) internal wave shear or strain variances because the ratio of horizontal kinetic energy and potential energy (KE/PE) is the same as that of the GM spectrum $(\text{KE/PE} = 3)$. On the basis of this HWF model, the shear-based finescale parameterization (Gregg 1989) and the strain-based finescale parameterization (Wijesekera et al. 1993) were proposed to infer turbulent dissipation rates.

In the proximity of mixing hotspots (localized regions of intense mixing) in the deep ocean, however, these shear-based and strain-based parameterizations are not expected to yield reliable estimates of turbulent dissipation rates, because, in comparison with the GM, the observed internal wave spectra are usually biased toward higher or lower frequencies $(\text{KE/PE} \neq 3)$. For example, the local internal wave spectra are biased toward lower frequencies near mixing hotspots associated with parametric subharmonic instability (Hibiya et al. 2006), while they are biased toward higher frequencies near rough bathymetry (Toole et al. 1997). Introducing the shear–strain variance ratio (or KE/PE) $R_v$ as a crude parameter for such spectral distortion, Polzin et al. (1995) and Gregg et al. (2003) applied a frequency-based...
correction to the HWF model and proposed the Gregg–Heney–Polzin (GHP) parameterization (see appendix A for the formulation).

Ijichi and Hibiya (2015, hereinafter IH), however, raised a question about the accuracy of the GHP parameterization, in which the single-wave approximation is applied to infer turbulent dissipation rates even in broadband internal wave spectra; the GHP parameterization overestimates the dissipation rates in the internal wave field dominated by near-inertial internal waves \((R_v \gg 3)\), because a multiplication factor of about 3 is used to adjust the predicted value at \(R_v = 3\) to the theoretical one for the GM. On the basis of the HWF model, IH reformulated the parameterization so as to become applicable to both a broadband spectrum like the GM \((R_v \sim 3)\) and a narrowband spectrum characterized by a prominent near-inertial peak \((R_v \gg 3)\); see appendix A for the formulation). Compared with the GHP parameterization, the IH parameterization exhibits stronger \(R_v\) dependence for \(R_v \sim O(1)\) and follows the single-wave formulation for \(R_v \sim O(10)\) (Fig. 1).

However, several simplifying assumptions are made in the HWF model so that the validity of the overall scaling of the turbulent dissipation rate should be numerically examined by removing such simplifying assumptions. The most promising approach is to use the 3D Navier–Stokes equations following Winters and D’Asaro (1997), who directly calculated the rate of energy transfer through the GM spectrum to confirm the HWF scaling: \(\varepsilon_{\text{HWF}} \propto E^2 N^2\). However, available resolution of such a numerical simulation is still far from adequate to cover the whole spectral range of distorted internal wave fields so that this approach is premature. Using the 3D eikonal equations (Lighthill 1978), in contrast, we can adequately reproduce distorted internal wave fields under the existing computational resources, although nonlinear wave–wave interactions are reduced to interactions between large-scale background waves and small-scale test waves (WKB scale-separated interactions).

The previous eikonal calculations (HWF; Sun and Kunze 1999b) examined the energy transfer rate through the GM spectrum at a fixed latitude of 30° to confirm \(\varepsilon_{\text{HWF}} \propto E^2 N^2\). Sun and Kunze (1999a,b) also pointed out that the horizontal WKB scale separation and the background vertical divergence both neglected in the HWF model play nonnegligible roles in the energy transfer through the GM spectrum. However, no previous studies have ever taken into account such neglected factors to examine the energy transfer rates through distorted internal wave spectra.

In this study, using the 3D eikonal equations, we calculate the rates of energy transfer through distorted internal wave spectra with various values of \(f, E,\) and \(R_v\) to assess their overall scaling (see section 2 for the experimental design and section 3 for the calculated results). We examine the sensitivity of the calculated results to the ambiguous factors in the eikonal calculations, namely, the degree of scale separation imposed between the test waves and the background waves (section 4a) and the effect of the background vertical divergence (section 4b). Finally, we discuss whether or not the results thus numerically obtained are consistent with the recent observational results (section 4c).

