Integration of Statistics- and Physics-Based Methods—A Feasibility Study on Accurate System Reliability Prediction

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Component reliability can be estimated by either statistics-based methods with data or physics-based methods with models. Both types of methods are usually independently applied, making it difficult to estimate the joint probability density of component states, which is a necessity for an accurate system reliability prediction. The objective of this study is to investigate the feasibility of integrating statistics- and physics-based methods for system reliability analysis. The proposed method employs the first-order reliability method (FORM) directly for a component whose reliability is estimated by a physics-based method. For a component whose reliability is estimated by a statistics-based method, the proposed method applies a supervised learning strategy through support vector machines (SVM) to infer a linear limit-state function that reveals the relationship between component states and basic random variables. With the integration of statistics- and physics-based methods, the limit-state functions of all the components in the system will then be available. As a result, it is possible to predict the system reliability accurately with all the limit-state functions obtained from both statistics- and physics-based reliability methods. [DOI: 10.1115/1.4039770]

1 Introduction

System reliability can be numerically measured by the probability that the system performs its intended function without failures. As the system state (safe or failed) is determined by the states of its components and it is hard to predict the system reliability directly, the system reliability is usually estimated based on component states. The accurate system reliability prediction requires the joint probability density of component states [1].

A physical component itself can be considered as a system since it may have multiple failure modes, and the reliability of the physical component is determined by the states of all the component failures. For this reason, we also consider a failure mode as a component and a physical component as a system if it has multiple failure modes.

Statistics-based methods [2] and physics-based methods [3] are two possible choices for component reliability analysis. A statistics-based method relies on field or testing data related to failures of a component. In this study, we consider only static reliability that does not change over time, which means the reliability estimation does not involve time. Physics-based reliability methods use a limit-state function, which is derived from physics principles, to predict the state of a component failure mode. The limit-state function is usually denoted by $y = g(\mathbf{X})$, where $\mathbf{X}$ is a vector of basic random variables, and y is the state variable. If $y < 0$, the state is failed. Then the probability of failure p_f with respect to this failure mode is given by

$$p_f = \Pr\{\text{state} = \text{failed}\} = \Pr\{y = g(\mathbf{X}) < 0\} \tag{1}$$

Since there is rarely a closed-form solution to Eq. (1), many approximation methods have been developed, such as the first-order reliability method (FORM) [4], the second-order reliability method (SORM) [5], the saddlepoint approximation method [6], Monte Carlo simulation [7], and matrix-based system reliability method [8]. Numerous applications of these methods have been reported for many systems, such as mechanical, automation, civil, and communication systems.

If the limit-state functions for all the failure modes of the components in the system are available, it is possible to estimate the system reliability for a given system configuration (series, parallel, mixed, and network). Next, we take a series system as an example because it is commonly encountered in mechanical applications. If one failure mode occurs or one component fails, the entire system will fail. Suppose the system consists of multiple components and there are totally m failure modes with each denoted by $P_i (i = 1, 2, \ldots, m)$, where P_i stands for failure event $g_i(\mathbf{X}) < 0$ and $g_i(\cdot)$ is the limit-state function for the ith failure mode. Then the probability of system failure is computed by

$$p_{fs} = \Pr\left(\bigcup_{i=1}^{m} P_i\right) = \Pr\left(\bigcup_{i=1}^{m} y_i = g_i(\mathbf{X}) < 0\right) \tag{2}$$

Equation (2) requires the joint distribution of $y_i (i = 1, 2, \ldots, m)$, but it is difficult to obtain such a joint distribution, which requires all the details about $g_i(\mathbf{X})$ and the dependency between components. As a result, the independence assumption is widely used in practice [9], where all the component states are assumed to be independent. For the above series system, the system reliability is calculated by

$$R_S = \prod_{i=1}^{m} R_i \tag{3}$$

Although this method is easy to use, it may result in an estimated reliability far smaller than the true value. Without the complete joint probability distribution of component states, it is difficult to evaluate the system reliability accurately, especially when component reliabilities are estimated by statistics- and physics-based methods independently.

Recently, Hu and Du [10,11] proposed a new method that reconstructs component limit-state functions with limited reliability information, making it possible to evaluate system reliability using Monte Carlo simulation. The method is effective for cases where component reliability data are provided with respect to system loads. A proof-of-concept method has also recently been proposed for systems with both in-house and outsourced components [12], where the reliabilities of in-house components are estimated with physics-based methods and those of the outsourced ones with statistics-based methods. The study has shown the feasibility of integrating statistics- and physics-based reliability approaches for special problems. The objective of this work is to further investigate the method proposed in Ref. [12]. For the statistics-based methods, samples of basic variables (loading, material properties, dimensions, etc.) and the component states, either safe or failed, are available. We adopt support vector machines (SVM) to build linear limit-state functions with respect to the basic variables since SVM is one of the best classification methods due to its high efficiency and accuracy. It has also been employed in many studies

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[13,14]. Then, with the limit-state functions generated by SVM and those from physics-based methods, the system reliability could be accurately estimated.

