A New Inversion Method to Obtain Upper-Ocean Current-Depth Profiles Using X-Band Observations of Deep-Water Waves

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ABSTRACT

A new method for estimating current-depth profiles from observations of wavenumber-dependent Doppler shifts of the overlying ocean wave field is presented. Consecutive scans of marine X-band backscatter provide wave field measurements in the time–space domain that transform into the directional wavenumber–frequency domain via a 3D fast Fourier transform (FFT). Subtracting the linear dispersion shell yields Doppler shift observations in the form of \((k_x, k_y, \Delta \omega)\) triplets. A constrained linear regression technique is used to extract the wavenumber-dependent effective velocities, which represent a weighted depth average of the Eulerian currents (Stewart and Joy). This new method estimates these Eulerian currents from the effective velocities via the inversion of the integral relationship, which was first derived by Stewart and Joy. To test the effectiveness of the method, the inverted current profiles are compared to concurrent ADCP measurements. The inversion method is found to successfully predict current behavior, with a depth-average root-mean-square (RMS) error less than 0.1 m s\(^{-1}\) for wind speeds greater than 5 m s\(^{-1}\) and a broad wave spectrum. The ability of the inversion process to capture the vertical structure of the currents is assessed using a time-average RMS error during these favorable conditions. The time-averaged RMS error is found to be less than 0.1 m s\(^{-1}\) for depths shallower than 20 m, approximately twice the depth of existing methods of estimating current shear from wave field measurements.

1. Introduction

Upper-ocean processes play a major role in the transfer of heat, momentum, and gases between the ocean and atmosphere, which in turn drive local and global climate. Specifically, near-surface currents are responsible for the horizontal advection of surfactants, such as pollution and nutrients. Furthermore, near-surface current shear aids in the vertical advection of these tracers, which have the potential to affect biological processes and human health (Wu 1969). However, measurements of currents in the upper ocean [including depths \(O(10)\) m] are historically difficult to make, as they are contaminated by wave orbital velocities and wave-induced platform motion (Alpers et al. 1981b; Davis et al. 1981). These negative effects of wave motion can be eliminated by using remote sensing, such as high frequency (HF) radar, X-band radar, and visual imaging. These techniques use different methods to provide a measurement of wave celerity. Then, by exploiting our understanding of ocean surface wave propagation and wave dispersion, the effects of surface currents can be extracted. HF radar, for example, has been shown to provide good estimates of the average current in the upper 1 m and is widely used to fill this gap in current measurements (Crombie 1955; Barrick et al. 1977; Teague et al. 1997; Terrill et al. 2006).

Radar, such as HF and marine X band, interacts with the ocean surface via Bragg scattering, which involves the electromagnetic (EM) waves interacting resonantly with ocean waves with wavelengths corresponding to half the EM wavelength. For example, an HF radar transmitting a 12-MHz EM wave has a wavelength of approximately 25 m and resonates with an ocean wave with a wavelength of 12.5 m. HF radar operates by directly measuring the Doppler shift of the reflected EM wave, thereby measuring the velocity of the Bragg ocean wave. Marine X-band radar, however, operates with a much higher frequency (e.g., 9410 MHz) and therefore is associated with Bragg waves in the capillary wave scale (approximately 3 cm). Rather than measuring the Doppler shift of the returned EM waves, marine X-band radar accurately measures the range and bearing of the Bragg scatterer via a time-of-flight measurement. These
capillary Bragg waves are modulated by the underlying gravity waves, causing them to collect just forward of the crest of the larger waves (Alpers et al. 1981a; Nieto Borge et al. 2004). With repeating pulses emitted from a rotating antenna, consecutive backscatter maps can be constructed that represent the location of the wave crests in directional space and time. Current information can then be extracted from the observed wave velocities using linear wave theory (Young et al. 1985).

In deep water, linear wave theory relates wavenumber magnitude $k = |k| = \sqrt{k_x^2 + k_z^2}$ and frequency $\omega$ via the dispersion relationship,

$$\omega = \sqrt{gk + k \cdot u_R(k)},$$

where $u_R(k)$ is the wavenumber-dependent velocity of encounter, which can be separated into two components, or $u_R(k) = \hat{u}(k) + u_d(k)$ (the velocity of the observer is assumed to be zero) (Senet et al. 2001; Dankert and Rosenthal 2004). The quantity $\hat{u}(k)$ is the effective velocity, which represents a weighted depth-average effect of Eulerian currents (Stewart and Joy 1974). The $u_d(k)$ is the filtered Stokes drift, which is a nonlinear wave interaction correction (Weber and Barrick 1977; Ardhuin et al. 2009; Lund et al. 2015). The effect of $u_d$ on a wave with wavenumber $k$, propagating with direction $\theta$, can be represented by

$$u_d(k, \theta) = u_d(f_i) + 4\pi ke \int_{f_i}^{\infty} f \cos(\theta - \theta_i) S(f, \theta) d\theta d\theta,$$

where $f_i = \sqrt{gk_i/(2\pi)}$, $S(f, \theta)$ is the directional wave spectrum, and $e_i$ is the unit vector in the direction of $\theta_i$. The quantity $u_d(f_i)$ represents the Stokes drift vector for waves with frequencies less than $f_i$ by

$$u_d(f_i) = 4\pi \int_{f_i}^{\infty} f k(\theta)f S(f, \theta) d\theta d\theta.$$