2. Eikonal calculations

a. Overview

Using the 3D eikonal equations, we trace the test waves adequately sampled from a background internal
wave field until their vertical wavenumber reaches a certain value (breaking limit). The energy flux of each test wave across the breaking limit is then calculated based on the conservation of wave action along the trajectory of each test wave. In this eikonal calculation, thus calculated energy flux for each test wave is summed to yield the spectral energy transfer rate toward dissipation scales.

b. Background internal wave field

Throughout this study, the ocean is assumed to have a constant depth of 2000 m and an exponential mean buoyancy frequency profile $N(z) = N_0 \exp(z/b)$ with $N_0 = 5.24 \times 10^{-3} \text{s}^{-1}$ and $b = 1300 \text{m}$. As a background internal wave field in the standard experiment, we incorporate a superposition of randomly phased linear internal waves with their amplitudes determined from the GM model (Munk 1981) into the model ocean at a latitude of $30^\circ$ ($f = f_0 = 7.29 \times 10^{-5} \text{s}^{-1}$). The energy density of the background internal waves $E_{BG}$ is specified as a function of frequency $\omega$ and vertical mode number $j$ in the separable form

$$\hat{E}_{BG}(\omega, \omega_j) = b^2 N_0^2 |\hat{N}(\omega)\hat{E}_0(\omega)|H(j),$$

where

$$\Omega(\omega) = \frac{\omega^{-p}(1 - f^2/\omega^2)^{-s}}{\int f} \quad \text{and} \quad \Omega(\omega) = \frac{\omega^{-p}(1 - f^2/\omega^2)^{-s}}{\int f} \,d\omega,$$

$$H(j) = \frac{(j^2 + j_0^2)^{-1}}{\sum_{j=1}^{n}(j^2 + j_0^2)^{-1}}.$$ (1)

with $E_0 = E_0^{GM} = 6.3 \times 10^{-5}$, $p = 2$, $s = 0.5$, $j_0 = 3$, and $j_{max}$ as the vertical mode number corresponding to the vertical wavenumber of 0.1 cpm. Based on the relation between $R_\omega$ and $\Omega(\omega)$ (Fofonoff 1969) given by

$$R_\omega = \frac{\int f (1 - \omega^2/\Omega^2)(1 + f^2/\omega^2)\Omega(\omega) \,d\omega}{\int f (1 - f^2/\omega^2)\Omega(\omega) \,d\omega}.$$ (2)

$R_\omega = 2.8$ is obtained for the standard experiment. The spatial and temporal structures of the background velocity $\mathbf{u}_{BG} = (u_{BG}, v_{BG}, w_{BG})$ and vertical isopycnal displacement $\xi_{BG}$ are given in appendix B. The obtained background internal wave spectra are pretty much the same as the GM model spectrum (Fig. 2). Along with this standard experiment, we carry out several experiments to see the sensitivity of energy transfer rates to the background parameters ($f$, $E_0$, and $R_\omega$).

c. Initial distribution of test waves

We determine the initial distribution of test waves in terms of frequency $\sigma_{i,\Omega, l, n}^{\text{initial}}$, vertical wavenumber $m_{i,\Omega, l, n}^{\text{initial}}$, and vertical position $z_{i,\Omega, l, n}^{\text{initial}}$ such that

$$\sigma_{i,\Omega, l, n}^{\text{initial}} = 1.2 \times 2^{(i-1)/2}f \quad (i = 1, 2, \ldots, i_{\text{max}}),$$

$$m_{i,\Omega, l, n}^{\text{initial}} = \begin{cases} \pm l/400 \text{ (cpm)} & (i = 1, 2, \ldots, 5), \\ \pm(l - 4)/100 \text{ (cpm)} & (i = 6, 7, \ldots, 14), \\ \pm 200n + 100 \text{ (m)} & (n = 1, 2, \ldots, 10), \end{cases}$$

where the subscript indices $(i, \pm l, n)$ denote the component numbers of frequency, vertical wavenumber, and vertical position, respectively, and $i_{\text{max}}$ is the maximum number of $i$ satisfying $\sigma_{i,\Omega, l, n}^{\text{initial}} < N(z_{i,\Omega, l, n})$. The initial horizontal position and horizontal wavenumbers of each test wave are randomly distributed. It should be noted that, unlike in the previous studies (HWF; Sun and Kunze 1999b), test waves are sampled also in the frequency domain to take into account the effect of the spectral distortion in estimating the energy transfer rate.

