Towards a Global Spectral Energy Budget for Internal Gravity Waves in the Ocean

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ABSTRACT

Energy transfers by internal gravity wave–wave interactions in spectral space are diagnosed from numerical model simulations initialized with realizations of the Garrett–Munk spectrum in physical space and compared with the predictions of the so-called scattering integral or kinetic equation. Averaging the random phase of the initialization, the energy transfers by wave–wave interactions in the model agree well with the predictions of the kinetic equation for certain ranges of frequency and wavenumbers. This validation allows now, in principle, the use of the energy transfers predicted by the kinetic equation to design a global spectral energy budget for internal gravity waves in the ocean where divergences of energy transports in physical and spectral space balance forcing, dissipation, the energy transfers by the wave–wave interactions, or the rate of change of the spectral wave energy. First global estimates show indeed accumulation of the wave energy in a range of latitude consistent with tidal waves at frequency \( v_T \) propagating toward the latitudinal window where \( 2 < \omega_T/f(\phi) < 3 \), as predicted by the kinetic equation.

1. Introduction

Internal gravity waves shape the ocean by mixing density and dissolved substances when they break. This mixing is a key player of Earth’s climate system since it can drive oceanwide flow such as the meridional overturning circulation (Munk and Wunsch 1998; Talley et al. 2003; Wunsch and Ferrari 2004). The wave forcing is mostly due to tides and localized in spectral space, while observations show a spread of energy over a wide range of wavenumbers. It has long been recognized that the spectral wave energy distribution is surprisingly similar at different locations. A series of studies (Garrett and Munk 1972, 1975; Cairns and Williams 1976; Munk 1981) formulated what is known today as the Garrett–Munk (GM) spectrum: a spectral energy distribution characterized by slopes close to minus two in the wavenumber and frequency range of internal gravity waves, superimposed by relatively weak spectral peaks near the inertial and tidal frequencies. Such a continuously populated energy spectrum is similar to isotropic turbulence with its famous \(-5/3\) slope. The apparently fixed slopes therefore point toward energy transfers from the wavenumbers and frequencies where forcing generates the waves, toward regions in spectral space where dissipative processes like wave breaking extract energy from the wave field. Wave–wave interactions by nonlinearities are thought to establish such an energy transfer.

The so-called scattering integral or kinetic equation (Hasselmann 1966; Nazarenko 2011) allows the prediction of the energy transfers by such wave–wave interactions for a given energy spectrum under the assumption of slowly changing wave amplitudes, that is, for weak interactions. Previous studies (Olbers 1976; McComas and Bretherton 1977; Pomphrey et al. 1980; Lvov et al. 2012; Eden et al. 2019b) have evaluated the kinetic equation for internal gravity waves for different versions of the GM spectrum, under different approximations, and using different numerical methods. They share some common features but are also in parts contradictory, which puts doubts on the validity of the method. Furthermore, Holloway in a series of comments (Holloway 1978, 1980, 1982) and Müller et al. (1986) argue that the assumption of weakly changing wave amplitudes might not be justified. It was proposed...
already by Holloway (1981) that numerical model simulations should be used to clarify the issue. Hibiya et al. (1998) find that forcing applied to a model initialized with the GM spectrum is most effective in the frequency range $2 < \omega/\sigma < 3$ to produce energy at high vertical wavenumber, but doubt is also cast on this study since the model is two dimensional but the wave–wave interaction is inherently three dimensional.

The aim of the study presented here is therefore to compare the prediction of the kinetic equation by the use of direct three-dimensional numerical simulations of the interaction of internal gravity waves. We show that the spectral energy transfers predicted by the kinetic equation are indeed similar to the numerical model predictions in section 3 with the predictions of the kinetic equation.

The domain size is $50 \text{ km} \times 50 \text{ km} \times 5 \text{ km}$ with 50-m horizontal and 5-m vertical resolution. The specific form of the wave spectrum which we use here is shown in Fig. 1a and is given by

$$
\mathcal{E}(m, \omega) = E_0 \frac{A(m/m_0)}{m_0} B(\omega),
$$

where $\mathcal{E}$ denotes the wave energy density in physical and wavenumber space, $m$ is vertical wavenumber, and $\omega$ is frequency, and with the shape functions $A(x) = n_b/(1 + x^2)^{1/2}$ and $B(x) = n_b/[x^{-1}(x^2 - \beta^2)]^{-1/2}$ normalized by $n_b$ and $n_a$ such that $\int_0^\infty A(x) \, dx = 1$ and $\int_0^\infty B(x) \, dx = 1$. The bandwidth parameter $m_0$ is given by $m_0^2(\omega) = (N^2 - \omega^2)/c_s^2$ with $c_s = 0.5 \text{ m s}^{-1}$, and the total energy level is set to $E_0 = 3 \times 10^{-3} \text{ m}^2 \text{s}^{-2}$. The parameter $r = 2$ denotes the (negative) spectral slope in $m$ for large wavenumbers. We show $\mathcal{E}(\omega, m)$ from Eq. (1) in Fig. 1a—and later $\partial \mathcal{E}$ in the following figures—in form of a content (or variance preserving) spectrum in which the volume under the surface $\omega m \mathcal{E}$ is proportional to the respective energy content with logarithmically spaced axis.