The effect of the combined currents therefore acts to produce a wavenumber-dependent Doppler shift with magnitude

$$\Delta \omega = k \cdot u_R(k) = k \cdot [\hat{u}(k) + u_d(k)].$$

The relationship between the Doppler shift and the underlying current profile $u(z)$ in deep water (i.e., $kh \ll 1$, where $h$ is the water depth) was derived by Stewart and Joy (1974),

$$\hat{u}(k) = \frac{\Delta \omega}{k} - u_d(k) = 2k \int_{-\infty}^{z} u(z) e^{2kz} dz.$$

In the case of a depth-uniform current $u(z) = u_0$, (4) suggests the effective velocity becomes independent of wavenumber, and $\hat{u}(k) = u_0$. To first order, $u_d = 0$, and a depth-uniform current can therefore be estimated with a linear fit to $\Delta \omega = k \cdot u_0$. In the presence of current shear, this linear fit results in a wavenumber-independent bulk velocity, which represents the bulk effect underlying currents have on waves. Although the bulk velocity lacks shear information, it supplies a first-order current estimate used by much of the remote sensing community to derive ocean currents from wave observations (e.g., Teague et al. 1997).

In the presence of currents that vary linearly with depth, or $u(z) = mz + u_0$, (4) implies the effective velocity can be expressed as

$$\hat{u}(k) = -\frac{m}{2k} + u_0 = u[z = -(2k)^{-1}].$$

This means that in the presence of a linear current profile, effective velocities of wavenumber $k$ are equal to the geophysical velocity at an effective depth of $z = -(2k)^{-1}$. For example, under the linear current profile assumption, the effective velocity measured by a 12-MHz HF radar that interacts resonantly with a Bragg wave of $k = 0.5 \text{ rad m}^{-1}$ is approximately equal to the current velocity at a depth of $z = -1 \text{ m}$.

Similarly, if a logarithmic current-depth profile is assumed, then

$$u(z) = u(z = 0) - u^* \log \left( \frac{z}{z_0} \right),$$

where $u^*$ is the friction velocity, $z_0$ is a roughness length, and $\kappa$ is the von Kármán constant, then evaluating (4) results in

$$\hat{u}(k) = u(z = 0) - \frac{u^*}{\kappa} \log \left( \frac{1}{2krz_0} \right),$$

where $r = 1.78$. This result suggests the corresponding effective depth for a logarithmic current profile is $z = -(3.56k)^{-1}$ (Ha 1979). Using these two effective depth definitions, estimates of effective velocities spanning multiple wavenumbers can supply information about the current-depth profile (Stewart and Joy 1974; Ha 1979; Fernandez et al. 1996; Teague et al. 2001; Lund et al. 2015). Although these methods supply a means to extract current shear from broadband effective velocity measurements, they are dependent on an assumption of the vertical structure of the currents, and current information is constrained to a small range of depths defined by the wavenumber bandwidth and the effective depth definitions.

Ha (1979) developed a method to estimate $u(z)$ from the inversion of (4) without the a priori assumption of a current profile shape. The input to his inversion was composed of effective velocity measurements of four different frequency waves using a unique multifrequency HF system. However, the inherent noise amplification of the exponential form of the inversion of (4) was not sufficiently constrained by the four unique frequency points, yielding noisy current estimates.
The aim of this study is to develop and test a new technique to estimate current-depth profiles from the inversion of (4) by taking advantage of the broad wavenumber sensing capability of marine X-band radar imaging. Our new inversion technique contains four primary steps (Fig. 1). First, we collect X-band backscatter data in the time–space domain, which we transform into wavenumber–frequency information via a three-dimensional fast Fourier transform (3D FFT). Second, we remove the linear dispersion shell to create \((k_x, k_y, \Delta \omega)\) triplet points. Third, effective velocity profiles are extracted from these Doppler shift observations using a constrained regression. Finally, a stabilized inversion technique is developed to invert the effective velocities in order to estimate current-depth profiles. To assess the effectiveness of this method, we compare these results with concurrent ADCP current measurements.

2. Data collection

X-band backscatter was collected as a part of the Office of Naval Research (ONR)-funded Southern California 2013 (SoCal2013) field campaign within the southern Channel Islands off the coast of Los Angeles, California, in November 2013 (Fig. 2). The X-band radar antenna was mounted at a height of approximately 20 m above the sea surface on Research Platform (R/P) Floating Instrument Platform (FLIP) (Fisher and Spiess 1963). Backscatter was collected for 9 days between 13 and 22 November 2013 with a range and resolution of approximately 3 km and 3 m, respectively, and a scan rate of 42 rpm (Table 1).

A moored Datawell buoy was deployed approximately 750 m from FLIP’s location to measure wave conditions, which included a mix of wind sea and swell (Fig. 3). Wind conditions were measured from FLIP 10 m above sea level. The signal-to-noise ratio (SNR) of X-band backscatter is a result of the combination of incident wind and wave conditions (Fig. 4). SNR was highest during the first 3 days, while wind speed was consistent and building. The second half of the period was less consistent, with a short duration of sustained SNR on 20 November 2013. To avoid the strong wind dependence of the backscatter, in this study we select only periods of wind speeds greater than 5 m s\(^{-1}\).