d. Ray tracing of each test wave

Once the background wave field and initial distribution of test waves are determined, both the position $\mathbf{r} = (x, y, z)$ and the wavenumber $\mathbf{k} = (k_x, k_y, m)$ of each

\footnote{In the standard experiment, the upper and lower values of $i_{\text{max}}$ are 12 at $z_{i,\Omega, l, n}^{\text{initial}} = -100 \text{ m}$ and 8 at $z_{i,\Omega, l, n}^{\text{initial}} = -1900 \text{ m}$, respectively, and the number of test waves amounts to 2828.}
test wave can be traced using the 3D eikonal equations:

$$\frac{D\sigma}{Dt} = \frac{\partial \sigma}{\partial k} + u_{BG} \cdot \nabla$$ and
$$\frac{Dk}{Dt} = -\frac{\partial \sigma}{\partial k} \nabla N^2 - \nabla u_{BG} \cdot k,$$

where $D/Dt$ is the time variation along the trajectory of each test wave, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, $N^2$ is the squared background wave-induced buoyancy frequency, and $\sigma$ is the intrinsic frequency of the test waves satisfying the dispersion relation

$$\sigma^2 = \frac{N^2(k_x^2 + k_y^2) + f^2 m^2}{k_x^2 + k_y^2 + m^2}.$$

Since the eikonal approach requires WKB scale-separated interactions, we restrict horizontal and vertical wavenumbers of the background waves ($k_x, k_y$) such that $k_H < \sqrt{k_x^2 + k_y^2}/2$ and $|k_z| < |m|/2$ at each time step (sensitivity to the degree of scale separation is discussed in section 4a). It should be noted, that to take into account the effect of wave-induced isopycnal displacements accurately, the background variables ($u_{BG}, \xi_{BG}$) should be set in a semi-Lagrangian frame $(x', y', z', t)$ in the semi-Lagrangian frame, the squared background buoyancy frequency can be expressed as $N^2(z) = N^2(z')/(1 + \partial \xi_{BG}/\partial z')$. Following Sun and Kunze (1999a), the eikonal equations (6) and (7) are transformed into the semi-Lagrangian frame (for detailed equations, see appendix C). These eikonal equations are integrated until the vertical wavenumber of each test wave reaches a breaking limit $m_{\text{break}}$ (the value is determined in section 3a), when the test wave is assumed to "break" and lose its energy to turbulence.

e. Estimate of the energy transfer rate

To calculate the energy flux at the breaking limit of each test wave, the wave action is assigned to each test wave such that

$$A_{i, z, l, n} = \frac{\Delta z}{2} \int_{\sigma_{\text{initial}}}^{\sigma_{\text{initial} + \Delta \sigma}} \int_{m_{\text{initial}}}^{m_{\text{initial} + \Delta m}} \frac{\dot{E}_{BG}(\sigma, m)}{\sigma} \, dm \, d\sigma,$$

where $\Delta z = 200$ m, and $\dot{E}_{BG}$ is specified as a function of frequency and vertical wavenumber. Since $A_{i, z, l, n}$ is conserved over the lifetime of each test wave $T_{i, z, l, n}$ (Henyey and Pomphrey 1983), the averaged wave action flux across the breaking limit is given by $A_{i, z, l, n}/T_{i, z, l, n}$, so that the corresponding energy flux becomes $\sigma_{\text{final}} A_{i, z, l, n}/T_{i, z, l, n}$ with $\sigma_{\text{final}}$ as the frequency of each test wave at its breaking limit. The vertical profile of the spectral energy transfer rate $\varepsilon(z)$ is thus evaluated every $\Delta z = 200$ m such that

$$\varepsilon(z_{n}) = \sum_{i, z, l, n} \sigma_{\text{final}} A_{i, z, l, n}/T_{i, z, l, n} \chi_{i, z, l, n},$$

where

$$\chi_{i, z, l, n} = \begin{cases} \Delta z^{-1} \cdots |z_{n} - z_{n-1}| \leq \Delta z/2, & |m_{\text{final}}| \geq m_{\text{break}}, \\ 0 \cdots & \text{otherwise} \end{cases}.$$

with $z_{n}$ as the vertical position and the vertical wavenumber of each test wave at its breaking limit, respectively. To obtain stable results, $\varepsilon(z)$ is averaged over 10 ensembles.