From the spectrum $\mathcal{E}$—transformed to $\mathcal{E}(\mathbf{k})$, where $\mathbf{k}$ denotes the wavenumber vector $\mathbf{k} = (k_x, k_y, m)$—we calculate the initial conditions for velocity $u, v, w$ and buoyancy $b$ in the model in the following way. After applying a three-dimensional Fourier transform, the velocity and buoyancy amplitudes $\hat{u}, \hat{v}, \hat{w}, \hat{b}$ can be represented as

$$
\hat{x} = \sum_{s=\pm} \alpha^s \mathbf{q}^s
$$

with $\hat{x}(\mathbf{k}) = (\hat{u}, \hat{v}, \hat{w}, \hat{b})^T$ and the right eigenvector of the linear discrete system $\mathbf{q}^s(\mathbf{k})$ given in appendix B.
The sum over the superscript $s = \pm$ corresponds to the positive and negative discrete wave frequencies $\omega (k)$ also given in appendix B. The wave amplitude $a^s (k) = p^\pm \cdot \mathbf{x}$, where $p^\pm$ is the left eigenvector of the linear discrete system, is related to wave energy by $E = (1/2) \sum_{s=\pm} |a^s|^2/n'$ with a normalization factor $n^s (k)$ also given in appendix B. The wave amplitudes at initial time are then given by

$$\dot{\mathbf{x}}(k)|_{t=0} = \sqrt{2E} \sum_{s=\pm} \sqrt{n'} \xi^s, \tag{3}$$

where $\xi^s (k)$ are realizations of a complex normal number with 0 mean and variance of 1. With different realizations of $\xi$ taken from a random number generator we generate an ensemble of 25 model integrations, which have initially the same spectral energy, but different phases of the waves. An example of the different initial conditions for the model ensemble is shown in Fig. 1b.

The mean wave energy $E = (1/2) \sum_{s=\pm} |a^s|^2/n'$ and the geostrophic energy $\mathcal{E}^0 = (1/2) |a^0|^2/n^0$ are diagnosed and averaged over the ensemble and $\partial_t \mathcal{E}$ is approximated using finite differences at different times. The difference of $\partial_t \mathcal{E}$ between the ensemble members tends to be larger than the common signal due to the wave–wave interactions, but the ensemble average effectively reduces the noise in $\dot{\mathcal{E}}$ introduced by the random initial conditions seen in individual ensemble members. The damping in the model by subgridscale operators and the time stepping (detailed in appendix A) is estimated from an identical ensemble simulation but without nonlinear terms [right-hand side of Eq. (A1)] and shown as the damping rate $\lambda_{num} = -\partial_t \mathcal{E}/E$ in Fig. 2a, with a maximum of 1.5 days. Only for the largest $m$, the damping time scale becomes comparable to the inertial period such that we can conclude that the simulated wave–wave interaction is not affected by the numerical damping for simulation periods smaller than a day.

From $t = 0$ to about $t = 0.1$ day there is an increase in the geostrophic energy $\mathcal{E}^0$ at large vertical wavenumbers and low frequencies (Fig. 2b), and a corresponding loss of wave energy (not shown). Since in the linear model there is no such initial increase in $\mathcal{E}^0$, this represents an inverse nonlinear geostrophic adjustment, similar to what is described in Eden et al. (2019b) (cf. their Fig. 6). For $t > 0.1$ day, however, $\partial_t \mathcal{E}^0$ becomes small while the wave energy transfer $\partial_t \mathcal{E}$ begins to show a consistent signal due to the wave–wave interactions: energy loss predominantly for $2 < \omega f < 3$ and energy gain at near-inertial frequencies but larger wavenumbers. Figure 2c shows $\partial_t \mathcal{E}$ approximated from $\mathcal{E}$ diagnosed at $t = 0.5$ day and $t = 0.1$ day; using estimates at other times gives similar results in shape, but the magnitude of $\partial_t \mathcal{E}$ decreases for later times, because of the dissipation in the model.

