Wave–Current Interaction: A Comparison of Radiation-Stress and Vortex-Force Representations

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ABSTRACT

The vortex-force representation of the wave-averaged effects on currents is compared to the radiation-stress representation in a scaling regime appropriate to coastal and shelf waters. Three-dimensional and vertically integrated expressions for the conservative current equations are obtained in both representations. The vortex-force representation decomposes the main wave-averaged effects into two physically understandable concepts—a vortex force and a Bernoulli head. The vortex force is shown to be the dominant wave-averaged effect on currents. This effect can occur at higher order than the apparent leading order for the radiation-stress representation. Excluding nonconservative effects such as wave breaking, the lowest-order radiation or interaction stress can be completely characterized in terms of wave setup, forcing of long (infragravity) waves, and an Eulerian current whose divergence cancels that of the primary wave Stokes drift. The leading-order, wave-averaged dynamical effects incorporate the vortex force together with material advection by Stokes drift, modified pressure-continuity and kinematic surface boundary conditions, and parameterized representations of wave generation by the wind and breaking near the shoreline.

1. Introduction

Surface gravity waves influence slowly evolving long waves, such as infragravity waves, as well as currents and material distributions in the ocean. This paper is an examination of the relation between two alternative representations of these wave-averaged effects in coastal waters in the absence of dissipative or forcing mechanisms. We denote the two representations as “radiation stress” and “vortex force.” The concept of radiation stress has helped to explain such phenomena as wave setup, surf beats, generation of alongshore currents in the surf zone, and nonlinear interactions within the wave field. We use the term radiation-stress representation to encompass equations that include wave effects on currents using the divergence of either a radiation or an interaction stress, or a three-dimensional analog, following the work of Longuet-Higgins and Stewart (1960, 1961, 1962, 1964, hereinafter collectively referred to as LHS) and Hasselmann (1971, hereinafter referred to as H71). The vortex-force representation arose to explain Langmuir circulations through wave vorticity generation by the currents and vortex stretching by the wave’s Lagrangian mean flow, the Stokes drift (Craik and Leibovich 1976), but the representation is more generally germane.

Formally, the radiation-stress and vortex-force representations are equivalent, related through two alternative representations of the inertial acceleration (i.e., advection). The radiation-stress representation arises from the identity

$$\mathbf{U} \cdot \nabla \mathbf{U} = \nabla \cdot (\mathbf{U} \mathbf{U}) + \mathbf{U} (\nabla \cdot \mathbf{U}),$$

(1)

together with incompressibility $\nabla \cdot \mathbf{U} = 0$, while the vortex-force representation comes from the identity
The important part of the surface wave field for coastal currents is the wave slope. This implies that advection is the Eulerian velocity vector. This leads to the assumption that the waves are slowly evolving in a coordinate system moving with the mean flow. The wave slope for the spectrum peak components is approximately 1, where $\varepsilon = O(1)$ for infragravity waves, $\varepsilon = O(10^{-2})$ for long gravity waves, $\varepsilon = O(10^{-4})$ for tides and inertial motions, and $\varepsilon = O(10^{-5})$ for long waves and variation of the wave quantities to the time scale of the waves.

- The wave slope for the spectrum peak components is small (again apart from breaking) (i.e., $\varepsilon \ll 1$, where $\varepsilon = Ak$ is the wave slope). This implies that advection plays a secondary role compared to propagation and its nonlinear dynamical effects occur on a much longer time scale than the wave period (e.g., 7–15 s).
- The current typical velocity ($0.1–0.5$ m s$^{-1}$) and sea level fluctuations ($0.05–0.3$ m) are smaller than the wave typical phase speed ($10–25$ m s$^{-1}$), orbital motion ($1–3$ m s$^{-1}$), and sea level fluctuation ($\sim 1–3$ m) (i.e., $\delta \ll 1$ and $\gamma \ll 1$, where $\delta$ is the ratio of the mean current to the wave orbital velocity and $\gamma$ is the long time scale associated with the mean current). This implies that there is a clear scale separation between the evolution rate for currents in association with internal gravity waves, tides, inertial motions, and advection and the primary wave frequency, allowing for a meaningful average over the primary wave fluctuations.
- The intended setting is beyond the shoreline wave-breaking zone (i.e., the littoral or surf zone), in water where the primary wavenumber times the typical depth is not small (i.e., $\mu = kH$ is not small and, indeed, could be large). For a 50-m peak wavelength, this condition is met once the depth is greater than a few meters.

Therefore, in many situations and over most coastal and open-ocean areas—with very strong tidal flows and the surf zone among the few exceptions—there is a clear separation between waves and currents in amplitude and horizontal space and time scales. This provides a basis for deriving the conservative wave–current interaction equations by avoiding closure assumptions. We adopt a particular set of the relation among the scaling parameters (section 2b) that is chosen to achieve the greatest generality for the resulting wave-averaged dynamical balances, even though in particular situations the actual parameter values will differ and the dynamical balance will involve a subset of dominant influences. Some alternative parameter relations are also analyzed in section 6b. Of course, any complete representation of wave–current interactions must also include their wind generation, spectrum distribution, and dissipative mechanisms, such as wave breaking.

Offshore, the wave breaking is a primary mechanism for conveying wind stress to currents, and near shore it is a principal cause of littoral currents. However, following MRL04, we focus here on conservative, unforced wave–current interactions.

In this paper we compare the outcomes of the adop-
station of radiation-stress and vortex-force representations in the derivation of mean Eulerian wave–current interaction equations in a scaling appropriate for coastal and shelf waters. In doing so, we make plain the discrepancies and similarities of these two representations under the asymptotic assumptions that we have made, as well as more generally. After a preliminary presentation of the fundamental equations (section 2), we relate and compare the vortex-force representation with the widely used radiation-stress representation. We compare the time-averaged equations derived through the two representations in section 3, and in section 4 we consider the vertically integrated, time-averaged equations. We also make an estimate of the energetics of the quasi-static response, long waves, and current. Section 5 considers two examples that highlight the importance of properly handling wave–current interactions, both with regard to derivational issues as well as scaling. Further commentary on these issues is made in section 6, which also touches upon some the difficulties with the radiation-stress representation in this scaling, and considers alternative scalings and other averaging frameworks. In section 7 we summarize our view of the important elements in wave–current interaction.

2. Preliminaries

a. Basic equations

In the absence of body forces and dissipation, the equations of motion are

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + \frac{1}{\rho_0} \nabla P + \frac{\rho g \mathbf{z}}{\rho_0} + f \mathbf{z} \times \mathbf{U} = 0 \quad \text{and} \quad \nabla \cdot \mathbf{U} = 0,$$

with tracer equation

$$\frac{\partial C}{\partial t} + \mathbf{U} \cdot \nabla C = 0.$$

The surface boundary conditions are

$$W = \frac{\partial E}{\partial t} + \mathbf{Q} \cdot \nabla_s E \quad \text{and} \quad P = P_a \quad \text{at} \quad z = E,$$

and the bottom boundary condition is

$$W = -\mathbf{Q} \cdot \nabla, H \quad \text{at} \quad z = -H.$$

The physics and well-posedness considerations determine the lateral boundary conditions. The Eulerian velocity vector is $\mathbf{U} = (\mathbf{Q}, W)$. The density is described by $\rho$ and $\rho_0$ is a reference value of the density; the Coriolis parameter is $f$ and the gravitational constant is $g$; the pressure is described by $P$ and $P_a$ is the atmospheric pressure; $C$ is the tracer concentration; $\nabla = (\nabla_x, \partial_z)$ by convention and $t$ is time; $z = E(x, t)$ is the surface displacement, $z = -H(x)$ describes the bottom topography, $z = 0$ is understood to be the quiescent level of the ocean, and $z$ is positive upward. The transverse coordinates are $x = (x_1, x_2)$. Greek subscripts run from 1 to 2. The standard summation convention on repeated indices is used.

We may redefine the pressure by subtracting the hydrostatic pressure from the kinematic pressure,

$$p = P - P_a + \rho_0 g z,$$

whereby (3) becomes

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} + \frac{1}{\rho_0} \nabla P - g \mathbf{z} + f \mathbf{z} \times \mathbf{U} = 0. \quad (9)$$

Here the buoyancy $g$ is defined as

$$g = \frac{\rho_0 - \rho}{\rho_0}.$$

The pressure surface boundary condition (7) becomes

$$p = \rho_0 g E \quad \text{at} \quad z = E.$$
Upon vertically integrating (9) and (5) using (4) and (6)–(8), we obtain
\[
\frac{\partial T_a}{\partial t} + \frac{\partial}{\partial x} \int_{-H}^{E} Q_a Q_z dz + \frac{\partial}{\partial x} \int_{-H}^{E} p dz = \frac{g}{\rho_0} \frac{\partial T_a}{\partial x} + \frac{g}{\rho_0} \frac{\partial E^2}{\partial x} - \frac{1}{\rho_0} p(-H) + (f x \times T)_a = 0, \tag{12}
\]
and
\[
\frac{\partial}{\partial t} \int_{-H}^{E} W dz + \frac{\partial}{\partial x} \int_{-H}^{E} W Q_a dz + gE = \frac{p(-H)}{\rho_0} - \int_{-H}^{E} B dz, \tag{13}
\]
Equations for the vertically averaged horizontal velocity \( \mathbf{T}/(H + E) \) may be obtained from (12). Similarly,
\[
\frac{\partial}{\partial t} \int_{-H}^{E} C dz + \frac{\partial}{\partial x} \int_{-H}^{E} c Q_a dz = 0. \tag{15}
\]

b. Nondimensionalization, asymptotic scaling, and flow decomposition

As in MRL04, we nondimensionalize according to
\[
(x, z) \rightarrow k_0^{-1}(x, z),
\]
\[
t \rightarrow \sigma_0^{-1} t,
\]
\[
E \rightarrow a_0 E = \varepsilon k_0^{-1} E,
\]
\[
\mathbf{U} = (Q, W) \rightarrow a_0 \sigma_0 (Q, W) = \varepsilon k_0^{-1} \sigma_0 (Q, W),
\]
\[
\Omega \rightarrow \varepsilon \sigma_0 \Omega,
\]
\[
P \rightarrow \varepsilon \rho_0 \sigma_0^2 k_0^{-2} P, \quad \text{and}
\]
\[
\mathcal{B} \rightarrow \mu \mathcal{B},
\]
where the quantities on the right are dimensionless: \( k_0 \) is the characteristic wavenumber, \( H_0 \) is the characteristic depth, \( \mu = k_0 H_0 \) is a nondimensional parameter that specifies the relative depth of the water; \( \sigma_0 = \sqrt{g k_0 L} \) is the characteristic wave frequency with \( L = \tan \mathbf{h} \); \( a_0 \) is the typical wave height and \( \varepsilon = a_0 k_0 \) the wave slope; \( f_0 \) is the Coriolis parameter with \( f_{\text{nd}} = f_0/\sigma_0 \); \( \mathcal{B} \) is a Rossby number based on wave time scales; \( \Omega \) is the vorticity.