In support of X-band-derived current estimates, an upward-looking 1200-kHz ADCP and a downward-looking 600-kHz ADCP were suspended from FLIP at a depth of 10 m and a distance of 10 m from the hull. The combination of the current measurements resulted in depth-current profile estimates from \(-50\) to \(-2\) m, representing 30-min ensemble-averaged currents. Depth bins above \(-2\) m were removed to avoid surface reflection and wave contamination. The evolution of the combined

**Fig. 1.** Flowchart describing the four main steps of the inversion process that begins with building the data cube of X-band backscatter and ends with the inversion of current profiles.
current profiles over time (Fig. 5) shows evidence of wind and tidal forcing throughout the experimental period.

3. Methods

To begin we defined a 750 m × 750 m inspection box, a region where we isolated backscatter data from each scan to create square subsections (Fig. 6). We then stacked 256 consecutive square subsections (representing 6 min of data) to create a space–time cube of backscatter. A 3D FFT was then used to transform these cubes into directional wavenumber–frequency space (Young et al. 1985). The result of three consecutive FFTs with 50% overlap was averaged to produce spectral energy in (kx, ky, ω) space representing 12 min of backscatter (Fig. 7).

The wavenumber bandwidth of the inversion process is defined by the dimensions and resolution of the backscatter cube, as well as the range of available observable wavenumbers. For this study, this wavenumber range was 0.05 rad m⁻¹ < k < 0.32 rad m⁻¹.

Because of the backscatter’s strong dependence on the wind, we assign the x and y directions as along- and crosswind, respectively. A dispersion mask was applied, defining “signal” as energy that did not deviate from the dispersion relationship (1) by more than an expected maximum effective current threshold of 1 m s⁻¹. The energy that was located far from the dispersion relationship was assumed to be noise, allowing us to define a signal-to-noise ratio SNR (Young et al. 1985). It has been shown that the strength of the marine X-band backscatter signal is proportional to wave height (Plant 1988). Retrieval of wave height information from X-band backscatter data typically requires the use of a modulation transfer function (MTF), which is constructed to remove radar-imaging effects (Plant 1988). This study, however, uses only the wavenumber–frequency information and therefore does not require the use of an MTF, thus simplifying the processing. However, the magnitude of the energy returned from the FFT process reflects that of the

FIG. 3. The periodogram from the moored Datawell wave buoy. The region bound by white dashed lines indicates the range of wave periods (4–10 s) used in the inversion process. Red and blue indicate high and low wave energy, respectively.

FIG. 4. Environmental conditions during the SoCal2013 experiment as a function of time. (a) Wind speed (solid) and direction (dotted) measured on R/P FLIP. (b) Significant wave height (solid) and peak wave period (dotted) from the moored Datawell buoy. (c) X-band SNR resulting from wind-wave conditions.

### Table 1. Configuration parameters for the X-band system used during SoCal2013.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radar type</td>
<td>Furuno 2117bb</td>
</tr>
<tr>
<td>Peak output power</td>
<td>12 kW</td>
</tr>
<tr>
<td>Antenna length</td>
<td>8 ft</td>
</tr>
<tr>
<td>Beamwidth</td>
<td>0.95° (horizontal)</td>
</tr>
<tr>
<td></td>
<td>20° (vertical)</td>
</tr>
<tr>
<td>Radar scan rate</td>
<td>42 rpm</td>
</tr>
<tr>
<td>Range resolution</td>
<td>3 m</td>
</tr>
<tr>
<td>Maximum range</td>
<td>~3 km</td>
</tr>
<tr>
<td>Radio wave frequency</td>
<td>9410 ± 30 MHz</td>
</tr>
<tr>
<td>Bragg scatterer wavelength</td>
<td>3.2 cm</td>
</tr>
<tr>
<td>Antenna height</td>
<td>30 m</td>
</tr>
<tr>
<td>ADCP resolution</td>
<td>12 bit</td>
</tr>
</tbody>
</table>
wave spectrum, with higher energy in the lower wavenumbers than higher wavenumbers. Removing noise by applying a uniform SNR cutoff would therefore bias toward the removal of higher wavenumber information and potentially leave unwanted noise in the lower wavenumbers. To avoid this problem, instead we applied a wavenumber-dependent SNR cutoff, selecting only \((k_x, k_y, v)\) points with the highest SNR in each wavenumber bin (Fig. 7). The Doppler shift of each of these selected points was calculated by subtracting the dispersion shell \([1]\) in the zero-current condition, or \(\Delta \omega = \omega_{\text{observed}} - (g k)^{1/2}\) (Fig. 8).

The process of estimating depth-current profiles from \((k_x, k_y, \Delta \omega)\) triplets involves two major steps (Fig. 1). First, wavenumber-dependent effective velocities are computed from the Doppler shifts using \((3)\). Second, current profiles are estimated from the effective velocities using an inversion of \((4)\).

### a. Estimation of the effective velocity profiles

Effective velocities were estimated from Doppler shift profiles using \((3)\) by first using a least squares process to...
estimate $\mathbf{u}_R(k)$, and then removing $\mathbf{u}_d(k)$ estimated from wave buoy observations. Constrained least squares techniques were employed to minimize noise in the effective velocities, which is inherently amplified in the later inversion of (4) (Ha 1979). This involved constraining the curvature of the $\mathbf{u}_R(k)$ profiles, which are expected to be smooth because they can be represented by integrals of inherently smooth functions (2) and (4).