3. Kinetic equation

We complement the result from the direct numerical simulation with a theoretical method in this section, which is detailed in Eden et al. (2019a) and only sketched here. Using the eigenvector representation, the Fourier transform of the equation of motions can be written in terms of the amplitudes $a^s$ as

$$\partial_t a^0_n = -i \sum_{s_1, s_2, s_3 = 0, \pm} \int \mathcal{D}k_1 \mathcal{D}k_2 \mathcal{D}k_3 a^{s_1}_{s_2} a^{s_2}_{s_3} \mathcal{E}(k_0, k_1, k_2, k_3) \times \delta(k_1 + k_2 - k_0) e^{i(s_1 \omega_1 + s_2 \omega_2 - s_3 \omega_3) t}, \tag{4}$$

where $s_0 = 0, \pm; s^0_n = a^0 \delta(k_n)$; and $\omega_n = \omega(k_n)$. The right-hand side of Eq. (4) represents the quadratic nonlinear terms in the equations of motion. They are given by triad interactions between amplitudes at wavenumbers $k_n$. 

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**Fig. 2.** (a) Numerical damping rate $\lambda_{num}$. The gray area corresponds to $\omega$ and $m$ values not covered by the model grid. (b) Ensemble mean geostrophic energy transfer $\partial_t \mathcal{E}^0$ in the numerical model after 0.1 day. (c) Wave energy transfer $\partial_t \mathcal{E}$ in the model after 0.5 day. The solid lines correspond to $\omega f = 1.25, 2, 3, 5, 10$, and 20, where $\omega$ denotes the wave frequencies on the numerical grid given by Eq. (B2). Dashed lines correspond to the analytical frequencies.
k₁, and k₂ satisfying k₀ = k₁ + k₂. Gravity waves with s₀ = ± are interacting only with each other choosing only the triads with s₁ = ± and s₂ = ±. Such triads represent the gravity wave–wave interaction, but in general triads for s₀ = ± can also be formed including the geostrophic mode s₁ = 0 or s₂ = 0. The same is true for the geostrophic mode with s₀ = 0; it can interact only with itself for s₁ = s₂ = 0, but mixed triads with s₁ = ± or s₂ = ± are also possible. Such mixed interactions are discussed in Eden et al. (2019a), and apparently also show up here (Fig. 2b). Different models can be represented by Eq. (4), with the interaction coefficients c as the only difference. Here we use the coefficients appropriate to the nonhydrostatic equations of motion given in Eden et al. (2019b). Neglecting the nonlinearities (c = 0), the wave amplitudes stay constant in time, while for c ≠ 0 nonlinear interactions will transfer energy between wave triads.

Being just another model representation, Eq. (4) is not yet useful. Using Hasselmann’s weak interaction assumption (Hasselmann 1966; Nazarenko 2011), however, it is possible to cast Eq. (4) into a tendency equation for the energy spectrum \( \dot{E}(kₐ) \). It consists of all possible triad interactions as Eq. (4), but involves \( \dot{E} = (1/2)|a'|²/n' \) instead of the amplitudes \( a' \) as in Eq. (4).

The resulting equation is called the scattering integral or the kinetic equation, and its derivation is discussed in more detail in Eden et al. (2019a; see also Hasselmann 1966; Nazarenko 2011). The assumption for the kinetic equation to hold is that wave amplitudes are only slowly changing with respect to the wave periods, a condition which is tested below. The time dependency in Eq. (4) transforms in the kinetic equation into a finite time response function \( \Delta(s₁ω₁ + s₂ω₂ - s₃ω₀, t) \) [given by Eq. (22) in Eden et al. (2019a)] which converges to \( \delta(s₁ω₁ + s₂ω₂ - s₃ω₀) \) for time \( t \to \infty \). Focusing on the gravity wave–wave interaction only, that is, for \( s₃, s₁, s₂ = ± \), only resonant triad interactions satisfying \( ω₀ = ω₁ ± ω₂ \) are thus important in the long run. For \( s₀, s₁, s₂ = ± \), the triad sum in Eq. (4), which stays the same in the kinetic equation, can be resolved into 1) the sum interaction with \( s₁ = s₂ = + \), for which \( ω₀ = ω₁ + ω₂ \) and similar for \( k \) holds, 2) two mixed terms with \( s₁ = +, s₂ = - \) or \( s₁ = -, s₂ = + \), for which \( ω₀ = ω₁ - ω₂ \) and similar for \( k \) holds, and 3) the remaining triad interaction, which never becomes resonant. The two mixed terms with \( s₁ = +, s₂ = - \) or \( s₁ = -, s₂ = + \) are identical and are denoted as difference interaction.

Here we use the same consistent numerical representation of our rederivation of the kinetic equation as detailed in Eden et al. (2019b). Both resonant and nonresonant interactions are calculated on a wavenumber grid corresponding to the model grid for \( Δ(ω, t) \) at different times \( t \), using method 2 of Eden et al. (2019b). We choose a grid with half as many grid points but the same domain as the model (the calculations take several days on several thousand processors of the DKRZ supercomputer in Hamburg, Germany), because we except the triads at higher wavenumbers to be affected by the damping in the model. We calculate \( \dot{E} = \partial E/∂t \) for gravity wave–wave interactions with \( s₀ = ± \), for the geostrophic mode \( \dot{E} = \partial E/∂t \) and also the mixed triad interactions between both. Figure 3a shows \( \dot{E} \) for \( t = 0.04 \) day, which is close to \( \dot{E} \) from the numerical model (Fig. 2b). A similar nonlinear adjustment of the GM spectrum has been found in Eden et al. (2019a) for the hydrostatic equations of motion. Their Fig. 6 shows that after about \( t = 0.1 \) day \( \dot{E} \) decreases and does not contribute anymore.