3 Results

a. Breaking limit in the eikonal calculations

First of all, we must choose a reasonable value of the breaking limit $m_{\text{break}}$ to yield the plausible energy transfer rate $\varepsilon(z)$ [(10)]. Figure 3 shows the total energy transfer rate

$$\varepsilon_{\text{total}} = \int_{-200m}^{0} \varepsilon(z) \, dz$$

FIG. 3. The total energy transfer rates $\varepsilon_{\text{total}}$ [(11)] for different values of the breaking limits $m_{\text{break}}$ in the standard experiment.
for different values of $m_{\text{break}}$ in the standard experiment, which is found to be saturated as $m_{\text{break}}$ increases. We can see that, although there is a certain amount of upscale energy transfer for $m_{\text{break}} \sim 0.1 \text{ cpm}$, downscale energy transfer begins to dominate the total beyond $m_{\text{break}} \sim 0.2 \text{ cpm}$. Since the calculated energy transfer rate is regarded as the turbulence production rate in this study, we carry out the following experiments using $m_{\text{break}} = 0.2 \text{ cpm}$.

### b. Standard experiment

We first briefly overview the results from the standard experiment for the GM background internal wave field at a latitude of 30°. The distributions of the assigned wave action $A_{i, z, l, 6}$, the calculated lifetime $T^{-1}_{i, z, l, 6}$, and the resulting wave action flux $A_{i, z, l, 6}/T_{i, z, l, 6}$ for each test wave with $m_{\text{initial}} = -1100 \text{ m}$. The white-shaded areas in (b) and (c) correspond to the test waves surviving for 40 inertial periods.

![Figure 4](image_url)

**FIG. 4.** The distributions of (a) the assigned wave action $A_{i, z, l, 6}$ (b) the calculated lifetime $T^{-1}_{i, z, l, 6}$, and (c) the wave action flux $A_{i, z, l, 6}/T_{i, z, l, 6}$ for each test wave with $m_{\text{initial}} = -1100 \text{ m}$. The white-shaded areas in (b) and (c) correspond to the test waves surviving for 40 inertial periods.

Figure 5 shows the vertical profile of the spectral energy transfer rate evaluated from (10), which is found to be consistent with the finescale parameterization:

$$e_{\text{GM}} = e_0 \frac{\cosh^{-1}(N/\nu)}{\cosh^{-1}(N_0/\nu_0)} \frac{N^2}{N_0^2}$$

(12)

with $e_0 = 6.73 \times 10^{-10}\text{ W kg}^{-1}$. It should be noted that the eikonal calculations in this study are different from previous ones in their much finer sampling of the test waves as well as the background waves. Sun and Kunze (1999b), for example, released only the test waves with an initial horizontal wavenumber of $10^{-3} \text{ cpm}$, and the number of internal waves consisting of the background field was much less than that in this study (for more details, see appendix B). Believing that the eikonal calculations in this study provide more reliable estimates of energy transfer rates...
than ever before, we next carry out several control experiments to examine the sensitivity of energy transfer rates to the background parameters ($f$, $E$, and $R_v$).

c. Control experiments

To examine the $f$ dependence of energy transfer rates, we first carry out the experiments for four different latitudes, $5^\circ$, $10^\circ$, $20^\circ$, and $40^\circ$, respectively, with the other parameters fixed as in the standard experiment. As shown by the green and black lines in Fig. 6, the probability distribution of the calculated lifetime $T$ of the test waves with $|m_{\text{initial}}| > 0.05$ cpm and $\sigma_{\text{initial}} < 4f$ can be well scaled by an inertial period $T_0 = 2\pi/f$. According to the HWF model, the Doppler shifting of test waves can be scaled by $Dm/Dt \approx f$, suggesting the obtained relation $T \approx T_0$. Consequently, the obtained energy transfer rates are found to be fairly consistent with the parameterization $\varepsilon/\varepsilon_{\text{GM}} \approx f/f_0$ (Fig. 7a). Note that, however, its applicability to the near-equatorial region is uncertain, because the above eikonal calculations are carried out under the $f$-plane approximation.