We adopt the following scalings for the primary waves, long waves, and currents: The nondimensional wave scale coordinates are \((x, z, t)\). The bottom topography varies over a longer horizontal spatial scale, \( X = \beta x \). This is also the horizontal spatial scale for the wave envelope, long waves, and the currents. The long waves and the currents are distinguished by their different temporal scales. Long waves evolve over the temporal scale \( \tau = \beta t \), as does the wave envelope. Thus, their scales are \((X, z, \tau)\). The current evolves over a longer time scale, \( T = \gamma t \), where \( \gamma \ll \beta \ll 1 \), giving it the scaling \((X, z, T)\). We also assume that the mean horizontal velocities are \( O(\delta) \) relative to the wave quantities, and long-wave velocities are \( O(\lambda) \) relative to the waves where \( \delta, \lambda \ll 1 \). Furthermore, incompressibility requirements demand that, in the vertical direction, the ratio of mean to wave velocity is \( \beta \delta \). We use the mutually consistent assumptions made in MRL04 that \( \beta = \varepsilon^4 \), \( \delta = \varepsilon \), \( \lambda = \varepsilon \), and \( \gamma = \varepsilon^4 \). If, on the other hand, \( \beta = \varepsilon \), dispersion enters the wave amplitude equation at the same order as the nonlinearity (Mei 1989, chapter 12).

While this affects the evolution of the waves and the nonlinear terms in the current equations, it does not alter the nature of the wave forcing of the mean equations. The essential results of this paper hold as long as \( \beta \ll 1 \), as required for leading-order wave (e.g., Wenzel–Kramers–Brillouin-like) solutions.

The scaling for tracers is different than the velocity scaling. It is discussed in MRL04 and McWilliams and Restrepo (1999). The largest component of the tracer concentration is due to the long-term currents. The primary waves and long waves cause smaller, faster-scale variations in the tracer concentration. The ratio of the wave concentration to the current concentration is \( \nu \) and that of the long-wave concentration to the current concentration is \( \lambda \nu \), where \( \nu \ll 1 \) and is \( \varepsilon \) in MRL04.

We decompose the flow into three components: current, long wave, and primary wave. The three-way decomposition for velocity is denoted as
\[
\mathbf{U} = (Q, W) = (\mathbf{q}', w') + (\mathbf{q}^{lw}, w^{lw}) + (\mathbf{q}^\mu, w^\mu)
\]
where \( \mathbf{q}' = (q'^1, w') \) is the mean current, \( \mathbf{q}^{lw} = (q^{lw1}, w^{lw}) \) is the long wave, and \( \mathbf{q}^\mu, w^\mu \) is the wave component. Waves are asymptotically expanded in small wave slope \( \varepsilon \), namely.
\[
(q^\mu, w^\mu) = (q^\mu_0, w^\mu_0) + \varepsilon(q^{\mu1}, w^{\mu1}) + \varepsilon^2(q^{\mu2}, w^{\mu2}) + \cdots
\]
the notation \((\mathbf{q}^\mu, w^\mu)\) is also used. Sea surface elevation and pressure are split into their mean, long-wave, and wave components according to \( E = \zeta + \zeta^{lw} + \zeta^\mu + p = \langle p \rangle + \mathbf{p} + p^\mu \). Likewise, tracers and buoyancy are decomposed and expressed as \( \mathbf{c} = c' + c^{lw} + c^\mu \) and \( \mathcal{B} = b' + b^{lw} + b^\mu \). At times we suppress the distinction between long waves and current by just defining waves and mean current. In this case we drop the superscript from the mean component.
As in MRL04, \( \langle \cdot \rangle \) signifies averaging over the wave time scale \( \tau \) and \( \langle \cdot \rangle \) signifies averaging over the longer \( \tau \) scale. Fluctuations are distinguished by
\[
(\cdot)' = (\cdot) - \langle \cdot \rangle \quad \text{and} \quad (\cdot)^\tau = (\cdot) - \langle \cdot \rangle.
\]
The long waves and the currents may be distinguished by averaging over the two different time scales so that \( \langle \mathbf{Q} \rangle = \mathbf{q}' \), \( \mathbf{Q} = \mathbf{q}' + \mathbf{q}^\tau \) and \( \mathbf{Q}' = \mathbf{q}'^\tau \).

3. Time-averaged equations of motion

a. MRL04’s current and long-wave equations

The mean Eulerian current equations in the vortex-force representation are
\[
\frac{\partial \mathbf{q}}{\partial T} + \left( \mathbf{q} \cdot \nabla \times \mathbf{w} \frac{\partial}{\partial z} \right) \mathbf{q} + \mathbf{f} \times \mathbf{q} + e^{-2z} \nabla_X \langle \mathbf{p} \rangle = -\nabla_X (e^{-2z} \mathcal{K}_1 + \mathcal{K}_2) + \mathbf{J},
\]
\[
e^{-2z} \frac{\partial \langle \mathbf{p} \rangle}{\partial z} = \frac{b}{L} - \frac{1}{\partial z} (e^{-2z} \mathcal{K}_1 + \mathcal{K}_2) + \mathbf{K},
\]
\[
\nabla_X \cdot \mathbf{q} + \frac{\partial w}{\partial z} = 0, \quad \text{and} \quad \frac{\partial c}{\partial T} = -(\mathbf{u} + \mathbf{w}^\tau) \cdot \nabla c.
\]

For full details, as well as the equations that capture the waves, the reader is referred to MRL04. The boundary conditions are
\[
\mathcal{K}_1 = \left\langle \frac{|\mathbf{u}_0|^2}{2} \right\rangle, \\
\mathcal{K}_2 = \left\langle \frac{|\mathbf{u}_0^w|^2}{2} + \mathbf{u}_0^w \cdot \mathbf{u}_0^w \right\rangle, \\
(J, K) = \left[ \langle (\mathbf{u} \times \mathbf{w})_0 \rangle, \langle (\mathbf{u} \times \mathbf{w})_0 \rangle \right], \\
(\mathbf{v}^\text{St}, w^\text{St}) = \left[ \left\langle \int_t^T \mathbf{u}_0^w \, dt \cdot \nabla \right\rangle \mathbf{u}_0, -\nabla_X \cdot \int_{-H}^z (\mathbf{v}^\text{St}) \, dz \right], \\
T^w = \left\langle \int_{-H}^z \mathbf{Q} \, dz \right\rangle = \int_{-H}^z \mathbf{Q} \, dz, \quad \text{and} \quad \frac{\mathcal{P}_0}{L} = \left\langle \frac{|\mathbf{Q}^w|^2}{2} \frac{\partial \langle \mathbf{p} \rangle}{\partial z} + \frac{e^{-1}}{2} \left( \frac{\xi}{L} \frac{\partial \langle \mathbf{p} \rangle}{\partial z} \right) \right\rangle,
\]

where \( \mathbf{w}^w = \nabla \times \mathbf{u}^w \) is the wave vorticity, which to lowest order is caused by the material advection of the mean vorticity field. The lowest-order momentum equations, (16)–(17), and the pressure boundary condition (20) represent a quasi-static balance between the wave stresses and the mean surface elevation and mean pressure. This is the phenomenon known as wave setup. Thus, we may separate the mean pressure and sea level into quasi-static and dynamic components as follows:
\[
\langle p \rangle = \tilde{\rho} + \nu^2 (\tilde{\rho}' + \tilde{\rho}^2) \quad \text{and} \\
\zeta = \tilde{\zeta} + \nu^2 (\tilde{\zeta}' + \tilde{\zeta}^2).
\]

Components with carets represent quasi-static balances while components with tildes are dynamic. The lowest-order quasi-static balance is
\[
\tilde{\rho} = -\tilde{\kappa}_1 - P_a = \left( \frac{\langle |A|^2 k \cosh[2k(z + H)] \rangle}{2L \sinh(2kH)} \right) - P_a
\]
and
\[
\tilde{\zeta} = L \left( \tilde{\rho} + \left( \tilde{\xi}^w \frac{\partial \tilde{\rho}^w}{\partial z} \right)_0 \right) = -\left( \frac{\langle |A|^2 k \rangle}{2 \sinh(kH)} \right) - LP_a,
\]
which comes in at \(O(e^3)\) in the horizontal momentum equations and \(O(e)\) in the vertical momentum equation. The higher-order balance is
\[
\tilde{\rho}' = -\tilde{\kappa}_2 = \frac{9}{64} \left( \frac{\langle |A|^2 \sigma^2 k^2 \cosh[4k(z + H)] \rangle}{\sinh^2(kH)} \right),
\]
and
\[
\tilde{\zeta}' = \left[ \frac{b}{L} - \frac{\partial}{\partial z} \tilde{\kappa}_2 + K \right] \left( \frac{\partial}{\partial z} \tilde{\rho}' + \left( \tilde{\xi}^w \frac{\partial \tilde{\rho}^w}{\partial z} \right)_0 \right) + \left( \frac{1}{L} \right) \tilde{\varphi}_0
\]
\[
\tilde{\zeta}' = L \left[ \tilde{\rho}' + \tilde{\zeta}' \frac{\partial \tilde{\rho}^w}{\partial z} + \left( \tilde{\xi}^w \frac{\partial \tilde{\rho}^w}{\partial z} \right)_0 \right] + \left( \frac{1}{L} \right) \tilde{\varphi}_0
\]
\[
(25)
\]
The pressure \(\tilde{\rho}\) differs from the historical wave-induced pressure \(p^w = -((w^w)^2)\) by \(\langle g|A|^2 k/[2 \sinh(2kH)] \rangle\), which is independent of \(z\). The advantage of using \(\tilde{\rho}\) rather than \(p^w\) is that \(\tilde{\rho}\) drops completely out of the wave-averaged momentum equations, even when the wave quantities vary over time.