Figure 3b shows \( \dot{E} \) for \( t = 0.5 \) day. As in the model, wave energy is transferred predominantly from \( 2 < \omega/\nu < 3 \) to lower frequencies but larger \( m \), and also with similar magnitudes as in the numerical model. For \( ω → N \) and \( m > (5–10)mₙ \), differences between model and kinetic equation can be noted. Considering only the resonant interactions in the kinetic equation allows for a larger spectral domain since fewer triad interactions need to be calculated. Figure 3c shows the resonant \( \dot{E} \) calculated following method 3 of Eden et al. (2019b) on a grid with ~17-m horizontal and ~1.8-m vertical grid resolution and the same domain size as the model. The result is in agreement with what was previously reported by Olbers (1976) (his Fig. 4, although for a different GM spectrum) and by McComas and Bretherton (1977) (their Fig. 10b, for the same spectrum as here but using the hydrostatic approximation). The normalized Boltzmann rate \( 2mₙT\dot{E}/(E\omega) \) (not shown) agrees with the one in Lvov et al. (2012) (their Fig. 5, for the same spectrum but also using the hydrostatic approximation). The Langevin rate [not shown; Eq. (35) in Müller et al. (1986)] agrees with the one shown in Müller et al. (1986) (their Fig. 6, for the same spectrum as here, but also using the hydrostatic approximation). A difference from the previous calculations is the large transfers showing up for \( ω → N \) (hardly seen under the gray shading), but a direct comparison is hampered by the hydrostatic approximation made in the previous studies or because the spectrum was truncated for large \( ω \).

4. Mechanism of resonant wave–wave interactions

McComas and Bretherton (1977) suggest three dominant kinds of resonant triad interactions in the scattering integral with different characteristics: elastic scattering (ES) with \( ω ≈ \omega' \gg \omega'' \), \( m = -m' \approx |m''/2| \);
induced diffusion (ID) with $\omega \approx \omega' \gg \omega''$, $m = m' \gg |m''|$; and parametric subharmonic instability (PSI) with $\omega' \approx \omega'' \approx \omega/2$. The triads satisfy the conditions $\omega = \omega' \pm \omega''$ and $m = m' \pm m''$, where all three frequencies and wavenumbers can become $\omega_{0,1,2}$ and $m_{0,1,2}$ in the scattering integral. Types ES and ID have their names from well-known processes in particle physics that are also described by a similar scattering integral. Types ES and ID have their names from well-known processes in particle physics that are also described by a similar scattering integral, whereas PSI is a classic wave--wave interaction.

To identify the mechanism, we first write the kinetic equation using the definition of wave action $A = E/\omega$ as

$$\delta_t E_0 = \int d{\bf k}_1 [S + D]$$

$$S = \delta(\omega_1 + \omega_2 - \omega_0)\omega_0 T^+(A_1 A_2 - A_0 A_1 A_0 A_2)$$

$$D = 2\delta(\omega_1 - \omega_2 - \omega_0)\omega_0 T^-(A_1 A_2 + A_0 A_1 A_0 A_2),$$

where $T^+$ and $T^-$ result from the coefficients $c$ in Eq. (4) and denote the scattering cross sections for sum and difference interaction, respectively, and are given in, for example, Olbers (1976) or Eden et al. (2019b), and where $A_0 = A({\bf k}_0)$, and so on, as in Eq. (4). We then calculate energy transfer weighted frequencies and wavenumbers for the sum interaction as

$$\overline{m}_{1,2} = \int d{\bf k}_1 |m_{1,2}| S / \int d{\bf k}_1 |S|,$$

$$\overline{m}_{1,2} = \int d{\bf k}_1 |m_{1,2}| S / \int d{\bf k}_1 |S|$$

and similar for the difference interaction using the weighting $D$ instead of $S$. Terms $\overline{m}_{1,2}$ and $\overline{m}_{1,2}$ are functions of $\omega$ and $m$ and represent the frequency and wavenumber of the dominant energy transfers. A part of $S$ (and $D$) with large magnitude but opposite sign could cancel in the integral in Eq. (5) but could largely affect $\overline{m}_{1,2}$ and $\overline{m}_{1,2}$ by weighting with $|S|$ and $|D|$. To test this effect, we also weighted with max$(0, S)$ and $-\min(0, S)$, and similar for $D$, and found no large differences in $\overline{m}_{1,2}$ and $\overline{m}_{1,2}$, such that we can exclude this effect.