We next examine the $E$ dependence of energy transfer rates by carrying out the experiments assuming three different dimensionless energy levels of $E_0 = 2E_0^\text{GM}$, $3E_0^\text{GM}$, and $4E_0^\text{GM}$, respectively, in (1). For each value of $E_0$, the rolloff of the vertical wavenumber spectrum (e.g., Duda and Cox 1989; Gregg...
et al. 1993) should be reproduced, taking into account nonlinear dynamics as done by Hibiya et al. (1996), but we create it here simply by adjusting the amplitudes of the background internal waves with vertical wavenumbers larger than the cutoff wavenumber of $0.1E_{0}^{GM}/E_0$ cpm (Fig. 8a). As shown by the red and black lines in Fig. 6, the calculated lifetime of the test waves tends to be shorter as $E_0$ becomes larger. This is quite reasonable because the stronger background shear promotes the Doppler shifting of test waves. The energy transfer rates thus calculated are consistent with the parameterization $\varepsilon/\varepsilon_{GM} \approx (E_0/E_{0}^{GM})^2$ (Fig. 7c).

Finally, the $R_w$ dependence of energy transfer rates is examined through the experiments assuming five different distorted frequency spectra $\Omega (\omega)$ [Eq. (2)] with $p = 1.5, 3, 4, 6,$ and $8,$ respectively, and $s = 0.5 \{ R_w = 2.0, 4.5, 6.2, 9.5,$ and $12.5 \{ (4) \}$. To isolate the assessment of the $R_w$ dependence of the IH parameterization, the value of $E_0$ is adjusted so that the shear spectral level assumed in the standard experiment can be maintained (Fig. 8b). As shown by the blue and black lines in Fig. 6, the calculated lifetime of the test waves tends to be longer as $R_w$ becomes larger. This is qualitatively consistent with the HWF model showing that test waves are not easily Doppler shifted in the internal wave field biased toward lower frequencies, as pointed out by IH. The obtained energy transfer rates thus calculated exhibit $R_w$ dependence close to that predicted from the IH parameterization $h_{IH}$ [(A4)] (Fig. 7b).

4. Discussion

a. Sensitivity to the degree of scale separation

The degree of scale separation imposed between the test waves and the background waves is one of the ambiguous factors in the eikonal calculations. We therefore carry out some additional experiments to examine the sensitivity of the calculated results to the degree of scale separations to $\kappa_H < \sqrt{k_z^2 + k_x^2/4}$ while keeping $|\kappa_z| < |m|/2$ (the blue bullets in Fig. 9), or extending the vertical scale separation to $|\kappa_z| < |m|$ while keeping $\kappa_H < \sqrt{k_z^2 + k_x^2/2}$ (the red bullets in Fig. 9). We can see that, although the calculated energy transfer rates somewhat vary depending on the degree of scale separation (Sun and Kunze 1999b), their parameter dependence is not largely affected by the choice of scale separations, following the overall scaling confirmed in section 3c.

b. Effect of the background vertical divergence

Although Sun and Kunze (1999a,b) pointed out that the background vertical divergence neglected in formulating the IH parameterization plays a nonnegligible role in the energy transfer through the GM spectrum, a good performance of the IH parameterization has been confirmed in section 3c. To see a possible explanation for this, we carry out some additional experiments removing the terms $m \nabla \psi_{BG}$ from (7). When the background vertical divergence is neglected, the calculated energy transfer rates decrease by a factor of about 2 for $R_w \sim 3$, whereas such a tendency is no more found for $R_w \gg 3$ (see the green diamonds in Fig. 9b and the green bullets in Fig. 7b). A possible explanation for this is that the IH parameterization takes into account the extra contribution from high-frequency background waves, which has been excluded under the horizontal WKB scale separation in the eikonal calculations in this study. When the background vertical divergence is taken into account, it compensates for the decreased energy transfer rates so that good performance of the IH parameterization results.

c. Comparison with the recent observational results

Observations offer some support for the $R_w$ dependence of the IH parameterization $h_{IH}$ [(A4)]. Using the available microstructure data mostly obtained near the Izu–Ogasawara Ridge where large values of $R_w$ presumably result from parametric subharmonic instability, IH showed that discrepancies between the predicted and observed dissipation rates can be successfully reduced by using the IH parameterization.