Once the quasi-static components are removed the three-dimensional current equations in (16)–(19) are
\[
\frac{\partial q}{\partial \tau} + \left( q \cdot \nabla + w \frac{\partial}{\partial z} \right) q + f x \times q + \nabla \tilde{\rho}' = -\nabla \tilde{\kappa}_2 + \mathbf{J},
\]
\[
(26)
\]
\[
\frac{\partial \tilde{\rho}^c}{\partial z} = \frac{b}{L} - \frac{\partial}{\partial z} \tilde{\kappa}_2 + K,
\]
\[
(27)
\]
\[
\nabla \times \cdot q + \frac{\partial w}{\partial z} = 0, \quad \text{and}
\]
\[
\frac{\partial c}{\partial \tau} = -\left( \mathbf{u} + \mathbf{u}^{Sw} \right) \cdot \nabla c.
\]
\[
(28)
\]
\[
(29)
\]
The vertical momentum equation shows that in this scaling the pressure is in a quasi-hydrostatic balance with the Bernoulli head, buoyancy, and vertical vortex force.

To lowest order, the boundary conditions are
\[
w(0) = \nabla \times \cdot \mathbf{T}^w,
\]
\[
\frac{\partial \tilde{\rho}^w(0)}{\partial z} - \frac{\tilde{\varphi}_0}{L} = -\frac{\tilde{\varphi}_0}{L}, \quad \text{and}
\]
\[
w(-H) + \mathbf{q}(-H) \cdot \nabla \chi H = 0.
\]
As derived in MRL04, \(\tilde{\kappa}_2, \mathbf{J}, K, \mathbf{T}^w\), and \(\tilde{\varphi}_0\) may be computed from the wave quantities:
\[
\tilde{\kappa}_2 = \frac{1}{4} \int \frac{\langle |A|^2 \sigma^2 k^2 \cosh[4k(z - z')] \rangle}{\sinh^2(kH)} dz' + \frac{1}{2} \langle \langle \mathbf{q}^w \rangle^2 \rangle,
\]
\[
(30)
\]
\[
(\mathbf{J}, K) = -z \times \langle \mathbf{v}^{Sw} \rangle (\chi^c + f) - \langle \mathbf{v}^{Sw} \rangle \mathbf{e}_x \cdot \langle \mathbf{v}^{Sw} \rangle \mathbf{e}_y ,
\]
where \(\langle \mathbf{e}_x, \mathbf{e}_y \rangle\) is the mean Eulerian vorticity, which, to lowest order, is
\[
\frac{\partial}{\partial \tau} - \frac{\partial}{\partial z} \left( \frac{\partial \tilde{\varphi}_w}{\partial x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial \tilde{\varphi}_w}{\partial y} \right).
\]

The Stokes velocity is
\[
\mathbf{w}^{Sw} = \frac{\langle |A|^2 \sigma \cosh[2k(z + H)] \rangle \mathbf{k}}{2 \sinh^2(kH)}
\]
\[
\mathbf{w}^{Sw}\] is as defined in (22), \(\tilde{\varphi}_0\) is given in MRL04’s (9.11), and \(\chi^c = \chi \cdot \mathbf{q}\). Note that the quantities in (30) differ slightly from those given previously in (22). We have used the gauge invariance to ensure that the definition of the vortex force is consistent with previous definitions (MRL04, their section 9.6).

The equations for the long waves are
\[
\frac{\partial q^w}{\partial \tau} + \nabla \times \cdot q^w = -\nabla \times \frac{1}{2} \langle \mathbf{u}^w \rangle^2,
\]
\[
(31)
\]
\[
\frac{\partial p^w}{\partial z} = -\frac{1}{\tilde{\kappa}_2} \langle \mathbf{u}^w \rangle^2,
\]
\[
\mathbf{w}^w(-H) + \mathbf{q}^w(-H) \cdot \nabla \chi (H) = 0,
\]
\[
\frac{\partial w^w(0)}{\partial \tau} - \frac{\partial \mathbf{w}^w}{\partial z} = \nabla \times \left( \mathbf{e}_x \cdot \mathbf{q}^w(0) \right),
\]
\[
\mathbf{p}^w(0) = -\frac{\tilde{\kappa}_2}{L} \mathbf{w}^w(0),
\]
\[
(31)
\]
where

\[
\bar{\zeta}^w = \left( \frac{1}{2} \left[ |A|^2 \sinh^2(k(z + H)) \right] + \frac{1}{2} \left[ \sinh^2(kH) \right] \right)^{\frac{1}{2}}.
\] (32)

We have included the long-wave material tracer equation (analogous to the long-wave buoyancy equation if buoyancy is taken to be a passive tracer), although it is not asymptotically coupled to the long-wave or current equations. It contains a vertical wave diffusion term that vanishes after long time averaging and thus does not appear in the corresponding current equation (19).

Once the quasi-static component is removed (as was done to the current equations above), it is apparent that the long waves are barotropic to lowest order, obeying

\[
\frac{\partial q^w}{\partial \tau} + \frac{1}{L} \nabla_X \tilde{q}^w = 0 \quad \text{and} \quad \frac{\partial \tilde{q}^w}{\partial \tau} + \nabla_X \cdot (Hq^w) = -Lg^w,
\] (33)

where

\[
g^w = \frac{1}{L} \left[ \nabla_X \cdot (T^w) \right] - \frac{\partial \tilde{q}^w}{\partial \tau}
\]

\[
= \frac{1}{2L} \left[ \nabla_X \cdot \left[ \frac{|A|^2 k}{k \tanh(2kH)} \right] + \frac{\partial}{\partial \tau} \left[ \frac{|A|^2 k}{\sinh(2kH)} \right] \right]^{\frac{1}{2}}.
\]

Accordingly, to lowest order, the only effects of the waves on the long waves are the wave setup and the influence of the waves on the dynamic surface elevation; the \(O(1)\) vertically integrated momentum equation is only affected by the rise and fall of the sea level due to the long waves themselves.

\[b. \text{Comparison of the equations in the different representations}\]

Given the wave decomposition, current scalings, and nondimensionalization in section 2b, the radiation-stress representation mean Eulerian equations derived from (9) and (4) are

\[
\varepsilon^3 \frac{\partial q^w}{\partial T} + \varepsilon^3 \frac{\partial q^w}{\partial X^\beta} + \varepsilon^3 w \frac{\partial q^w}{\partial z} + \varepsilon^3 \frac{\partial (p)}{\partial X^\alpha} + \varepsilon^3 (fz \times q^w) = -\varepsilon^3 \frac{\partial}{\partial X^\beta} (q^w q^\alpha) - \varepsilon \frac{\partial}{\partial z} (q^w q^w) + \varepsilon^3 \frac{\partial (p)}{\partial X^\alpha} + \varepsilon^3 (fz \times q^w),
\] (34)

\[
\varepsilon^4 \frac{\partial w}{\partial T} + \varepsilon^4 q^\beta \frac{\partial w}{\partial X^\beta} + \varepsilon^4 w \frac{\partial w}{\partial z} + \varepsilon^4 \frac{\partial (p)}{\partial X^\alpha} - \varepsilon \frac{\partial}{\partial z} \left( \frac{1}{L} \varepsilon^3 \right) = -\varepsilon^4 \frac{\partial}{\partial X^\beta} (q^w q^\alpha) - \varepsilon \frac{\partial}{\partial z} (w^w), \quad \text{and} \quad (35)
\]

Similarly, the mean Eulerian tracer equation derived from (5) becomes

\[
\varepsilon^3 \frac{\partial q^w}{\partial T} + \varepsilon^3 q^\beta \frac{\partial q^w}{\partial X^\beta} + \varepsilon^3 w \frac{\partial q^w}{\partial z} + \varepsilon^3 \frac{\partial (p)}{\partial X^\alpha} + \varepsilon^3 (fz \times q^w) = -\varepsilon^3 \frac{\partial}{\partial X^\beta} (q^w q^\alpha) - \varepsilon^3 \frac{\partial}{\partial z} (q^w q^w) + \varepsilon^3 \frac{\partial (p)}{\partial X^\alpha} + \varepsilon^3 (fz \times q^w).
\] (36)

The lowest order in (34) is actually \(O(\varepsilon)\) as \(q^w q^w \sim O(\varepsilon^2)\). Likewise, \(q^w w^w \sim O(\varepsilon^2)\), and thus (37) occurs at \(O(\varepsilon^4)\). We keep this notation, however, to indicate how many wave terms need to be calculated in order to evaluate these quantities.

Following MR104, we consider the horizontal momentum equations up to \(O(\varepsilon^4)\), the vertical momentum equation up to \(O(\varepsilon^3)\), and the mass and tracer equations to their lowest respective orders. The boundary conditions are the same as given for the vortex-force representation (20)–(21).

Equating the wave forcing in the radiation-stress representation (34)–(37) with that in the vortex-force representation (16)–(19), we can see that

\[
-\varepsilon^2 \left( \nabla_X \cdot (q^w q^w) + \frac{\partial}{\partial z} (q^w q^w) \right) = -\nabla_X (\varepsilon^2 K_1 + K_2) + J.
\] (38)

Likewise, comparing the two representations of the vertical momentum equations leads to

\[
-\left( \nabla_X \cdot (w^w q^w) + \frac{\partial}{\partial z} (w^w q^w) \right) = -\frac{\partial}{\partial z} (\varepsilon^2 K_1 + K_2) + K.
\] (39)

Thus, the lowest-order radiation-stress divergence terms balance the quasi-static pressure gradient. We also see that for the tracers

\[
\nabla \cdot (\varepsilon^w u^w) = u^{\alpha \beta} \cdot \nabla \zeta;
\]

that is, divergence of the wave tracer flux is equal to advection of the mean tracer by the three-dimensional Stokes velocity.