First, observations were distributed into $m$ wavenumber magnitude bins, each with a unique $\mathbf{u}_R$. The $i$th velocity of encounter can be expressed with the matrix equation

$$
\begin{pmatrix}
\Delta \omega_{i,j} \\
\Delta \omega_{i,j+1} \\
\vdots \\
\Delta \omega_{i,n}
\end{pmatrix} =
\begin{bmatrix}
k_{x,i,j} & k_{y,i,j} \\
k_{x,i,j+1} & k_{y,i,j+1} \\
\vdots \\
k_{x,i,n} & k_{y,i,n}
\end{bmatrix}
\begin{pmatrix}
u_{R,i,j} \\
u_{R,i,j+1} \\
\vdots \\
u_{R,i,n}
\end{pmatrix}
$$

or

$$
\Delta \omega = \mathbf{K} \mathbf{u}_R,
$$

where $\Delta \omega_{i,j}(k_{x,i,j}, k_{y,i,j})$ represents the $j$th of $n$ observations within the $i$th wavenumber bin. Combining all wavenumber bins into one matrix equation yields

$$
\begin{pmatrix}
\Delta \omega_{1,j} \\
\Delta \omega_{1,j+1} \\
\vdots \\
\Delta \omega_{m,j}
\end{pmatrix} =
\begin{bmatrix}
\mathbf{K}_1 & \mathbf{K}_j & \cdots & \mathbf{K}_m
\end{bmatrix}
\begin{pmatrix}
u_{R,1,j} \\
u_{R,1,j+1} \\
\vdots \\
u_{R,m,j}
\end{pmatrix}
$$

Equation (6) is in the appropriate form to estimate $\mathbf{u}_R$ using least squares techniques. Because there are more observation points than wavenumber bins, (6) is an overdetermined system. Therefore, the unconstrained least squares solution would result in smoother results than solving (3) for each observation point at a cost of fewer unique wavenumbers. However, the unconstrained result does not take advantage of neighboring solutions to constrain the smoothness of $\mathbf{u}_R(k)$. So, instead the smoothness was constrained by minimizing the second derivative of the solution. The second derivative of $\mathbf{u}_R$ can be estimated using a Taylor expansion by

$$
\frac{\partial^2 \mathbf{u}_R}{\partial k^2} \approx \frac{1}{\Delta k^2} \begin{bmatrix}
1 & 0 & -2 & 0 & 1 & 0 \\
0 & 1 & 0 & -2 & 0 & 1 \\
0 & 0 & 1 & 0 & -2 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}\begin{pmatrix}
u_{R,1,j} \\
u_{R,1,j+1} \\
u_{R,2,j+2} \\
u_{R,3,j+3} \\
\vdots
\end{pmatrix}
$$

The curvature of the combined $\mathbf{u}_R(k)$ profiles is therefore

$$
\frac{\partial^2 \mathbf{u}_R}{\partial k^2} \approx \mathbf{C} \mathbf{u}_R,
$$

To further constrain the solution to the inversion, the wavenumber-independent bulk current was used as a first guess, representing a first-order depth-uniform current assumption. The bulk current was calculated from a linear regression,

$$
\Delta \omega = \mathbf{k} \cdot \mathbf{u}_{\text{bulk}}.
$$

The bulk velocity vector was then reformatted to match $\mathbf{u}_R$. Solving (6) while constraining the curvature (7) yielded the least squares result

$$
\mathbf{d} = \mathbf{A} \mathbf{u}_R,
$$

where $\lambda$ is a tunable scalar representing the extent to which the curvature is constrained. The effective velocities are then

$$
\mathbf{u} = \mathbf{u}_{\text{bulk}} + (\mathbf{A}^T \mathbf{A} + \lambda \mathbf{C}^T \mathbf{C})^{-1} \mathbf{A}^T (\mathbf{d} - \mathbf{A} \mathbf{u}_{\text{bulk}}),
$$

Following Arduhin et al. (2009), the Stokes drift vector was approximated by the nondirectional Stokes drift $u_s(f) = (16\pi^2 g) \int_0^\infty E(f) df$, where $E(f)$ is the one-dimensional...
energy spectrum. The filtered Stokes drift contribution in (2) was calculated using \$u_i(f)\$ and the directional wave spectrum from the moored wave buoy. Because of the conditions observed during this experiment, along with the range of wavenumbers used in the inversion process, the contribution of the filtered Stokes drift to the velocity of encounter was no more than 5\% of the effective velocity. The along-wind and crosswind components \$u_x(z)\$ and \$u_y(z)\$ individually.

b. Current profile inversion

An inversion method was developed to estimate current profiles from the wavenumber-dependent effective velocities \$\bar{u}(k)\$. Previous work in deep water (Ha 1979) showed that the inversion of (4) lead to amplification of measurement noise. The inversion process was therefore stabilized by 1) applying Legendre quadrature to approximate (4) as a finite sum (Cohen 2007) and 2) using multiple constrained least squares techniques to estimate current profiles (Twomey 1977; Wunsch 1996).