Recently, Whalen et al. (2015) showed that the strain-based parameterization works well through a wide range of field observations. Although this seems
to be incompatible with the good performance of the IH parameterization, we can give a possible explanation for such a good performance of the strain-based parameterization for a relatively wide range of $R_v$; using the relation between the shear-based parameterization $\varepsilon_{\text{shear}}$ [(A2)] and the strain-based parameterization $\varepsilon_{\text{strain}}$ [(A6)], we can rewrite the IH parameterization as

$$\varepsilon = \varepsilon_{\text{shear}} h_{\text{IH}} = \varepsilon_{\text{strain}} \left( \frac{R_v}{3} \right)^2 h_{\text{IH}}^2,$$

where $h_{\text{IH}}^2$ is the frequency-based correction to the strain-based parameterization (A7). As shown in Fig. 10, for $R_v < 9$, the $R_v$ dependence of $h_{\text{IH}}^2$ is weak so that the strain-based parameterization is expected to perform relatively well. For $R_v > 9$, however, $h_{\text{IH}}^2$ rapidly increases with $R_v$ so that the strain-based parameterization is thought to underestimate turbulent dissipation rates. The best performance of the IH parameterization is therefore expected for the internal wave field significantly biased toward lower frequencies with large values of $R_v$.

5. Concluding remarks

IH recently proposed the revised finescale parameterization for turbulent dissipation rates near mixing hotspots in the deep ocean where internal wave spectra are usually distorted from the Garrett–Munk spectrum. In this study, extending the previous eikonal calculations (HWF; Sun and Kunze 1999b), we have calculated the rates of energy transfer through the distorted internal wave spectra to assess the validity of the revised parameterization where the background vertical divergence and the horizontal WKB scale separation between the test waves and the background waves are neglected. Although such factors...
neglected in formulating the parameterization are found to affect the calculated results to a certain degree, their parameter dependence has been confirmed as expected; \(\varepsilon\) follows the basic scaling \(\varepsilon \propto E^2 N^2 f\) and exhibits strong \(R_v\) dependence close to that predicted from the IH parameterization. However, \(\varepsilon\) does not take into account solitary waves, hydraulic jumps, and double diffusion. To infer turbulent mixing associated with such physical mechanisms, other parameterizations should be developed and utilized.

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APPENDIX A

Finescale Parameterizations of Turbulent Dissipation

The GHP parameterization predicts turbulent dissipation rates \(\varepsilon\) in the form

\[
\varepsilon = \varepsilon_{\text{shear}} h_{\text{GHP}}(\varepsilon, N/f).
\]

Here, \(\varepsilon_{\text{shear}}\) is the shear-based parameterization given by

\[
\varepsilon_{\text{shear}} = \varepsilon_0 \left(\frac{U_z^2}{U_z^{2}_{\text{GM}}}\right) N^2 f \frac{N^2}{f_0}.
\]

where \(\langle U_z^2 \rangle\) and \(\langle U_z^2 \rangle_{\text{GM}}\) are the shear variances for the observed spectrum and the GM spectrum, respectively; \(\varepsilon_0 = 6.73 \times 10^{-10}\) W kg\(^{-1}\), \(N_0 = 5.24 \times 10^{-3}\) s\(^{-1}\); and \(f_0 = 7.29 \times 10^{-5}\) s\(^{-1}\). In terms of \(R_v\), the frequency-based correction of the GHP parameterization \(h_{\text{GHP}}\) is introduced such that