If, instead of averaging over time scales characteristic of the currents, we average over long-wave scales, we obtain the following equation in the radiation-stress representation:
\[
\frac{\partial q_{\alpha}^{lw}}{\partial \tau} + \frac{\partial p_{\alpha}^{lw}}{\partial X_{\beta}} = -\frac{\partial}{\partial X_{\beta}} (q_{\alpha}^{w} q_{\beta}^{w})^\dagger + e^{-\gamma} \frac{\partial}{\partial z} (q_{\alpha}^{w} w_{\alpha}^{w})^\dagger, \\
\frac{\partial p_{\alpha}^{lw}}{\partial z} = -\frac{\partial}{\partial z} [(w_{\alpha}^{w})^2]^\dagger, \quad \text{and} \\
\frac{\partial c_{\alpha}^{lw}}{\partial \tau} + \frac{\partial c_{\alpha}}{\partial X_{\beta}} + w_{\alpha}^{w} \frac{\partial c_{\alpha}}{\partial z} = -\frac{\partial}{\partial X_{\beta}} (c_{\alpha}^{w} q_{\alpha}^{w})^\dagger - e^{-\gamma} \frac{\partial}{\partial z} (c_{\alpha}^{w} w_{\alpha}^{w})^\dagger, \quad (40)
\]

with the mass equation and boundary conditions still being given by (31). Similarly to the currents, \((q_{\alpha}^{w} w_{\alpha}^{w})^\dagger \sim O(e^2)\), so these terms come in at order one.

The radiation-stress representation, (40), is not clear as to the effect of the waves. In contrast, it is easy to identify the quasi-static pressure and wave setup in (31). With the quasi-static components removed the vortex-force representation (33) makes it readily apparent that the long waves are barotropic. Furthermore, the wave effects on the tracer equation are identified as Stokes advection and vertical wave diffusion.

4. Vertically integrated, time-averaged equations

It is in the context of depth-integrated circulation equations that the concept of radiation stress was conceived and studied by LHS. Here we derive the transport equations using the vortex-force and radiation-stress representations. We use the approach adopted in H71 of separating the total transport \(\langle T \rangle\) into \(T^c\) and \(T^{\text{w}}\), the corresponding current and wave terms. We compare our results with H71, given our scaling assumptions.

a. Transport

In H71 the transport is decomposed into three quantities—the total transport \(\langle T \rangle\) (this is \(\overline{M}\) in H71), the vertically integrated momentum of the mean flow \(T^c\) (\(M^m\) in H71), and the vertically integrated momentum of the surface layer \(T^{\text{w}}\) (\(M^w\) in H71). The three quantities are related by

\[
\langle T \rangle = \left\langle \int_{-H}^{E} Q \, dz \right\rangle = \int_{-H}^{E} q \, dz \\
+ \left\langle \int_{E}^{E} Q \, dz \right\rangle = T^c + \langle T^{\text{w}} \rangle, \quad (41)
\]

where \(\langle T \rangle\) is the average of the total velocity over the total sea depth and, as such, contains a wave component. It can likewise be interpreted as the Lagrangian mean velocity integrated over the mean sea depth. In the same manner, \(T^{\text{w}}\) can be interpreted as a purely surface phenomenon occurring between the mean and fluctuating sea levels, or it can be interpreted as the Stokes drift velocity integrated over the mean sea depth.

The asymptotic treatment in MRL04 naturally leads to the flow being decomposed into primary waves, long waves, and current. When using this decomposition, we denote the total vertically integrated, time-averaged momentum as \(\overline{T}\). Thus, the decomposition given by \(\overline{T}\) is

\[
\overline{T} = T^c + T^{\text{w}} + \overline{T^{\text{w}}}, \quad (42)
\]

where \(\overline{T}\) as defined above and

\[
\overline{T^{\text{w}}} = \int_{-H}^{E} \int_{E}^{E} Q \, dz \, dz + \int_{-H}^{E} \int_{-H}^{E} Q \, dz \, dz,
\]

For LHS, the radiation stress \(S_{\alpha\beta}\) is the excess momentum flux in the presence of waves. A slightly different definition of radiation stress is used in H71, wherein it is denoted by \(\tau_{\alpha\beta}^{\text{rad}}\). The main difference is the sign, but there are several other subtle differences. These are enumerated in appendix A. Radiation stress can be further decomposed into an interaction stress \(\tau_{\alpha\beta}^{\text{int}}\) which acts on the bulk of the fluid, and a surface layer stress \(\tau_{\alpha\beta}^{\text{sl}}\). These three stresses are related by

\[
\tau_{\alpha\beta}^{\text{rad}} = \tau_{\alpha\beta}^{\text{int}} + \tau_{\alpha\beta}^{\text{sl}}.
\]

where

\[
\tau_{\alpha\beta}^{\text{int}} = -\left\langle \int_{-H}^{E} (q_{\alpha} q_{\beta}^{w} + \delta_{\alpha\beta} p^{w}) \, dz \right\rangle \quad \text{and} \quad \tau_{\alpha\beta}^{\text{sl}} = -\left\langle \int_{E}^{E} (Q_{\alpha} Q_{\beta} + \delta_{\alpha\beta} p) \, dz \right\rangle.
\]

The wave effects on the evolution of \(\langle T \rangle\) are obtained by taking the divergence of the radiation stress. In H71 it is shown that the divergence of the interaction stress accounts for the wave effects on the evolution of \(T^c\). Note that we use the term radiation-stress representation to refer to equations of either type.

The mean current stress is defined in H71 to be

\[
\tau_{\alpha\beta}^{m} = -\left\langle \int_{-H}^{E} (q_{\alpha} q_{\beta}^{w} + \delta_{\alpha\beta} p^{w}) \, dz \right\rangle.
\]

H71’s radiation stress and interaction stress can also be related via
\[ \tau_{\alpha\beta} = - \left( \int_{-H}^{X} (Q_{\alpha} Q_{\beta} + \delta_{\alpha\beta} \rho) \, dz \right) - \tau_{\alpha\beta}^{m}, \]

where \( X = E \) for the radiation stress and \( X = \zeta \) for the interaction stress.

The dynamic mean pressure is decomposed into a wave contribution \( \rho^{w} \) and a current contribution \( \rho^{m} \) in H71. This is done by requiring that \( \rho^{m} \) be the dynamic pressure in the absence of waves. We differ from H71 in this regard. We decompose the mean pressure into quasi-static (owing to the waves) \( \bar{\rho} + \epsilon^{2} \rho^{w} \), and dynamic mean pressure \( \bar{\rho}^{c} \), as defined in section 3. Although these two decompositions are defined differently, in the limit of deep water and low Rossby number they are similar. Also, as discussed in appendix A, the quasi-static and dynamic mean pressure used here are easily related to numerical or field data and have a clear ordering within our scaling regime.

b. **Vortex-force representation**

Because our vortex-force representation gives equations for mean Eulerian quantities, it makes sense to compare the evolution equation of \( T^{c} \) in each of the frameworks; \( T^{c} \) represents the mean Eulerian velocity integrated over the mean ocean depth. As such, the vortex force and Bernoulli head are most immediately comparable to the divergence of H71’s interaction force.

We obtain the time evolution equation for \( T^{c} \) in vortex-force representation by vertically integrating (26) and making use of the quasi-static pressure relationship (27):

\[ \frac{\partial T^{c}}{\partial T} - \frac{\partial \tau_{\alpha\beta}^{m}}{\partial X_{\beta}} \right\} = \epsilon^{-2} \frac{\partial \tau_{\alpha\beta}^{m}}{\partial X_{\beta}} + \left\{ \epsilon^{-2} (q_{\alpha}^{0}(w^{0}(0)) - (q_{\alpha}^{1w}(0)) + \delta_{\alpha\beta} \rho) \right\}.

\]

where the wave terms are given by (22) or (30). The effect of the waves can be broken into three parts. The first term in the square brackets is a momentum transfer owing to mass influx from the surface layer, the second term is a Bernoulli head evaluated at the mean surface, and the third term is a vertically integrated vortex force.

\[ \frac{\partial T^{c}}{\partial T} = \frac{\partial \tau_{\alpha\beta}^{m}}{\partial X_{\beta}} \right\} = \epsilon^{-2} \frac{\partial \tau_{\alpha\beta}^{m}}{\partial X_{\beta}} + \left\{ \epsilon^{-2} (q_{\alpha}^{0}(w^{0}(0)) - (q_{\alpha}^{1w}(0)) + \delta_{\alpha\beta} \rho) \right\}.

\]

c. **Comparison of the different representations**

We find the evolution equation for \( T^{c} \) in radiation-stress representation by vertically integrating the phase-averaged (34) over the mean vertical depth \([-H, 0]\).

Assuming the scaling given in section 2b, the evolution equation for \( T^{c} \) reads

\[ \frac{\partial T^{c}}{\partial T} = \frac{\partial \tau_{\alpha\beta}^{m}}{\partial X_{\beta}} \right\} = \epsilon^{-2} \frac{\partial \tau_{\alpha\beta}^{m}}{\partial X_{\beta}} + \left\{ \epsilon^{-2} (q_{\alpha}^{0}(w^{0}(0)) - (q_{\alpha}^{1w}(0)) + \delta_{\alpha\beta} \rho) \right\}.

\]

Note that \( (q_{\alpha}^{0}(0)w^{0}(0)) \) is \( O(\epsilon^{2}) \). Thus, the lowest-order terms are \( O(\epsilon^{-2}) \).

To facilitate comparison with (44) we rewrite (43) as retaining the lowest-order terms.

The wave effects appearing within the square brackets on the right-hand sides of (44) and (45) are equal:
recalling \((q_a(0)w_a(0)) \sim O(\varepsilon^2)\). The left-hand side is the radiation-stress representation and the right-hand side is the vortex-force representation. The last term on the left-hand side and the first term on the right-hand side of (46) cancel trivially. The three terms on the left-hand side with order lower than one cancel, giving an order-one residual. In the vortex-force representation, however, \(\hat{\rho}_l\) and \(X_l\) cancel exactly, leaving no lower order terms.