Legendre quadrature suggests that an integral of a smooth function \$f(\xi)\$ can be estimated as the weighted sum,

\[
\int_{-1}^{1} f(\xi) \, d\xi \approx \sum_{i=1}^{n} f(\xi_i) w_i, \tag{9}
\]

where \$\xi\$ are the zeros of the Legendre polynomial of order \$n\$, and \$w\$ are their weights, both of which are tabulated (Golub and Welsch 1969; Cohen 2007). To change the form of (4) to match (9), the substitution

\[
\xi = 2e^{-2k_0z} - 1
\]

was made, where \$k_0\$ is a reference wavenumber chosen to minimize quadrature error (Ha 1979). The resulting integral is

\[
\frac{2\bar{u}(k)}{k/k_0} = \int_{-1}^{1} \bar{u}(z) \left( \frac{1}{2} \xi + \frac{1}{2} \right)^{(k/k_0)-1} d\xi, \tag{10}
\]

where

\[
\xi = \frac{\ln \left( \frac{1}{2} \xi + \frac{1}{2} \right)}{-2k_0}. \tag{11}
\]

Applying (9) to (10) and (11) yields

\[
\frac{2\bar{u}(k)}{k/k_0} \approx \sum_{i=1}^{n} \left( \frac{1}{2} \xi_i + \frac{1}{2} \right)^{(k/k_0)-1} \bar{u}(z_i) w_i, \tag{12}
\]

or

\[
f = \bar{G}u, \tag{12}
\]

where

\[
f_i = \frac{2\bar{u}(k_i)}{k_i/k_0}
\]

\[
G_{ij} = \left( \frac{1}{2} \xi_i + \frac{1}{2} \right)^{(k_i/k_0)-1} w_i
\]

\[
u_i = u(z_i)
\]

\[
z_i = \frac{\ln \left( \frac{1}{2} \xi_i + \frac{1}{2} \right)}{-2k_0}.
\]

Similar to (6), the form of (12) is appropriate for using least squares techniques to estimate \$u(z)\$. Again, because the solution is expected to be smooth, the curvature was constrained using the second derivative approximation

\[
\frac{\partial^2 f}{\partial z^2} \approx \frac{1}{\Delta z^2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -2 & 4 & -1 & 0 \\ 0 & -2 & 4 & -1 \\ 0 & 0 & -2 & 4 \end{bmatrix} u = Df, \tag{13}
\]

To further constrain the inversion result, the observations were weighted by their variances with

\[
S = \boldsymbol{I} \delta,
\]

where \$\delta_i = \sigma_i^{-2}\$, the inverse variance of the Doppler shifts in the \$i\$th wavenumber bin.

As previously, we used a uniform depth current as an initial guess,

\[
u_0 = u_{\text{Bulk}}.
\]

Combining (12)–(14) resulted in the cost function

\[
\Lambda = (\bar{G}u - f)^T (D^TD)(\bar{G}u - f) + \eta \nabla^T (S^T S) f + \gamma (u - u_0)^T (u - u_0), \tag{15}
\]

where \$\eta$ and \$\gamma$ are tunable parameters, specifying the relative extent to which curvature minimization, distance from the first guess, and the weighted model are constrained. Minimizing (15) with respect to \$u\$ yields the least squares result

\[
u = [\bar{G}^T (S^T S) \bar{G} + \eta (D^TD) + \gamma I]^{-1} [\bar{G}^T (S^T S) f + \gamma u_0]. \tag{16}
\]

This result was used to estimate current profiles \$u_x(z)\$ and \$u_y(z)\$ from effective velocities \$\bar{u}_x(k)\$ and \$\bar{u}_y(k)\$, respectively.
To test the robustness of the inversion results, we compared our current profile estimates to those measured by the acoustic instruments. The root-mean-square error (RMSE) was used as the error metric to evaluate the success of the inversion model, where we define RMSE as

$$\text{RMSE} = \sqrt{\left(\bar{u}_{\text{inverted}} - \bar{u}_{\text{measured}}\right)^2},$$ (17)

and $\bar{\cdot}$ denotes an average either over depth (to see time dependence) or time (to see depth dependence).

4. Results

Current profiles were estimated from X-band-derived Doppler shift estimates using (8) and (16) using 10 days of data and the model parameters displayed in Table 2. To increase the signal-to-noise ratio of the Doppler shift signal, one hour of $(k_x, k_y, \Delta \omega)$ triplets was combined to estimate the effective velocities from (8).

To tune the weighting parameter $\lambda$ in (8) and to assess the result of the effective velocity estimation, a forward solution of (4) was found, which transformed ADCP current profiles into effective velocity–wavenumber profiles. This forward solution supplied useful insight into the relationship between current profiles and the overlying wave field. For example, the exponential weighting of near-surface currents in (4) and the truncation of ADCP profiles above 2-m depth resulted in a bias between the forward solution and the estimated effective velocities. To remove this bias, a surface current of magnitude $u_0$ was added to the ADCP current profiles. The forward problem was then carried out to estimate the effective velocity. Here, $\lambda$ was chosen to minimize the difference between the estimated effective velocities from (8) and the calculated effective velocity from the forward problem.

To compare the results of the forward problem with the modeled effective velocity estimates from (8), (3) was used to calculate $(k_x, k_y, \Delta \omega_{\text{forward}})$ and $(k_x, k_y, \Delta \omega_{\text{XBand}})$ triplets, where $(k_x, k_y)$ were chosen to span all directions (Fig. 9). As indicated by (4), a depth-uniform current $u_0$ would result in a Doppler shift–wavenumber relationship $\Delta \omega \approx k \cdot u_0$, or a plane in $(k_x, k_y, \Delta \omega)$ space. Because our results show nonlinear contour lines (Fig. 9), this is an indication that current shear is present. The strength and direction of the gradient of the Doppler shift–wavenumber relationships and their contour line shape behave similarly, indicating the estimated effective velocities from (8) agree with the forward solution of (4) in magnitude, direction, and current shear content.