Finally, it should be noted that the scaling cannot be applied to areas where physical mechanisms other than wave–wave interactions among linear internal waves play essential roles in the downscale energy cascade (Polzin et al. 2014). In the Southern Ocean, for example, interactions between the Antarctic Circumpolar Current (ACC) and high-frequency lee waves generated by ACC impinging on a rough ocean bottom should play crucial roles. Furthermore, internal wave scattering and reflection at the rough ocean bottom should affect the energy cascade just over the boundary. The scaling, of course, does not take into account solitary waves, hydraulic jumps, and double diffusion. To infer turbulent mixing associated with such physical mechanisms, other parameterizations should be developed and utilized.
\[
\begin{align*}
    h_{\text{GHP}}(R_w,N/f) &= \frac{1 + 1/R_w}{4/3} \frac{L_1}{L_0} \sqrt{\frac{2}{R_w - 1}}, \quad (R_w < 9) \\
    &\quad + \frac{1 + 1/R_w}{4/3} \frac{1}{L_0} \sqrt{\frac{2}{R_w - 1}} \quad (R_w \geq 9)
\end{align*}
\]

with \( L_1 = 2L_2^2 \) and \( L_2 = \log_3(2L) \). The \( R_w \) dependence of each parameterization is shown in Fig. 1. The IH parameterization can be also expressed in terms of the strain-based parameterization \( \varepsilon_{\text{strain}} \) as

\[
e = \varepsilon_{\text{strain}} h'_{\text{IH}}(R_w,N/f),
\]

where

\[
\varepsilon_{\text{strain}} = \frac{(\xi_z^2)^2}{(\xi^2)'_{\text{GM}}} \frac{N^2}{f \beta_0}, \quad \text{and}
\]

\[
h'_{\text{IH}}(R_w,N/f) = \begin{cases} 
\frac{R_w^2 + 1/R_w}{4/3} \frac{L_1}{L_0} R_w - L_1 & (R_w < 9) \\
\frac{R_w^2 + 1/R_w}{4/3} \frac{1}{L_0} \sqrt{\frac{2}{R_w - 1}} & (R_w \geq 9)
\end{cases}
\]

with \((\xi_z^2)\) and \((\xi^2)_{\text{GM}}\) as the strain variances for the observed spectrum and the GM spectrum, respectively (Fig. 10).

**APPENDIX B**

Spatial and Temporal Structures of Background Internal Waves

In this study, the background velocity \( u_{BG}(\omega) \) and vertical isopycnal displacement \( \xi_{BG} \) are reproduced by summing over 400 frequencies \( \omega \) and 2000 vertical modes \( j \), in contrast to the previous study by Sun and Kunze (1999a), which summed over only 100 random sets of \( \omega \) and \( j \). Based on the polarization relations for linear internal waves, the decomposed velocity \( (u_{w,j}, v_{w,j}, w_{w,j}) \) and vertical isopycnal displacement \( \xi_{w,j} \) are expressed in a semi-Lagrangian frame \((x', y', z', t')\) such that

\[
\begin{align*}
    u_{w,j} &= c_{w,j} \frac{dV_j}{dz'} \sqrt{\frac{\omega^2 - f^2}{\kappa^2 + \kappa_y^2}} \left( -\kappa_x \sin \theta - \kappa \frac{f}{\omega} \cos \theta \right), \\
    v_{w,j} &= c_{w,j} \frac{dV_j}{dz'} \sqrt{\frac{\omega^2 - f^2}{\kappa^2 + \kappa_y^2}} \left( -\kappa_y \sin \theta + \kappa \frac{f}{\omega} \cos \theta \right), \\
    w_{w,j} &= c_{w,j} V_j \sqrt{\omega^2 - f^2} \cos \theta, \quad \text{and}
\end{align*}
\]

where \((\kappa_x, \kappa_y)\) is the corresponding horizontal wave number with the direction randomly determined to ensure horizontal isotropy, \( \theta \) is the wave phase given by \( \theta = \kappa_x x' + \kappa_y y' - \omega t' + \delta_{w,j} \) with \( \delta_{w,j} \) a random value ranging from 0 to \( 2\pi \), \( V_j \) is the vertical structure function, and \( c_{w,j} \) is the amplitude coefficient. Note that \( V_j(z') \) must satisfy

\[
\frac{d^2 V_j}{dz'^2} + \frac{\bar{N}(z')^2}{\omega^2 - f^2}(\kappa_x^2 + \kappa_y^2)V_j = 0,
\]

with \( V_j(0 \text{m}) = V_j(-2000 \text{m}) = 0 \) and \( \bar{N}(z') = N_0 \exp(z'/b) \).