We may also compare (45) with the equation derived by Garrett (1976) for wave effects on currents in deep water [his (3.9) and (3.11), noting the erratum that (3.11) should read \(-q\mathbf{\nabla} \cdot \mathbf{T}^w + \mathbf{T}^w \times (\mathbf{V} \times \mathbf{q})\) when given in our notation]. Garrett does not include rotational effects, assumes the mean current is independent of depth, and, because of the deep water assumption, includes no topographic variation. Thus, (45) becomes

\[
\frac{\partial T_a}{\partial t} = \frac{\partial T_a}{\partial X_a} + q_a(0)(w^S(0)) - \frac{\partial}{\partial X_a} \int_{-H}^{0} \hat{X}_2 dz' + \int_{-H}^{0} J_a dz'.
\]

The only difference between this and Garrett’s equation is the pressure adjustment involving \(\hat{X}_2\), but as noted earlier the pressure decomposition used by H71 (and also by Garrett) is not the same one used here; furthermore, \(\hat{X}_2\) is higher order than what Garrett evaluated. Taking this difference into account the two equations concur. Thus, the equations derived in MRL04 are a more general version of what is derived in Garrett (1976).

d. Energetics

In H71 an equation is derived for the evolution of the current energy that includes the wave contribution. We repeat this derivation within the context of the vortex-force representation. We derive wave-averaged energy conversion terms for the quasi-static response due to pressure and sea level changes (i.e., wave setup), the long waves, and the currents. In each case we respect the asymptotic scaling by not including terms at higher order than the magnitude of the energy changes for the wave-averaged components. This is only a partial view of the total energetics since the sum of the component energies is not the total energy, even if the energy of the primary waves is included (i.e., there are cross terms among the components in the total energy balance). Only energy exchange through conservative mechanisms is considered.

We define the quasi-static response, \(\hat{\rho}(z)\) and \(\hat{\xi}\), as the sum of the contributions from the long waves (MRL04, section 6) and currents [(23)] that are quasi-statically balanced with the wave-averaged terms that do not include long-wave velocity, currents, or Stokes drift. Given this definition, the only contribution to a quasi-static energy \(E^\xi\) is through the sea level potential energy,

\[
E^\xi = \int_G \frac{g}{2} \xi^2 d\mathbf{x}.
\]

The associated energy balance equation has a forcing equal to that of the area-integrated time derivative of the diagnostic forcing terms for \(\hat{\xi}\).

The long-wave dynamics (MRL04, their section 6) imply the following energy balance:

\[
\frac{dE^{lw}}{dt} = \frac{d}{dt} \int_G \left[ \frac{H}{2} |q|^{lw^2} + \frac{g}{2} (\xi^{lw})^2 \right] d\mathbf{x} = -\int_G \tilde{\xi}^{lw} \tilde{\xi}^{lw} d\mathbf{x},
\]

excluding the quasi-static component.

While we derive the energy balance for the currents, noting that both the three-dimensional \(\mathbf{u}\) and \(\mathbf{u}^S\) (augmented with the vertical Stokes pseudovelocity) are nondivergent.

The energy equation associated with the current is
assuming appropriate lateral boundary conditions that preclude energy flux. The first term on the right-hand side is due to the Coriolis effect. The second and third conversion terms are analogous to familiar shear production and potential energy conversion terms except here they involve the Stokes drift. The surface conversion terms are unfamiliar but appear to be due to mass influx from the waves.

The energy equation associated with the currents can also be written as

\[
\frac{d\mathcal{E}}{dt} = \int_G \int_{-H}^{0} \left\{ \begin{array}{l}
\frac{1}{2} |\mathbf{q}(0)|^2 - \frac{1}{2} |\mathbf{q}(z)|^2 \cdot \mathbf{b} + \frac{1}{2} \left[ (\mathbf{u}' \times \mathbf{u})^T \right] (\mathbf{v} \times \mathbf{u}) - b' w^2 \right\} dz \ dx
\end{array} \right. + \int_G w^2(0) \left[ \frac{1}{2} |\mathbf{q}(0)|^2 + g\tilde{\zeta} + \tilde{\zeta}_0 + \tilde{\chi}(0) \right] dx.
\]

(49)

This equation shows that, if the current and the Stokes drift are parallel or antiparallel, the first term vanishes and only the potential energy conversion term and the surface conversion term remain.

\[
\frac{d\mathcal{E}}{dt} = \int_G \int_{-H}^{0} \left\{ -w(0) \left[ \frac{|\mathbf{q}|^2}{2} + e^{-2}(p) + (w^2) \right] - e^{-2} \mathbf{q}(0) \cdot (\mathbf{q} w^2) + \int_{-H}^{0} (w^2 \mathbf{q}^T) \cdot \nabla w + e^{-2} (\mathbf{q} w^2) \cdot \frac{\partial \mathbf{q}}{\partial z} \right\} dx.
\]

(51)

This is essentially (22) in H71 given the assumption of the scaling described in section 2b. The terms up to the plus sign before the single integral are the work done by the surface interaction stress. The next four terms make up what H71 called the “dissipation” term. The final term is the potential energy conversion term, which is the same as in (49).

A comparison between (49) and (51) [noting \( w^2(0) = -w(0) \)] shows that

\[
\int_G \int_{-H}^{0} \left\{ (w^2 \mathbf{q}^T) \cdot \nabla w + e^{-2} (\mathbf{q} w^2) \cdot \frac{\partial \mathbf{q}}{\partial z} \right\} dz \ dx
\]

\[
- e^{-2} \int_G w(0)(p)_{z=0} + \mathbf{q}(0) \cdot (\mathbf{q} w^2)_{x=0} dx.
\]

(52)
5. Illustrative examples

In MRL04 (their section 13) we show how a barotropic vortical current and waves interact on a finite-depth water sloped basin for \( \varepsilon = 0.04 \). While a great many simplifications went into producing that example, the vortex-force representation makes it very easy to identify which aspects of the interaction are consequential for the current evolution. In Fig. 1 we illustrate the differences between the two representations by showing the radiation-stress and vortex-force terms that would appear in the \( x \)-momentum equation. The gradient of the radiation stress divided by the depth is shown in Fig. 1a. If we compare this with Fig. 1b, the gradient of the wave setup due to the lowest-order Bernoulli head, we can see that these quantities cancel each other at lowest order in the steady-state case [cf. (53)]. By contrast, the vortex force is shown in Fig. 1c (excluding Coriolis force) and Fig. 1d (including Coriolis force).

The radiation stress clearly does not capture these effects to its lowest order. These examples show how the mean wave effects (exemplified by the vortex force) are tied to the vorticity of the currents. In the absence of Coriolis force the wave effect is basically the advection of the vorticity by the Stokes drift. When planetary rotation is included in the model, the vortex force includes a sort of Stokes–Coriolis term (Huang 1979). In this sense the vortex-force representation might be seen as concisely identifying the wave effects on the currents. By comparison, the radiation-stress representation may incorporate a variety of phenomena, including wave effects on waves as well as currents, and it does not distinguish between the Bernoulli head and the vortex force and to lowest order cancels out with the quasi-static wave setup due to the Bernoulli head.

In what follows we consider two other examples that highlight important consequences related to differences between the radiation-stress and vortex-force represen-
tations as well as issues related to the relative asymptotic balances. Complementary commentary will be followed in the discussion section.

a. Lowest-order radiation and interaction stress

To illustrate the role played by the lowest-order radiation and interaction stresses in the vertically integrated momentum equations we consider their divergences. The divergence of LHS’s radiation stress (A5) is

\[
\frac{\partial S_{a\beta}}{\partial X_\beta} = -\frac{\partial (T'_{a\alpha})}{\partial T} \frac{H}{L} \frac{\partial \tilde{\xi}}{\partial X_a} + O(\varepsilon^2),
\]

(53)

where \(\langle T'_{a\alpha}\rangle\) is defined in section 4a and makes use of the wave evolution equations in MRL04 [(5.22) and (5.37)]. A similar result is obtained when H71’s radiation stress \(\tau_{a\beta}^{rad}\) is used (appendix A). Thus, the dominant dynamic role of the radiation-stress divergence is in balancing the evolution of the wave momentum, while its static effect is balancing the wave setup, as seen in Fig. 1.

To lowest order the interaction stress is

\[
\tau_{a\beta}^{int} = \left( \sum_{\mu} \frac{1}{4\tilde{L}} \right) \left( \left| \phi \right|^2 \right) \sinh(2kH) + \delta_{a\beta} \left( \frac{1}{4\tilde{L}} P_c + P_{cH} \right).
\]

(54)

The divergence of (54) is

\[
\frac{\partial \tau_{a\beta}^{int}}{\partial X_\beta} = \frac{\partial (T'_{a\alpha})}{\partial T} + \frac{1}{4\tilde{L}} \left( \frac{\partial |\phi|^2}{\partial X_a} - \frac{\partial H}{\partial X_a} \right) + O(\varepsilon^2).
\]

(55)

Like the radiation stress, the divergence of the interaction stress is equal to the time evolution of the \(T'_{a\alpha}\) plus boundary terms. The evolution equation for \(T'\) (44), does not involve the time evolution of \(T'_{a\alpha}\). Using wave quantities we may also evaluate

\[
-\varepsilon^2 \langle q_{a\alpha}^{(0)}w\rangle = -\frac{\partial (T'_{a\alpha})}{\partial T} - \frac{1}{4L} \frac{\partial (|\phi|^2)}{\partial X_a} + O(\varepsilon^2).
\]

(56)

Substituting (55) and (56) into (44) shows that the lowest-order divergence of the interaction stress cancels out with other boundary terms, leaving only higher order effects.

If we account for the long waves separately, there is an interaction stress associated with them to lowest order. Its only effect is the static pressure adjustment (i.e., wave setup). Once the static component is removed, there is no other contribution by the interaction stress to the long-wave dynamics. If we consider the combined effect of the long waves and the primary gravity waves on the current, the long waves do not contribute to either stress at lowest order. Their effects are only seen in the stresses at higher orders.