![Fig. 9. A comparison of Doppler shifts calculated from (a) the effective velocity from (8) and (b) those from the forward problem. In the absence of current shear, the counters would map as straight lines, so here the nonlinearity of the contours indicate the presence of current shear. Green indicates no Doppler shift, whereas red indicates positive and blue indicates negative Doppler shifts.](http://journals.ametsoc.org/doi/pdf/10.1175/JTECH-D-16-0120.1)
To ensure $\hat{u}(k)$ estimated from (8) were not biased by the least squares process, the triplets $(k_x, k_y, \Delta \omega_{XBand})$ were compared to those calculated by the FFT. The difference between this new set of Doppler shift–wavenumber pairs and those observed (e.g., Fig. 8) represents the residual from (8) (Fig. 10). The residual field is made up of small magnitude (green) values with no coherent structure in wavenumber space. This indicates the least squares technique (8) does not introduce significant bias into the effective velocity estimates.

Currents were estimated from the effective velocity profiles using (16) in the along- and crosswind directions (Fig. 11). For these data, we excluded periods when the wind speed fell below 5 m s$^{-1}$. Although the inversion process results in current profile estimates extending to the surface, the top 2 m of the results were discarded to be directly comparable to the concurrent ADCP measurements. Because the integral in (4) accounts for currents at infinite depth, the depths to which the current inversion attempts to estimate currents is bounded only by the order of the Legendre polynomial in (9). However, it is expected that the inversion has an effective depth to which inverted currents yield reliable results. Furthermore, this effective depth is expected to be dependent on the wavenumber bandwidth used in the inversion. The results show that both orthogonal current estimates approach zero at a depth of approximately 20 m, below which they rapidly become unrealistically large. Because this structure is not seen in the ADCP current profiles, this suggests the effective depth of this inversion process is approximately $-20$ m.

Based on our finding of $-20$ m for the effective depth of the inversion, we calculated the depth-averaged RMS error between inverted and ADCP currents to only $-20$-m depth (Fig. 12). The average RMS errors during the first 3 days of the experiment were primarily below 0.1 m s$^{-1}$, with slightly higher errors in the crosswind direction. The error in the second half of the experiment, however, was typically larger than 0.1 m s$^{-1}$ despite similar wind speeds. We attribute the higher error in the second half of the experiment to the distribution of energy in the wave field. The periodogram from the moored Datawell buoy (Fig. 3) shows that the location of wave energy in spectral space varied throughout the experiment. During the first half of the experiment, wave energy was broadly spread throughout the band of

**Figure 10.** The Doppler shift residuals, $\Delta \omega - \hat{u}(k) \cdot k$, derived from estimating effective velocities. Term $\hat{u}(k)$ is from (8). Green denotes zero residual, red denotes positive residual, and blue indicates negative residual.

**Figure 11.** Results of current inversion in (a) along-wind and (b) crosswind directions. White spaces indicate times of excluded data because the wind speed fell below the threshold value of 5 m s$^{-1}$. Red represents higher velocities in the (a) along-wind and (b) crosswind directions, and blue represents higher velocities in the opposite direction.
frequencies used in the X-band inversion (i.e., 0.7–1.7 rad s\(^{-2}\)). Later in the experiment, however, there was a lack of energy in either the higher (e.g., 19 November 2013) or lower (e.g., 20 November 2013) parts of the inversion’s frequency band (Fig. 3). A comparison of the wave spectra from 1300 UTC 13 November 2013 and 1300 UTC 20 November 2013 (Fig. 14) shows the lack of wave energy in the lower part of the inversion frequency band on the later date. The higher RMS error during periods when there are gaps in the energy spectrum implies the inversion is not only dependent on wind speed but also on the available observable wavenumbers determined by the wave spectrum.

The tunable parameters \(\eta\) and \(g\) in (17) were chosen to minimize RMS error. This process showed little sensitivity to fluctuations in \(g\) between 0.1 and 1. The selection of \(\eta\), however, was found to have a larger effect on the inversion. In general, a large \(\eta\) resulted in a smooth \(u(z)\) profile, which effectively constrained the noise amplification with the cost of neglecting current shear. During times of high SNR, therefore, a smaller value of \(\eta\) could be used to attempt to capture more of the current shear structure. Data with lower SNR however contained more noise, requiring a larger value of \(\eta\) to constrain the noise amplification. Thus, we devised this simple empirical relationship between SNR and \(\eta\):

\[
\eta(SNR) = 0.25 + 0.3 \left(1 - \frac{\text{SNR}}{\text{SNR}_{\text{max}}}ight),
\]

leading to a value of \(\eta\) varying between 0.25 and 0.35.