Figure 4 shows $\overline{m}_1$ and $\overline{m}_1$ for the difference interaction. Terms $\overline{m}_1$ and $\overline{m}_1$ are almost identical and thus are not shown. Since $2\overline{\omega}_1 \approx 2\overline{\omega}_2 \approx \omega$, the PSI mechanism clearly dominates the sum interaction. For small $\omega$ and/or small $|m|$, $\overline{m}_{1,2} \gg |m|$, and if the energy transfer is negative, that is, from $\omega$ directed toward $\overline{m}_{1,2}$, the energy transfer is toward larger $|m|$. For the difference interaction (Fig. 5), we find for large $\omega$ and large $|m|$ (and thus large $k$) that $\omega \approx \overline{\omega}_1 \gg \overline{\omega}_2$ and $|m| \approx \overline{m}_1 > \overline{m}_2$, which indicates the importance of ES and ID triads for the difference interactions in this region. Only for the largest $\omega$ and $|m|$ do we find the condition for ID $|m| > \overline{m}_2$. For smaller $\omega$ and large enough $|m|$, we find PSI again, since here $\overline{\omega}_1 \approx 2\overline{\omega}_2 \approx 2\omega$, but with frequencies exchanged relative to the sum interaction. For small $\omega$ and small $|m|$ we find $\overline{\omega}_1 \approx \overline{\omega}_2 \approx \omega$ and $\overline{m}_1 \approx \overline{m}_2 \gg |m|$ and thus ID again.

The energy transfer at low frequencies is dominated by the PSI-like interactions. Figure 4c shows $\delta_t E$ allowing only for approximate PSI interaction using only the negative sum interactions for $|\omega_1 - \omega_2| < 30\%$ of $(\omega_1 + \omega_2)/2$, and only the positive difference interactions $|\omega_0 - \omega_2| < 30\%$ of $(\omega_0 + \omega_2)/2$. This implies that most of the energy transfers for $m < (5-10)$ $m_0$ seen in the model are also generated by PSI-like mechanisms.

If time scales of wave--wave interactions become shorter than the wave periods, the assumption of weak interaction becomes invalid. This applies for PSI at large $m$ but in particular for ID at large frequencies. Holloway (1978, 1980, 1982) and Muller et al. (1986) argue that the Bolzmann rate $\delta_t E/\epsilon$ or the Langevin rate are
insufficient to determine the time scales of the interactions, since individual triads in the scattering integral with much shorter time scales violating the weak interaction assumption can balance each other, leading to a much larger time scale in the integrated effect. To account for this effect, we therefore define “worst case” rates as

$$n_{S} = \int d\mathbf{k} |S|/\mathcal{E} \quad \text{and} \quad n_{D} = \int d\mathbf{k} |D|/\mathcal{E} \quad (7)$$

for sum and difference interaction, respectively, which should be smaller than the wave period for the weak interaction assumption to hold. The condition $\max(n_{S,D}) < \omega_{0}/(2\pi)$ is violated (transparent shading in Fig. 3c) for $\omega \to N$ and for $m > (5-10)m_{\ast}$, which corresponds for $m_{\ast} = 0.01 \text{ m}^{-1}$ to a vertical wavelength of $\sim 125-60 \text{ m}$. The differences between model and kinetic equation for $m > (5-10)m_{\ast}$ can therefore be explained by the kinetic equation becoming invalid.

5. Summary and discussion

We find good agreement between the predictions of the kinetic equation for the spectral energy transfers in the GM spectrum Eq. (1) and direct simulations of internal gravity wave-wave interactions in a three-dimensional, nonhydrostatic numerical model for a certain range of frequencies and wavenumbers. Such an agreement between the kinetic equation for gravity waves and numerical model simulations has not been shown before to our knowledge. Because the signal by wave-wave interactions is relatively small in the model, it is necessary for this validation to remove the effect of the random phase of the initial conditions in physical space by using an ensemble model integration. We also find it necessary to use eigenvalues and eigenvectors appropriate to the discrete numerical model for the initialization with the GM spectrum, for the model diagnostic, and for the calculation of the interaction coefficients for the kinetic equation. Part of the wave energy generates initially the geostrophic mode. After that inverse nonlinear geostrophic adjustment, wave energy is decreased at $2/\omega f$, and increased at smaller $\omega$ and larger $m$ by the nonlinear terms. The initial increase in the geostrophic mode and the transport pattern of wave energy for small $m$ is reproduced by the kinetic equation including nonresonant interactions. The latter is similar to the result including only resonant interactions, but differences show up for both between the kinetic equation and the model simulations for $m > (5-10)m_{\ast}$, where $m_{\ast}$

![Fig. 4.](http://journals.ametsoc.org/jpo/article-pdf/50/4/935/4927655/jpod190022.pdf)

**Fig. 4.** (a) Energy transfer weighted frequency $\overline{m}_1$ for the sum interaction scaled by the frequency $\omega$. (b) As in (a), but wavenumber $\overline{m}_1$ scaled by $m$. (c) Energy transfer $\hat{\omega}$ by approximate PSI interactions only. The solid lines denote $\omega f = 1.25, 2, 3, 5, 10,$ and 20. Gray areas correspond to $\omega$ and $m$ values not covered by the grid.