As first done by Garrett (1976), \(\partial T'/\partial T\) may be derived as the difference,

\[
\frac{\partial T}{\partial T} - \frac{\partial (T'_{a\alpha})}{\partial T}.
\]

From (B5) and (53),

\[
\frac{\partial T}{\partial T} = [\nabla \times T' + f \times T + p'(-H) \nabla \times H] - \nabla \times S^{(2)} - f \times T' = \frac{1}{2L} \nabla \times \tilde{s}_2^2 - \frac{\partial}{\partial X_\mu} [q(0)T'_{\mu} + q_{\mu}(0)T'_{\gamma}],
\]

(57)

where \(S^{(2)}\) is the \(O(\varepsilon^2)\) radiation stress as calculated with our scalings in (A6) and (A7). As in (43), only the \(O(\varepsilon^2)\) wave components are required for the evaluation of (57). However, the challenge of calculating higher-order radiation-stress terms persists. Furthermore, the result is simply that given by the vortex-force representation as seen by comparison of (57) with (45).

b. Linear, rotating, stratified wave-averaged dynamics

To further expose what is forced by waves in the radiation-stress representation, apart from the elements closely tied to the vortex force, we now analyze a problem with certain nonlinear effects suppressed. Specifically, we step outside the asymptotic scaling (and accordingly revert to dimensional equations) and form a composite of the preceding long-wave and current dynamical balances without the quadratic terms related to Eulerian and Stokes advection. This is a type of linearized dynamics about a rotating, stratified, resting state with \(B(z)\) its buoyancy stratification. After removing quasi-static sea level \(\xi\) and pressure \(\bar{p}\) fields (with the latter augmented by \(\rho_0 \int B \, dz\)), the wave-averaged equations are
\[
\frac{\partial \mathbf{q}}{\partial t} + f \mathbf{z} \times \mathbf{q} + \frac{1}{\rho_0} \nabla p = -f \mathbf{z} \times \mathbf{v}^S, \\
\frac{1}{\rho_0} \frac{\partial p}{\partial z} - b = 0, \\
\nabla \cdot \mathbf{q} + \frac{\partial \mathbf{w}}{\partial z} = 0, \\
\frac{\partial b}{\partial t} + w \frac{\partial B}{\partial z} = -w \frac{\partial \mathbf{w}}{\partial z} + \frac{1}{2} \frac{\partial}{\partial z} \left[ \left( \frac{\partial}{\partial t} \mathbf{v} \right) \cdot \mathbf{w} \right].
\]
\[
w(-H) + \mathbf{q}(-H) \cdot \nabla H = 0, \\
w(0) - \frac{\partial \mathbf{z}}{\partial t} = \nabla \cdot \mathbf{T}^{\mathbf{w}} + \frac{\partial \mathbf{z}}{\partial t}, \quad \text{and} \\
p(0) - g \mathbf{z} = 0. \tag{58}
\]

Strikingly, this system has no wave forcing. For constant \(f\) its solutions are of the form
\[
(q', w') = \left[ \mathbf{z} \times \frac{1}{f \rho_0} \nabla (g \mathbf{z} - \int_0^z b \, dz'), 0 \right]
\]
and
\[
\mathbf{q}'(-H) \cdot \nabla H = 0.
\]

These are determined by the boundary conditions of the problem.

Thus, the only steady, wave-forced current is the anti-Stokes flow represented by the first components in (59). Note that this local anti-Stokes flow is linked with finite rotation and stratification. In the special situation where \(f = B = 0\), the equations are essentially those given by Bühler and McIntyre (2003). In this case \(\mathbf{q}\) is barotropic and, assuming irrotationality for both waves and currents, the steady state of (58) is found through
\[
\nabla \cdot (H \mathbf{q}) = -\nabla \cdot \mathbf{T}^{\mathbf{w}},
\]
which is a nonlocal form of anti-Stokes flow. This in not the generic case, however, since the existence of even very small nonzero values of \(f\) or \(B\) breaks the irrotationality assumption, reverting to the local steady-state solution.

In summary, the linearized dynamical response to the conservative wave effects that are contained in the radiation-stress representation, not including the vortex force, is limited to the following classes of phenomena: wave setup, forced propagating (long) waves, and steady anti-Stokes flow. Only the last of these is a “current” in the sense of a flow with a slow evolutionary time scale. Of course, in a more general situation neither the waves nor the currents are steady in time and advective dynamics are not negligible, so the currents

The notation should be interpretable from the previous sections; \(\mathbf{v}^r\) is defined in (32). The wave-averaged effects appear on the right-hand side and thus serve as forcing terms for the long-wave and current responses.

When the wave fields vary in time on an intermediate time scale (previously denoted by \(\tau\)), the principal response in (58) is forced propagating waves. The low-frequency wave behavior implicit in the left-hand side are a combination of long, shallow-water surface gravity waves, internal waves on the stratification \(B(z)\), and rotational waves due to \(f\).

To focus instead on the long-time response, we assume that the wave field is in steady state and set \(\partial_t = 0\) in (58). We further decompose the steady velocity field by
\[
\mathbf{q} = -\mathbf{v}^S(z) + \mathbf{q}', w = \nabla \cdot \int_{-H}^z \mathbf{v}^S(z') \, dz' + w'. \tag{59}
\]

The resulting equations for \((\mathbf{q}', w')\) are the following:
\[
\begin{align*}
&f \mathbf{z} \times \mathbf{q}' + \frac{1}{\rho_0} \nabla p = 0, \\
&\frac{1}{\rho_0} \frac{\partial p}{\partial z} - b = 0, \\
&\nabla \cdot \mathbf{q}' + \frac{\partial \mathbf{w}'}{\partial z} = 0, \\
&w' \frac{\partial B}{\partial z} = 0, \\
&w'(-H) + \mathbf{q}'(-H) \cdot \nabla H = 0, \\
w'(0) = 0, \quad \text{and} \\
p(0) - g \mathbf{z} = 0. \tag{60}
\end{align*}
\]
will exhibit a broader range of wave-influenced behavior than anti-Stokes flow.

Though certain nonlinear effects and pressure adjustments have been suppressed in this example, the vortex-force representation makes it clear that the crucial aspects of wave effects on currents is complete: see (38) and (39). Both representations address the same physics; however, it is less obvious from the divergence of a radiation stress what aspects of the wave/current interaction are important. Moreover, if the leading-order radiation or interaction stress does not include the vortex force, it captures only a limited part of the wave-averaged dynamical effects. This could be the case if our scaling is applicable to the physical situation at hand.

6. Discussion

a. Asymptotic consistency

Although both radiation-stress and vortex-force representations give valid expressions for the conservative effects of waves on currents, the vortex force and the Bernoulli head provide a simpler and more specific physical interpretation than the divergence of the radiation or interaction stress. Furthermore, comparisons with the vortex-force representation in (38), (39), and (46) show that, in both the three-dimensional and vertically integrated current equations, the lowest-order stress divergence simply balances the quasi-static pressure gradient; it does not include vortex-force effects. The same is true in the current energy equation in (52). The vortex-force effects must be evaluated as the higher-order difference between stress divergence and boundary terms. In this way it can be said that the radiation-stress representation is asymptotically inconsistent.

To obtain meaningful results from the radiation-stress representation in this scaling, the primary waves need to be calculated up to $O(\varepsilon^4)$. In the vortex-force representation we circumvent this requirement by calculating the wave vorticity to $O(\varepsilon^2)$ and making use of the irrotationality to $O(\varepsilon^2)$. Although there are other ways to circumvent this requirement [e.g., see (57)], the asymptotic analysis has already been done in MRL04.

In contrast, the long-wave equations do not include the vortex force; thus the radiation-stress representation does capture the leading-order dynamics [see (33); cf. (40)], but its usage is still potentially confusing. The vortex-force representation clearly separates the quasi-static and dynamic response components, making it obvious that the long-wave response is barotropic. This is not as evident using the radiation-stress representation.

b. An alternative scaling

Although we have concentrated on the specific scaling used in MRL04 in this paper, the applicability of the vortex-force representation is wider. To see this we revert to a more general scaling, using $\delta$ and $\beta$ as defined in section 2b. We assume that the waves are irrotational to lowest order, $\varepsilon \ll 1$, and that their phase-averaged properties only vary slowly; that is, $\beta \ll 1$ [N.B., $\delta = O(\varepsilon)$ and $\beta = O(\varepsilon^2)$, with MRL04 scaling].

The wave vorticity comes from either the rotational currents or planetary rotation. If the currents are the source of vorticity, then, to lowest order,

$$\xi^w = \varepsilon \delta z \times (\nabla_x z \cdot \xi^w) \times \int t q^w \, dt$$

and

$$\chi^w = -\varepsilon \delta \nabla_x \cdot (\xi^w \int t w^w \, dt),$$

where $\xi$ and $\chi$ are the horizontal vector and vertical vorticities, respectively. The vortex-force term is given by $\varepsilon(\xi^w \chi^w \times u^w)$—the wave-averaged cross product of the wave vorticity and the wave velocity. Because the wave vorticity and velocity are in quadrature, the average of the horizontal component of the vortex force is demoted from its apparent order by a factor of $\eta$ [N.B., in MRL04, $\eta = O(\varepsilon^2)$]. This is also the nondimensional order of the term $(q^w w^w)$ that appears in the radiation stress. If it is planetary rotation that provides the vorticity, then

$$(\xi^w, \chi^w) = -\nabla \times \left( f^{rad} z \times \int t q^w \, dt \right),$$

with the vertical component denoted by $\eta$.

Consider the order of the vortex-force and radiation-stress (and thus also interaction stress) terms in the horizontal and vertical directions. In the horizontal velocity equation, the vortex force enters at $O[\max(\varepsilon^2 \delta, \varepsilon f^{rad})]$, the Bernoulli head at $O(\varepsilon \beta)$, and the radiation stress at $O[\max(\varepsilon \beta, \eta)]$. In the vertical direction the orders are $O[\max(\varepsilon^2 \delta, \varepsilon f^{rad} \eta)]$ for the vortex force, $O(\varepsilon)$ for the Bernoulli head, and $O[\max(\varepsilon, \varepsilon \beta \eta) = \varepsilon]$ for the radiation stress. From this we can see that, if the current strength is similar to or weaker than the waves (i.e., $\delta \lesssim 1$) and planetary vorticity is weak (i.e., $f^{rad} \ll \varepsilon^2$), then the radiation stress is asymptotically inconsistent with respect to the vortex force. As with the MRL04 scaling, its lowest-order divergence is the wave setup and long-wave forcing, but higher orders are needed to capture the vortex force. Only if either the currents are stronger than the waves (e.g., $\delta = 1/\varepsilon$) or the Rossby number is large ($f^{rad} \gtrsim \varepsilon^2$) do the radiation stress and vortex force enter the wave-averaged equations at the same order.
In the case $f^\text{ind} \sim O(e^2)$, the currents obey (58). For $\delta = 1/e$ the current enters into the wave dispersion relation at its leading order and the waves are no longer irrotational at any order. Nevertheless, even in this regime, which is atypical for the ocean, the vortex-force representation retains its advantage of distinguishing between pressure adjustment and vortex force.

7. Conclusions

This paper shows that the conservative effects of waves on currents can be characterized in terms of

- quasi-static wave setup,
- long- (infragravity) wave forcing,
- mass and other material transport by Stokes drift, and
- vortex force and other asymptotically comparable momentum effects.