For \( \omega \) \( N_0/4 = \bar{N}(-2000 \text{m}/4), V_j(z') \) is given by the WKB solution of (B1) under the hydrostatic approximation:

\[
V_j(z') = \frac{1}{\sqrt{N_0 \bar{N}(z')}} \sin \left[ j\pi \frac{\bar{N}(z') - N_b}{N_0 - N_b} \right].
\]

The corresponding vertical wavenumber \( \kappa_z \) is

\[
\kappa_z = \frac{j \pi}{b} \frac{\bar{N}(z')}{N_0 - N_b},
\]

and \( c_{w,j} \) is expressed in terms of \( E_{BG} \) \([1]\) as \( c_{w,j} = 2 \sqrt{E_{BG} N_0 \bar{N} \Delta \omega} \) with \( \Delta \omega \) as a difference between two adjacent decomposed frequencies. For \( \omega > N_b/4 \), \( V_j(z') \) is given by the exact solution of (B1) expressed in Bessel functions (Garrett and Munk 1972). In this case, \( \kappa_z \) is given by the eigenvalue of (B1). The expected background internal wave spectra are successfully obtained through a superposition of such decomposed waves (Figs. 2, 8).

**APPENDIX C**

Eikonal Equations in a Semi-Lagrangian Frame

Following Sun and Kunze (1999a), the eikonal equations (6) and (7) are transformed into a semi-Lagrangian frame \((x', y', z', t') = (x, y, z - \xi_{BG}, t)\) such that
\[
\frac{D'x'}{D't} = \frac{\partial \sigma}{\partial k_x} + u_{BG}', \quad \frac{D'y'}{D't} = \frac{\partial \sigma}{\partial k_y} + v_{BG}', \quad \frac{D'z'}{D't} = \frac{\partial \sigma}{\partial m},
\]

\[
\frac{D'k_x}{D't} = -\frac{\partial \sigma}{\partial N^2} \frac{\partial^2 N^2}{\partial \xi'^2} - k_x \frac{\partial u_{BG}'}{\partial \xi'} - k_y \frac{\partial v_{BG}'}{\partial \xi'} - m \frac{\partial w_{BG}'}{\partial \xi'} + \frac{\partial \xi_{BG}'}{\partial \xi'} \left( \frac{\partial \sigma}{\partial N^2} \frac{\partial N^2}{\partial \xi'} + k_x \frac{\partial u_{BG}'}{\partial \xi'} + k_y \frac{\partial v_{BG}'}{\partial \xi'} + m \frac{\partial w_{BG}'}{\partial \xi'} \right),
\]

\[
\frac{D'k_y}{D't} = -\frac{\partial \sigma}{\partial N^2} \frac{\partial^2 N^2}{\partial \eta'^2} - k_x \frac{\partial u_{BG}'}{\partial \eta'} - k_y \frac{\partial v_{BG}'}{\partial \eta'} - m \frac{\partial w_{BG}'}{\partial \eta'} + \frac{\partial \xi_{BG}'}{\partial \eta'} \left( \frac{\partial \sigma}{\partial N^2} \frac{\partial N^2}{\partial \eta'} + k_x \frac{\partial u_{BG}'}{\partial \eta'} + k_y \frac{\partial v_{BG}'}{\partial \eta'} + m \frac{\partial w_{BG}'}{\partial \eta'} \right),
\]

\[
\frac{D'm}{D't} = \frac{1}{1 + \xi_{BG}' / \eta'} \left( \frac{\partial \sigma}{\partial N^2} \frac{\partial N^2}{\partial \xi'} + k_x \frac{\partial u_{BG}'}{\partial \xi'} + k_y \frac{\partial v_{BG}'}{\partial \xi'} + m \frac{\partial w_{BG}'}{\partial \xi'} \right),
\]

where \(D'/D't\) is the ray-following time derivative in the semi-Lagrangian frame, \(N^2(z')/(1 + \xi_{BG}' / \eta')\) is the squared background wave-induced buoyancy frequency with \(N(z') = N_0 \exp(z'/b)\) the mean buoyancy frequency.

REFERENCES