![Fig. 5.](http://journals.ametsoc.org/jpo/article-pdf/50/4/935/4927655/jpod190022.pdf)

**Fig. 5.** Energy transfer weighted frequency and wavenumber for the difference interaction. Note that $\overline{m}_1$ is scaled with $\omega$ and that $\overline{m}_2$ is scaled with $\overline{m}_1$, and likewise for $\overline{m}_1$ and $\overline{m}_2$. Note also the different color scales. The solid lines denote $\omega f = 1.25, 2, 3, 5, 10,$ and 20. Gray areas correspond to $\omega$ and $m$ values not covered by the grid.
represents the dominant vertical wavenumber in the GM spectrum.

The resonant energy transfers predicted by the kinetic equation are in agreement with previous studies, although here calculated without using the hydrostatic approximation and with considerably larger numerical effort. Energy transfer weighted frequencies and wavenumbers show that parametric subharmonic instability (PSI) triad interactions are mostly responsible for the energy transfers at low frequencies toward large $m$, but that ID triad interactions become important for larger $\omega$ and large $m$, but also for small $\omega$ and very small $m$, as suggested by McComas and Bretherton (1977). The triad interactions for $m > (5-10) m_p$, however, are found to violate the weak interaction assumption inherent to the kinetic equation, as anticipated by Holloway (1978, 1980, 1982) and Müller et al. (1986), which might explain the difference to the model simulations.

Due to the large computational costs, we were not able to increase the model resolution to fully cover the large scale separation important for ID. However, in the range of $\omega$ and $m$ which we do cover (with considerable computational effort) and which is not affected too much by numerical damping and grid dispersion errors, there are no indications of significant energy transfers beyond the dominant energy transfers from $2 < \omega f < 3$ toward smaller $\omega$ and larger $m$ generated by PSI triad interactions. Our model results thus suggest no large role of ID for energy transfers in the GM spectrum, but this needs to be checked with models with higher resolution in the future. Sugiyama et al. (2009) find in a forced model simulation of internal wave interaction also predominantly energy transfers to near inertial frequencies by PSI triad interactions, but there the model is two dimensional and of coarser resolution as here.

Ignoring the caveat of model resolution and the role of ID, the observed GM spectrum demands energy transfers from $2 < \omega f < 3$ to smaller $\omega$ and larger $m$. Note that it was shown in Eden et al. (2019b) that varying parameters in the GM spectrum Eq. (1) like the Coriolis parameter $f$, the bandwidth $c_n$, or the spectral slope $r$, this coherent energy transfer pattern stays very similar. In steady state, these energy transfers need to be balanced by other terms in a more complete spectral energy balance in physical and spectral space. We believe that this question forms the major challenge to understand the global gravity wave field in the ocean and its effects. Already in the early studies by Müller and Olbers (1975), Olbers (1976), McComas and Bretherton (1977), and Pompfrey et al. (1980) it was envisioned that spectral regions of $\partial_t E < 0$ or $\partial_t E > 0$ are balanced locally by wave forcing and dissipation, respectively. Knowing the form of $\partial_t E$ for a given GM spectrum would then allow for an estimation of dissipation and forcing of the wave field in the ocean. Since we found a forward energy cascade toward larger $m$ where waves can break and dissipate this viewpoint is at least partly satisfied.

However, we also know today that the wave field is predominantly generated by the interaction of the barotropic tide with topography, forcing waves with fixed frequency $\omega_f$, at a rate of 1–2 TW (Wunsch and Ferrari 2004). Recent estimates by Falahat et al. (2014) and Vic et al. (2019) point toward somewhat lower values of 0.5 TW for the first 10 baroclinic modes.

Other forcing processes as inertial pumping at the bottom of the surface mixed layer induced by winds appear to be much smaller (e.g., Rimac et al. 2013). Internal tidal waves can be refracted and scattered at the bottom or the balanced flow (Müller and Xu 1992; Savva and Vanneste 2018), changing their wavenumbers while their frequency $\omega_f$ remains constant. As long as $m$ remains small, the tidal waves cannot be directly dissipated by these processes. This route can then only be established by the interaction of the internal tidal waves with the continuum described by the GM spectrum.