The depth dependence of the inversion result was evaluated with a time-average RMS error from (17) between inverted and measured velocity profiles (Fig. 13). To assess this error separately from the depth-averaged

![Fig. 12. The time series of the RMSE of inverted currents between 2- and 20-m depth for along- and crosswind directions. Blank periods indicate times of wind speeds < 5 m s\(^{-1}\).](image)

![Fig. 13. The depth dependence of RMSE of the (a) along-wind and (b) crosswind current-depth profiles estimated by the inversion (solid), a linear assumption (dashed), a logarithmic assumption (asterisk–dotted), and depth uniform (dotted). RMSE was calculated for times between 0000 UTC 13 Nov 2013 and 0800 UTC 16 Nov 2013.](image)
error, only periods during which the depth-average RMS error was less than 0.1 m s$^{-1}$ were used. The inversion performs well, with an RMSE $> 0.1$ m s$^{-1}$ in the top 20 m, as was qualitatively expected from Fig. 11. RMS error in the along-wind direction is smaller on average, and it remains small deeper than the crosswind direction. The depth dependence of the RMS error was also compared to the RMS error of currents estimated under the linear, logarithmic, and depth-uniform current profile assumptions [e.g., from (5)]. In the along-wind direction, where current shear was highest, all three methods that took current shear into account performed better than the depth-uniform assumption (Fig. 13a). The tidally dominated crosswind currents were more appropriately approximated by the depth-uniform assumption, which resulted in RMS errors that were similar to the methods that took shear into account (Fig. 13b). For both directions, currents estimated under the logarithmic and linear assumptions exhibited very similar behavior to the inversion at the depths at which they are defined, with RMS errors within 0.02 m s$^{-1}$ of each other. However, the inversion continued to perform with similar errors down to a depth of approximately $-20$ m.

5. Discussion

The inversion method summarized by (8) and (16) successfully captured time and depth fluctuations of along- and crosswind currents (Figs. 11–13). These results proved to be strongly dependent on environmental conditions, specifically wind speed and wave spectra. In favorable conditions (e.g., 13–16 November 2013), the estimated currents had RMS errors less than 0.1 m s$^{-1}$. This error is similar to other reported radar-derived current estimates, which range between 0.07 and 0.2 m s$^{-1}$ (e.g., Paduan and Rosenfeld 1996; Graber et al. 1997; Teague et al. 2001; Kelly et al. 2003). The smaller errors in the along-wind direction than in the crosswind direction can be attributed to two factors. First, there is a strong dependence between winds and X-band backscatter energy, where the backscatter is strongest in the direction of the wind (Dankert et al. 2003). This leads to less uncertainty in Doppler shift measurements of waves traveling in the wind direction compared to crosswind waves. Second, because wind and wave directions were similar throughout the experiment (Fig. 4a), both winds and waves contribute to the clustering of $(k_x, k_y, \Delta \omega)$ triplet points in the wind direction (Fig. 7), leading to larger error in (8) in the crosswind direction.

The success of the current inversion technique showed strong dependence on environmental conditions. As expected, the envelope of environmental conditions in which the inversion performed successfully is defined by the wind and wave field. Specifically, wind speeds greater than 5 m s$^{-1}$ and a wave spectrum with significant energy spanning the entire frequency range used in the inversion process lead to the lowest RMS error (Fig. 12 and Fig. 14). Environmental variables that did not vary appreciably during this experiment but would be expected to play a role in the inversion include wave directional spread, split sea-swell events, and wind-wave direction differences. Furthermore, it is expected that the role of Stokes drift would become more important under different wave conditions. With a stronger Stokes drift influence, the propagation of uncertainties of approximations such as the unidirectional Stokes drift estimation and the filtered Stokes drift calculation could negatively influence the inversion result without proper care.

Because of the inversion’s sensitivity to the wave spectrum, one possible improvement to this inversion model is to bandpass the frequency of waves used in the inversion based on the measured wave spectrum. Changes in the frequency band however would require changes in the weighting parameters and would result in fluctuations of the model’s effective depth.

Although the RMS error suggests a close relationship exists between the inversion error and the environmental conditions, there is not a clear way to quantify the individual effect wind and waves have on error. Instead, the source of error can be broken into components, such that the Doppler shift in (3) is

$$\Delta \omega_{\text{observed}}(k) = \Delta \omega_{\text{true}}(k) + \Delta \omega'(k) + \epsilon_{\text{radar}}(k),$$  \hspace{1cm} (18)

where $\Delta \omega_{\text{true}}$ is the error-free Doppler shift caused by the underlying currents, the quantity $\Delta \omega'$ is the wavenumber-dependent random noise in the observations based on data...
resolution, small nonlinear interactions, etc., the quantity $\varepsilon_{\text{radar}}$ is a bias introduced by the assumption that the radar backscatter perfectly represents the spectral quantities of the wave field. The inversion result, therefore, can be expressed as

$$u_{\text{Inverted}}(z) = u_{\text{True}}(z) + u'(z) + \delta_{\text{radar}}(z) + \delta_{E}(z) + \delta_{I}(z),$$

where $u_{\text{True}}$ is the true current profile, $u'$ represents the propagation of the random error in the Doppler shifts, and $\delta_{\text{radar}}$ represents the radar bias that is propagated through the inversion process. The $\delta_{E}$ is the bias introduced during the estimation of effective velocities in (8). Because of the form of the kernel in (6) [i.e., geometrically, (6) describes an overdetermined planar fit for each wavenumber bin], $\delta_{E}$ is expected to be very small, which is supported by the low bias of the residuals in Fig. 10. The $\delta_{I}$ is the depth-dependent bias introduced by the inversion process (16), which is to be determined. The error associated with the inversion process can therefore be approximated by estimating the terms $u'(z)$ and $\delta_{I}(z)$. This requires separating the effects of the input noise in (18).