Both the radiation-stress and the vortex-force representations encompass all of these effects. In the radiation-stress representation the vortex force may be hidden by lower-order effects. The vortex-force representation, however, cleanly decomposes the physics into a Bernoulli head and a vortex force. The Bernoulli head is a pressure adjustment owing to the wave effects associated with the well-known wave setup effect. The combination of transient mass transport and wave setup forces long waves. The vortex force is an interaction between the wave velocity and the wave vorticity. In the asymptotic regime, the wave vorticity arises from wave advection of the current vorticity, so the vortex force is equal to the curl of the Stokes drift and the current vorticity. Given an irrotational background current, there would not be a vortex force (most ocean currents have nonzero vorticity). Material properties in the ocean, including the buoyancy, follow wave-averaged trajectories that move with the sum of the Eulerian and Stokes drift velocities (including the vertical Stokes pseudovelocity). By decomposing the wave effects in this fashion, we are able to cleanly remove the wave setup, revealing the underlying wave effects on the long waves and currents that are obscured by the radiation-stress representation.

The wave effects on the energy associated with the current can be characterized by four terms: a Coriolis–Stokes drift term, Stokes drift terms analogous to the well-known shear production and potential energy conversion terms, and a surface layer term.

In addition to lacking a meaningful physical decomposition, the radiation-stress representation suffers from being asymptotically inconsistent in this scaling. The apparent order of the radiation stress is that of the wave setup. The currents evolve at a higher order with significant advective and Coriolis forces. Although we give these results for a specific asymptotic regime, section 6 suggests that unless the currents are strong compared to the wave orbital velocities—the opposite of typical shelf conditions—the lowest-order radiation or interaction stress divergence will not encompass the vortex force. The divergence of the radiation or interaction stress and vortex force only come in at the same order in the case of strong currents (i.e., $\delta \sim 1/e$).

Wave–current interactions have also been cast using the generalized Lagrangian mean (GLM) formalism (Andrews and McIntyre 1978a,b; Gjaja and Holm 1996; Groeneweg and Klopman 1998), Lagrangian mean (Weber 1983; Jenkins 1987), and other averaging variants of a Lagrangian flavor (Mellor 2003, 2005). (Mellor only follows the vertical wave motions. Furthermore, his work is not completely asymptotically consistent.) In MRL04, we chose the Eulerian frame because of the conceptual simplicity of the Eulerian mean and because Eulerian means are usually more practical in relating to oceanic measurements and large-scale oceanic numerical models (e.g., pressure). While a detailed comparison between frameworks is beyond the scope of this paper, an essential start is a clear understanding of how averages in the Eulerian and the Lagrangian frames are to be compared. The GLM velocity is equivalent to the Eulerian mean velocity plus a residual velocity, often identified as the Stokes velocity. Thus, the GLM velocity may be recovered from the mean Eulerian velocity to lowest order if desired. When the GLM equations are written in terms of the Eulerian mean velocity, the relationship to the vortex-force representation becomes apparent. Some work is required to interpret terms such as the GLM pressure in an Eulerian sense, however. In general, it is crucial to remember that 1) the choice of formalism has no bearing on the physics and 2) a direct comparison of the results presented here to those of a Lagrangian-framework-based derivation can only be made by the adoption of a common scaling. These two points are obvious; nevertheless, they are at the heart of the confusion with regard to the characterization of stresses in wave–current interactions. In this paper we adhere to these two principles. In doing so we are able to make clear how the vortex-force and radiation-stress representations compare to each other. We feel we have succeeded in this regard.

The vortex-force representation given here provides an interpretation of the effect of radiation or interaction stress in conservative dynamics with nearly irrotational waves. Within this regime, the lowest order vorticity in the waves is provided by the currents and planetary vorticity. The effects of wave breaking would need to be considered separately, as this is a different
source of vorticity for the waves. Conventional practice is to parameterize the effects of wave breaking. Offshore wave breaking is often represented as surface wind stress, though this approach has only limited validity on the scale of the surface boundary layer compared to modeling the breaking as stochastically distributed impulses (Sullivan et al. 2004). In the surf zone the shoreward decrease of wave amplitude though breaking is a primary cause of littoral currents (Longuet-Higgins 1970) and the parameterization of surf-zone breaking is a necessary element of the wave-averaged dynamics; this is often expressed as a radiation-stress divergence. The conservative vortex force might also play a role in this region. Under certain circumstances this could be included in an additive way, although the use of asymptotically derived vortex force in the surf zone would be semiempirical at best.

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APPENDIX A

Definitions of Radiation Stress

As mentioned in section 4, Hasselmann (H71) identifies his radiation stress with that of LHS with the exception of a sign difference. However, other differences arise from the two definitions of radiation stress.

Given our scalings the radiation stress in H71 is

$$\tau_{\alpha\beta}^{\text{rad}} = \tau_{\alpha\beta}^{\text{int}} + \tau_{\alpha\beta}^{\text{sl}}$$

$$= - \int_{-H}^{\varepsilon \zeta} \left( \langle q_{\alpha} q_{\beta} \rangle_{\mu} + \frac{1}{e} \delta_{\alpha\beta} \rho \right) dz$$

$$- \left\langle \int_{\varepsilon \zeta}^{\varepsilon E} Q_{\alpha} Q_{\alpha} + \frac{1}{e} \delta_{\alpha\beta} P \right\rangle dz;$$ \quad (A1)

whereas LHS’s definition is

$$S_{\alpha\beta} = \int_{-H}^{\varepsilon \zeta} \left( \langle q_{\alpha} q_{\beta} \rangle_{\mu} + \frac{1}{e} \delta_{\alpha\beta} \langle P \rangle - P_{c} \right) dz$$

$$- \left\langle \int_{\varepsilon \zeta}^{\varepsilon E} q_{\alpha} q_{\alpha} + \frac{1}{e} \delta_{\alpha\beta} P_{c} \right\rangle dz.$$ \quad (A2)

Here $P_{c}$ is taken to be the pressure of the current without any wave effects; $P$ and $P_{c}$ are assumed to include the hydrostatic pressure.

A comparison between (A1) and (A2) suggests that $\tilde{p}^{m} = \langle P \rangle - P_{c}$. Thus,

$$\tilde{p}^{m} = P_{c} + \frac{z}{L},$$

e.g., $\tilde{p}^{m}$ is the dynamical part of $P_{c}$. If we assume that $\varepsilon^{2} \zeta$ is the sea elevation in the absence of waves, then

$$\tilde{p}^{m}(\varepsilon^{2} \zeta) = P_{c}(\varepsilon^{2} \zeta) + \frac{2}{e \zeta} = P_{c} + \frac{2}{e \zeta},$$ \quad (A3)

as required by H71.

In addition to the sign difference between the two formulations, the expression in H71 includes the term

$$\left\langle \int_{\varepsilon \zeta}^{\varepsilon E} Q_{\alpha} Q_{\beta} dz \right\rangle,$$

in the surface layer stress, $\tau_{\alpha\beta}^{sl}$, whereas LHS only include the term

$$\left\langle \int_{\varepsilon \zeta}^{\varepsilon E} q_{\alpha} q_{\alpha} dw dz \right\rangle.$$

This means that the radiation stress in H71 includes the additional terms

$$q_{\alpha}(\varepsilon^{2} \zeta) T_{\beta}^{m} + q_{\beta}(\varepsilon^{2} \zeta) T_{\alpha}^{m}.$$

In H71 these two terms are neglected when calculating $\tau_{\alpha\beta}^{sl}$ in H71’s (17a), as noted by Garrett (1976). Furthermore, LHS use the total pressure when calculating the radiation stress, whereas H71 uses the dynamical pressure. This does not make any difference in the bulk of the fluid because only the pressure difference is calculated. In the surface layer this differences adds a term

$$- \left\langle \frac{q^{m2}}{\zeta_{c}^{2}} + 2 \frac{q^{m2} \zeta_{c}^{2}}{\zeta_{c}^{2}} \right\rangle,$$

to the LHS version of radiation stress.

Thus, H71’s definition of radiation stress may be related to that of LHS by

$$\tau_{\alpha\beta}^{\text{rad}} = -S_{\alpha\beta} - q_{\alpha}(\varepsilon^{2} \zeta) T_{\beta}^{m} + q_{\beta}(\varepsilon^{2} \zeta) T_{\alpha}^{m}$$

$$- \left\langle \frac{q^{m2}}{\zeta_{c}^{2}} + 2 \frac{q^{m2} \zeta_{c}^{2}}{\zeta_{c}^{2}} \right\rangle,$$ \quad (A4)

There are still difficulties that arise in the interpretation of the pressure. Respectively $P_{c}$ and $\tilde{p}^{m}$ are defined as the pressure and dynamic pressure of the cur-
rent in the absence of waves. In fact, when LHS first defined radiation stress, they assumed that the fluid was at rest when wave effects were not present; thus \( P_r \) was just the hydrostatic pressure. When currents were considered, they were simple ones. Likewise, the sea elevation \( \xi \) is defined as the sea elevation of the currents without waves. While this may be appropriate when considering a specific current and asking what effects waves would have on this current, at other times this approach is less convenient. Furthermore, wave setup may significantly change the mean sea level (notably in shallow or finite depth water; this effect becomes insignificant in deep water). If the mean sea level is altered by the addition of waves, then (A3) is no longer generally true.

To alleviate these problems and provide a consistent, computable interpretation of the radiation stress, we identify the wave mean pressure with \( \bar{\rho} \), the static mean pressure balance from MRL04. The remaining part of the mean pressure is \( \rho' \), which we identify with the current mean pressure. Unlike \( \bar{\rho}^m \) in H71, the surface boundary condition on \( \rho' \) contains wave forcing terms. As stated in H71 the surface boundary condition on \( \bar{\rho}^m \) is chosen to make it look like the usual pressure boundary condition rather than being derived from the wave-averaged equations. We note that in the limit of deep water and small Rossby number the wave forcing terms in the surface mean pressure boundary condition become negligible.