We find that the GM spectrum in steady state demands an energy source in the spectral region of $2 < \omega f < 3$. To adjust the Coriolis parameter $f$ in order to locate the tidal forcing frequency $\omega_f$ in this frequency interval implies for the most important half-daily tide roughly that $20^\circ < |\phi| < 30^\circ$ (or $10^\circ < |\phi| < 15^\circ$ for the daily tide), where $\phi$ denotes geographical latitude. To obtain a balanced energy spectrum in steady state, we could invoke therefore a nonlocal spectral energy balance, in which the tidal wave energy needs to propagate toward the latitudinal window where $2 < \omega_f f(\phi) < 3$. Such a nonlocal spectral energy balance is given by the radiative transfer equation for gravity waves discussed in, for example, Olbers et al. (2012), where divergences of energy transports in physical and spectral space balance forcing, dissipation, the energy transfers by the wave–wave interactions, or the rate of change of the spectral energy $E$. In such a hypothetical nonlocal energy balance, the divergence of the energy transport in physical or spectral space of the tidal waves could balance the energy transfers by wave–wave interactions as forcing term, instead of the local forcing in physical space as implied by the earlier studies. It is clear that to establish such a nonlocal spectral wave energy balance we need to better understand the interaction of a spectral peak represented by the tidal wave with the GM-like continuum,
as discussed in, for example, Olbers (1974, 1976) and Onuki and Hibiya (2018). Based on the coherent spectral shape seen above, we could speculate that the tidal decay by such interactions is most effective in the spectral window of \(2<\omega f/|f|<3\). We might then also expect an accumulation of wave energy in the corresponding latitudinal window.

To test the prediction of energy accumulation from our simple hypothetical nonlocal energy balance, Fig. 6 shows a global estimate of wave energy and dissipation rates. Argo float profiles (Argo 2019) collected between 2006 and 2017 are used to calculate vertical wavenumber spectra of strain

\[ \xi(z) = \frac{(N^2 - N_{fit}^2)}{N^2}, \]

where \(N_{fit}\) is a smooth quadratic fit to the buoyancy frequency profile \(N(z)\) and \(N^2\) is its vertical mean. Wave energy spectra \(E(m)\) are obtained from \(\xi(z)\) using the polarization relation of internal gravity waves. Integrating \(E(m)\) over a suitable wavenumber range allows to estimate the total wave energy as described in Pollmann et al. (2017). Energy transfer due to wave dissipation is estimated from the integrated strain spectra \(\xi(z)\) following the so-called finestructure method (e.g., Gregg 1989; Kunze et al. 2006; Polzin et al. 2014). The figure shows that both energy and dissipation feature large meridional variations. In the Pacific Ocean, maxima in the expected latitudinal range can be seen in energy and dissipation, although this is not so clear for the other ocean basins. We suggest that further work using the kinetic equation and observations is needed to explain these large regional variations and to understand how the wave–wave interactions together with propagation, dissipation and forcing shape the global gravity wave energy and its dissipation in the ocean.

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APPENDIX A

Numerical Model

We use a model based on the incompressible equations of motions for a rotating and stratified fluid. The Earth’s rotation and stratification stability frequencies, \(f = 10^{-4}\) s\(^{-1}\) and \(N = 50 f\), respectively, are constant and representative for the midlatitude interior ocean. The numerical discretization is given in Eq. (A1). The model domain is triple periodic with 1001 grid points in all directions and 50 km \(\times\) 50 km \(\times\) 5 km extent and 50-m lateral and 5-m vertical grid resolution. Dissipation is given by explicit harmonic diffusion and friction with coefficients of \(2.5 \times 10^{-5}\) m\(^2\) s\(^{-1}\) (\(2 \times 10^{-5}\) m\(^2\) s\(^{-1}\)) in lateral (vertical) direction, which can be seen to parameterize the effect of smaller-scale turbulent motions. There is also implicit damping by the time stepping scheme which is chosen as a quasi-second-order Adam–Bashforth interpolation with adjusted weights to allow for stable simulations of gravity
waves of highest frequency $N$ with a time step of 20 s. The explicit damping affects predominantly the high wave numbers, while the implicit damping affects the high frequencies, and both effects are proportional to $\varepsilon$. The discrete model equations are given by

$$\begin{align*}
\partial_t u_{i,j,k} &- f \mathbf{k} \cdot \mathbf{u}_{i,j,k} = -\delta_x^+ (\mathbf{u}^+ \cdot \mathbf{v}^+) - \delta_y^+ (\mathbf{v}^+ \cdot \mathbf{u}^+) - \delta_z^+ (\mathbf{w}^+ \cdot \mathbf{w}^+), \\
\partial_t v_{i,j,k} &- f \mathbf{k} \cdot \mathbf{v}_{i,j,k} = -\delta_y^+ (\mathbf{u}^+ \cdot \mathbf{v}^+) - \delta_z^+ (\mathbf{v}^+ \cdot \mathbf{w}^+) - \delta_z^+ (\mathbf{w}^+ \cdot \mathbf{w}^+), \\
\partial_t w_{i,j,k} - \mathbf{b} \cdot \mathbf{w}_{i,j,k} + \bar{b}_{i,j,k} = -\delta_x^+ (\mathbf{u}^+ \cdot \mathbf{w}^+) - \delta_y^+ (\mathbf{v}^+ \cdot \mathbf{w}^+) - \delta_z^+ (\mathbf{w}^+ \cdot \mathbf{w}^+), \\
\partial_t b_{i,j,k} + N^2 \bar{w}_{i,j,k} = -\delta_x^+ (u \bar{b}^+) - \delta_y^+ (v \bar{b}^+) - \delta_z^+ (w \bar{b}^+).
\end{align*}$$