The random noise in the radar-derived Doppler shift, $\Delta\omega(k)$, was approximated using the scatter of the residuals in Fig. 10. The standard deviation of the spread of these residuals was azimuthally averaged to produce profiles of $\sigma_{\Delta\omega}(k)$ for each time point for which the environmental envelope described above was filled (0000 UTC 13 November 2013–0300 UTC 16 November 2013) (Fig. 15). Because of the small variation among these profiles, the average $\sigma_{\Delta\omega}(k)$ profile is a good representation of the random noise for this period.

There are many individual error sources that contribute to $\varepsilon_{\text{radar}}$, including the various effects related to the geometry of the radar imaging. The shadowing of smaller waves by larger waves, for example, affects the spectrum of the wave field observation. By imaging only smaller waves located near the crests of larger waves where orbital velocities are largest, shadowing also introduces bias into the Doppler shift of the smaller waves. Furthermore, the effect of shadowing increases with range from the radar as the imaging angle between the radar antenna and the sea surface decreases. This introduces heterogeneity within the inspection square, which is spatially averaged by the FFT. Wave height and wave age also play roles in the geometry of the radar imaging, affecting wave shadowing. Additionally, temporal and spatial fluctuations in wind speed also introduce heterogeneity in the wave field imaging. These combined effects make it impractical to formulate an analytical solution to the influence of $\varepsilon_{\text{radar}}$ on the Doppler shift estimate. However, if it is assumed that the forward solution to (4) using ADCP currents approximates $\Delta\omega_{\text{True}}$, then a hybrid set of Doppler shifts can be constructed such that $\Delta\omega_{\text{True}} = \Delta\omega_{\text{True}} + \Delta\omega'$, where $\Delta\omega'(k)$ is made up of random noise with wavenumber-dependent standard deviation $\sigma_{\Delta\omega}(k)$.

We constructed a hybrid Doppler shift profile for currents measured on 0000 UTC 15 November 2013. A Monte Carlo analysis was used to estimate $u'(z)$ and $\delta_{I}(z)$ by inverting the constructed Doppler shifts and comparing them to the ADCP current profile. The results of 100 iterations of this process (Fig. 16) show that both $\delta_{I}(z)$ and $u'(z)$ increase with depth, as expected from the exponential weighting in (4). The maximum of their combined values remains less than 0.05 m s$^{-1}$ in the top 20 m, suggesting it makes up approximately half of the error in the inversion, with the other half contained in $\varepsilon_{\text{radar}}$. The value of the inversion bias $\delta_{I}(z)$ is dominated by the exponential shape of the kernel of (12), leading to its increase with depth. However, $\delta_{I}(z)$ is also sensitive to strong current shear, as seen in Fig. 16a, between −15 and −20-m depth. This bias is introduced by the necessity of enforcing a smooth current profile via the constraint in (13) and is directly related to the selection of $\eta$ as discussed above.

A potential way to minimize the enhanced noise observed at low wavenumbers (Fig. 15) is to increase the size of the inspection square, thereby sampling a higher number of longer waves. This could also provide signal at wavenumbers lower than the 0.05 rad m$^{-1}$ cutoff used in this study. However, (4) shows that the integral express that is inverted in this work is proportional to $\Delta\omega k^{-2}$. The use of lower wavenumbers therefore increases the amplification of noise in the Doppler shift.
measurement. Therefore, if lower wavenumbers (i.e., \( k < 0.05 \text{ rad m}^{-1} \)) are used in the inversion process, then care must be taken to reduce the noise to a lower level than was observed in this study. There has been much work on the reduction of error in the inversion of surface currents from radar measurements (e.g., Gangeskar 2002), which could be combined with this inversion process to enhance accuracy.

6. Conclusions

This work has presented the development and application of a new current inversion method, using the wavenumber-dependent Doppler shift measurements of X-band radar to estimate depth-dependent currents. The results of the current inversion showed RMS error less than 0.1 m s\(^{-1}\) for wind speeds greater than 5 m s\(^{-1}\) and a wave spectrum containing wave energy throughout the inverted wavenumber band. For the wavenumber range used for this study (0.05 < \( k < 0.3 \text{ rad m}^{-1} \)), the RMS error of inverted currents was below 0.1 m s\(^{-1}\) to a depth of approximately -20 m. The time and depth dependences of the RMS error of the current inversions indicate the inversion process is capable of estimating currents with a similar error as other radar-based methods, but with approximately twice the depth range (Fig. 13) and without an a priori assumption of the current-depth profile shape.

Although the X-band backscatter used in this study was collected from a stationary floating platform, X band can operate with similar performance from moving vessels. This inversion process could therefore be used with a ship-based system with the requirement that the inspection square be illuminated by the radar for the required FFT window (6 min).

The results of the inversion process (Fig. 11) indicate an amplification of the near-surface current in the alongwind direction, but not in the crosswind direction. Although a detailed analysis of this result is left for future work, this is qualitatively consistent with models of wind-driven currents. This result holds promise for the inversion process developed here to provide current

![Diagram](image_url)
shear information in the upper few meters, a regime in which it has been historically difficult to make current measurements.

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