Using these definitions we write the radiation stress as

\[
S_{\alpha\beta} = \int_{-H}^{z} \left[ \langle q_{\alpha}^w q_{\beta}^w \rangle + \epsilon^2 \langle q_{\alpha}^w q_{\beta}^w \rangle + \delta_{\alpha\beta} \left( \bar{\rho} - \frac{\xi}{L} \right) \right] dz + \left\langle \int_{-H}^{z} q_{\alpha}^w q_{\beta}^w + \epsilon^2 q_{\alpha}^w q_{\beta}^w + \frac{1}{e} \delta_{\alpha\beta} \left( p - \frac{eE - z}{eL} \right) \right\rangle dz,
\]

which, in terms of linear gravity waves, is

\[
S_{\alpha\beta} = \left\langle \frac{|A|^2 k_{\alpha} k_{\beta}}{4k^2 L} \left[ 1 + \frac{2kH}{\sinh(2kH)} \right] \right\rangle + \epsilon^2 \delta_{\alpha\beta} \left\langle \frac{|A|^2 2kH}{4L \sinh(2kH)} \right\rangle + \epsilon^2 S^{2a}_{\alpha\beta} + \epsilon^2 \delta_{\alpha\beta} S^{2b}. \tag{A5}
\]

We can see that, to lowest order, this agrees with LHS’s radiation stress calculated for linear gravity waves. The higher-order terms in (A5) are given by

\[
S^{2a}_{\alpha\beta} = \left\langle \frac{|A|^2 k_{\alpha} k_{\beta}}{64L} \left( 9 \frac{\cosh(2kH)}{\sinh^2(kH)} \right) \left[ 1 + \frac{4kH}{\sinh(4kH)} \right] + 56 + \frac{60}{\sinh^2(kH)} \frac{36}{\sinh^2(kH)} \right\rangle
\]

\[
- \left\langle |A|^2 \sigma k_{\alpha} k_{\beta} \left( \int_{-H}^{z} \frac{\sinh(2k(z + H))}{k^2 \sinh^2(kH)} \right) dz \right\rangle - \frac{1}{2} \int_{-H}^{0} \left( \cosh(2k(z + H)) + \cosh(2kz) \right) dz
\]

\[
+ \frac{1}{4} \left( \cosh(-H) \left[ 1 + \frac{2kH}{\sinh(2kH)} \right] \right) + \frac{1}{2} \int_{-H}^{0} \left( q_{\alpha}^w q_{\beta}^w + q_{\alpha} q_{\beta}^w \right) dz + \int_{-H}^{0} \langle q_{\alpha}^w q_{\beta}^w \rangle - \frac{1}{e} \langle q_{\alpha} q_{\beta}^w \rangle - \left\langle \frac{|A|^2 k_{\alpha} k_{\beta}}{2k \sinh(kH)} \right\rangle
\]

(A6)

and

\[
S^{2b} = \left\langle \frac{1}{2k} P + \frac{1}{4L} \left\langle \frac{|A|^2 k}{\sinh(2kH)} \right\rangle \right\rangle \left\langle |A|^2 k \tanh(kH) \right\rangle + \left\langle \frac{|A|^4 k^2}{64L} \left[ 9 - \frac{9}{\sinh^2(kH)} - \frac{9}{\sinh^2(kH)} - \frac{8}{\sinh^2(kH)} \right] \right\rangle.
\]

(A7)
APPENDIX B

Total Vertically Integrated Momentum

Here \( \mathbf{T} \) may be calculated by decomposing (12) into mean and wave components and then averaging according to section 2b. This gives the following horizontal equation:

\[
\frac{\partial}{\partial t} \langle T_\alpha \rangle = - \frac{\partial}{\partial x_\beta} \left( \int_{-H}^{E} q_\alpha q_\beta \, dz \right) + \int_{-H}^{E} \langle q_\alpha q_\beta \rangle \, dz - \int_{-H}^{E} (h \times \langle \mathbf{T} \rangle)_\alpha + \int_{-H}^{E} \langle q_\alpha q_\beta \rangle \, dz + \int_{-H}^{E} \langle q_\alpha q_\beta \rangle \, dz
\]

\[
\frac{1}{2L} \frac{\partial}{\partial x_\alpha} \left( \int_{-H}^{E} (q_\alpha q_\beta) \, dz + \int_{-H}^{E} (q_\alpha q_\beta) \, dz \right)
\]

\[
\frac{1}{2L} \frac{\partial}{\partial x_\alpha} \left( \int_{-H}^{E} (p(-H)) \, dz + \int_{-H}^{E} (p(-H)) \, dz \right).
\]

The other vertically integrated equations are

\[
\frac{\partial}{\partial t} \left( \int_{-H}^{E} W \, dz \right) = - \frac{\zeta}{L} + \frac{\partial}{\partial x_\beta} \left( \int_{-H}^{E} w q_\beta \, dz + \int_{-H}^{E} (w q_\beta) \, dz + \langle \int_{-H}^{E} W Q_\beta \, dz \rangle \right),
\]

\[
\frac{\partial}{\partial t} \left( \int_{-H}^{E} \mathbf{C} \, dz \right) = - \frac{\partial}{\partial x_\beta} \left( \int_{-H}^{E} c q_\beta \, dz + \int_{-H}^{E} (c q_\beta) \, dz + \langle \int_{-H}^{E} \mathbf{C} Q_\beta \, dz \rangle \right).
\]

If we split the mean pressure into a static and a dynamic component, we can rewrite the mean vertically integrated momentum as

\[
\frac{\partial}{\partial t} \langle T_\alpha \rangle = \frac{\partial \tau_{m}^{\alpha}}{\partial x_\beta} + \frac{1}{2L} \frac{\partial \xi^{2}}{\partial x_\alpha} + p^s(-H) \frac{\partial H}{\partial x_\alpha} - f(z \times \mathbf{T})_\alpha + \left[ \frac{\partial \tau_{ad}^{\alpha}}{\partial x_\beta} + \frac{1}{2L} \frac{\partial \xi^{2}}{\partial x_\alpha} + \rho(-H) \frac{\partial H}{\partial x_\alpha} - f(z \times \mathbf{T})_\alpha \right].
\]

(B3)

where \( \tau^m \) and \( \tau_{ad} \) are defined in section 4a.

Introducing the scalings from section 2b, (B3) becomes

\[
\frac{\partial}{\partial t} \langle T_\alpha \rangle = \frac{\partial \tau_{m}^{\alpha}}{\partial x_\beta} + p^s(-H) \frac{\partial H}{\partial x_\alpha} - f(z \times \mathbf{T})_\alpha + \left[ \frac{\partial \tau_{ad}^{\alpha}}{\partial x_\beta} + \frac{1}{2L} \frac{\partial \xi^{2}}{\partial x_\alpha} + \frac{\xi^{2}}{2L} \frac{\partial \xi^{2}}{\partial x_\alpha} + \rho(-H) \frac{\partial H}{\partial x_\alpha} - f(z \times \mathbf{T})_\alpha \right].
\]

(B4)

As is the case with the equation for \( \mathbf{T} \), the wave terms at \( O(\varepsilon^{-2}) \) cancel, leaving an order-one residual. This may make practical evaluation difficult since it depends on the \( O(\varepsilon^{-2}) \) difference between two \( O(1) \) quantities. Written using LHS's version of radiation stress, (B4) becomes

\[
\frac{\partial}{\partial t} \langle T_\alpha \rangle = \left[ \frac{\partial \tau_{m}^{\alpha}}{\partial x_\beta} + p^s(-H) \frac{\partial H}{\partial x_\alpha} - f(z \times \mathbf{T})_\alpha \right] - \frac{\varepsilon^{-2} H + H}{\varepsilon^{2} L} \frac{\partial \xi^{2}}{\partial x_\alpha} - \left[ \frac{\partial \tau_{ad}^{\alpha}}{\partial x_\beta} + \frac{1}{2L} \frac{\partial \xi^{2}}{\partial x_\alpha} + \frac{\xi^{2}}{2L} \frac{\partial \xi^{2}}{\partial x_\alpha} + \rho(-H) \frac{\partial H}{\partial x_\alpha} \right].
\]

(B5)
Writing \( \mathbf{T} \) in vortex-force representation gives

\[
\frac{\partial (T_\alpha)}{\partial T} + \frac{\partial}{\partial x_\alpha} \left[ \int_{-H}^{z} \frac{\mathbf{q} \cdot \mathbf{q}}{2} + p^f \, dz + f(\mathbf{z} \times \mathbf{T})_\alpha \right] + \int_{-H}^{z} \left( \mathbf{u} \times \mathbf{u} \right)_\alpha \, dz \quad - \quad q_\alpha (\mathbf{E} \mathbf{u}^f) \quad \left( \frac{\mathbf{q} \cdot \mathbf{q}}{2} + p^f \right) \left. \frac{\partial H}{\partial x_\alpha} \right|_{z=-H} \\
= - \frac{\partial}{\partial x_\alpha} \left[ \int_{-H}^{z} \mathbf{X}_\alpha \, dz \right] + \frac{1}{2L} \frac{\partial}{\partial x_\alpha} \left[ -e^{-2} \frac{\partial}{\partial z} (\mathbf{w}^f_{\alpha}) + \mathbf{Q}_{\alpha} a^f \right] \\
\quad \left( \frac{\mathbf{w}^f_{\alpha}}{2} + e^{-1} p \, dz \right) + \int_{-H}^{z} \left( \mathbf{J}_\alpha \, dz - f(\mathbf{z} \times \mathbf{T}^f) \right) \\
\quad + \left( a_{\alpha w} w^f + e^{-2} (q_{\alpha w} w^f) \right) + e^{-3} \left( \frac{\mathbf{U}^f}{2} \frac{\partial E}{\partial x_\alpha} \right) - e^{-3} \left( Q_{\alpha} Q_{\beta} \frac{\partial E}{\partial \beta} \right) \left. \frac{\partial E}{\partial x_\alpha} \right|_{z=-H}. \tag{B6}
\]

Again, this can be shown to be all \( O(1) \), with the lower-order terms canceling. The first two rows are the contributions of the currents and these can be shown to be identical to the first row of (B4) to \( O(\mathbf{e}^4) \). To evaluate (B6) directly to \( O(1) \), it is necessary to know the waves to \( O(\mathbf{e}^4) \). This is two orders higher than is needed to evaluate (B4). Thus, the vortex-force representation is not useful when evaluating the evolution equations of \( \mathbf{T} \), because the free surface terms are onerous to evaluate. The vortex-force value is in the interpretation of the mean Eulerian momentum equations and, because \( \mathbf{T} \) can be seen as the vertically integrated Lagrangian mean momentum, we do not gain anything by expressing the equations in this form. Nevertheless, if \( \mathbf{T} \) is required, it can formally be written down in a more tidy form as the sum of the equations for \( \mathbf{T}^f \) and \( \mathbf{T}^m \).