(A1)

together with the diagnostic relation $\delta_x^+ u_{i,j,k} + \delta_y^+ v_{i,j,k} + \delta_z^+ w_{i,j,k} = 0$. The indices $i$, $j$, and $k$ denote discretization in the $x$, $y$, and $z$ directions, and finite-differencing operators $\delta_x^+ h_{i,j,k} = (h_{i+1,j,k} - h_{i,j,k})/\Delta x$ and $\delta_y^+ h_{i,j,k} = (h_{i,j,k+1} - h_{i,j,k})/\Delta y$, where $\Delta x$ is the grid spacing in the $x$ direction, and averaging operators $\bar{h}_{i,j,k} = h_{i,j,k} + h_{i+1,j,k}/2$ and $\bar{h}_{i,j,k} = h_{i,j,k} + h_{i,j,k-1}/2$, and similar for the other directions, are introduced. Dissipative terms and the time discretization are omitted in Eq. (A1) but are included in the numerical model.

**APPENDIX B**

Eigenvalues and Eigenvectors

With $u_{i,j,k} = \int dk \hat{u}(k)e^{ik \cdot x}$, $k = (k_x, k_y, k_z)$, $x = (x, y, z) = (i\Delta x, j\Delta y, k\Delta z)$, and similar for the other variables, the Fourier transform of Eq. (A1) becomes

$$\begin{align*}
\partial_t \hat{u} &- f \mathbf{k} \cdot \hat{u} + i\hat{k}_z \hat{p} = N, \\
\partial_t \hat{v} &+ i\hat{k}_y \hat{p} - i\hat{k}_x \hat{b} = N, \\
\partial_t \hat{w} &+ i\hat{k}_y \hat{p} + i\hat{k}_x \hat{b} = N.
\end{align*}$$

(B1)

with $i\hat{k}_z (k_z) = (e^{i \Delta z} - 1)/\Delta z$, $i\hat{k}_y (k_y) = (e^{i \Delta y} + 1)/2i$, $\hat{k}_x = (k_x)^*$, $\hat{k}_y = (k_y)^*$, and similar for $\hat{k}_z$, $\hat{k}_y$, and so on. Note that $\hat{k}_z \to k_z$ and $\hat{k}_x \to 1$ for $\Delta z \to 0$. The left-hand side is linear in the wave amplitudes, and the nonlinear right-hand side involves products of amplitudes integrated over all wave numbers that are not explicitly shown here.

The linear system formed by the left-hand sides of Eq. (B1) has three real eigenvalues

$$\omega^{(0)} = 0,$$

$$\omega^{\pm} = \pm \sqrt{\frac{1 + 1}{4} N^2(k_x^2 + k_y^2 + k_z^2) \pm \frac{1}{2} (1 + 1) f}.$$  

(B2)

The vanishing eigenvalue corresponds to the balanced mode, the two roots represent gravity waves.

For $\Delta_x, \Delta_y, \Delta_z \to 0$, the finite roots converge to the internal gravity wave dispersion relation. The corresponding right and left eigenvectors are given by

$$q^p = \frac{\begin{pmatrix}
-1^+ \hat{k}_z \hat{p} & -1^+ \hat{k}_z \hat{s} \omega \\
\hat{k}_y \hat{p} & \hat{k}_y \hat{s} \omega
\end{pmatrix}}{1^+ \hat{k}_x \hat{s} \omega}, \quad q^r = \begin{pmatrix}
\hat{k}_y \hat{p} & \hat{k}_y \hat{s} \omega \\
1^+ \hat{k}_z \hat{p} & 1^+ \hat{k}_z \hat{s} \omega
\end{pmatrix}.$$  

(B3)

for $s = 0, \pm$ and with the normalization

$$n^r = \frac{|(1^+ \hat{k}_x \hat{s} \omega - 1^+ \hat{k}_z \hat{s} \omega)^2|}{(1 + s^2)(1 + 1^+ \hat{k}_y^2 \hat{k}_x^2 + 1^+ \hat{k}_y^2 \hat{k}_z^2 + 1^+ \hat{k}_x^2 \hat{k}_z^2 \hat{f}^2)}.$$  

(B4)

The eigenvectors converge to their continuous counterpart for $\Delta_x, \Delta_y, \Delta_z \to 0$.

**REFERENCES**