The scope and assumptions of the new method are as follows: components fail due to excessive loads. For components whose reliability is estimated by a statistics-based method, observations of basic random variables are available. Distributions of all basic random variables are known. The study focuses on series systems although it can be extended to parallel systems.

The proposed method has the following advantages: (1) limit-state functions built from statistical data can be easily integrated with those derived from physics. This helps system designers understand the dependency between component failures and enables them to construct a complete joint distribution of component states; (2) the proposed method does not restrict the number of basic random variables (such as loads) shared by components. Hence, it has a broader application scope than the previously proposed methods [10,11] that can accommodate only one common system load.

We provide a brief review of methodologies used in this work, including SVM and first-order reliability method in Sec. 2. In Sec. 3, the SVM method for building limit-state functions and the procedure of system reliability analysis with the proposed method are introduced. Three examples are discussed in Sec. 4. Conclusions and future work are presented in Sec. 5.

2 Methodology Review

In this work, we use FORM for physics-based component reliability analysis and use SVM to construct limit-state functions for failure modes (components) whose reliabilities are estimated by a statistics-based method. Both of the methods are briefly reviewed below.

2.1 Support Vector Machine.

Given a set of training points

$$(\mathbf{x}_1, y_1), (\mathbf{x}_2, y_2), \ldots, (\mathbf{x}_k, y_k), \mathbf{x} \in R^n, y \in (+1, -1) \tag{4}$$

in which $\mathbf{x}_i$ is a training point, and y is the class label for $\mathbf{x}_i$. Note that $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})(i = 1, 2, \ldots, k)$ is an n-dimensional row vector. The value of y depends on whether the point belongs to the first class or the second one. In this work, if the point falls in the safe region, then $y = +1$; otherwise, $y = -1$. The objective of SVM is to separate the training points into two classes with a hyperplane, as shown in Fig. 1, which is given by

$$\boldsymbol{\omega}\mathbf{X}^{\mathrm{T}} + b = 0 \tag{5}$$

where $\boldsymbol{\omega}$ is a weight vector, and b is the bias. The shaded points passed by these hyperplanes are called support vectors (SVs), and there are no points between these hyperplanes.

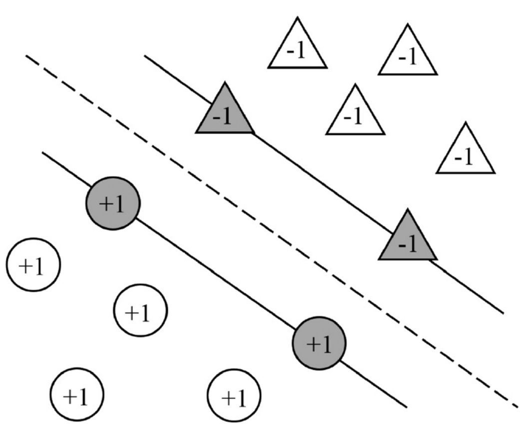

Fig. 1 Marginal classifiers along with SVs

The optimal separating hyperplane appears in the center and can be obtained by solving the following quadratic optimal problem:

$$\begin{cases} \min\limits_{\boldsymbol{\omega}} \dfrac{1}{2}\boldsymbol{\omega}\boldsymbol{\omega}^{\mathrm{T}} \\ \text{s.t. } y_i(\boldsymbol{\omega}\mathbf{x}_i^{\mathrm{T}} + b) \geq 1, (i = 1, 2, \ldots, k) \end{cases} \tag{6}$$

It can be converted into a dual problem according to the Lagrange principle and is given by

$$\begin{cases} \max\limits_{\lambda} L = \sum\limits_{i=1}^{k} \lambda_i - \dfrac{1}{2} \sum\limits_{i=1,j=1}^{k} \lambda_i \lambda_j y_i y_j \mathbf{x}_i \mathbf{x}_j^{\mathrm{T}} \\ \text{s.t. } \sum\limits_{i=1}^{k} y_i \lambda_i = 0,\ \lambda_i \geq 0, \quad (i = 1, 2, \ldots, k) \end{cases} \tag{7}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ are the Lagrange multipliers. The weight vector is then calculated by

$$\boldsymbol{\omega} = \sum_{i=1}^{k} \lambda_i y_i \mathbf{x}_i \tag{8}$$

According to the Karush–Kuhn–Tucker conditions, only the SVs lead to $\lambda_i \neq 0$. This means that only the SVs appear in the optimal result.

2.2 First-Order Reliability Method.

First-order reliability method is a physics-based reliability method, which linearizes the limit-state function $g(\mathbf{X})$ at the most probable point using the first-order Taylor expansion. Three steps are involved.

First, assume that all the random variables in $\mathbf{X}$ (in the X-space) are independent. The original random variables $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ are transformed into standard normal random variables $\mathbf{U} = (U_1, U_2, \ldots, U_n)$ in the U-space. The transformation is given by [15]

$$x_i = F_i^{-1}(\Phi(u_i)) = T(u_i) \quad (i = 1, 2, \ldots, n) \tag{9}$$

where $F_i(\cdot)$ and $\Phi(\cdot)$ are the cumulative distribution functions of X_i and a standard normal variable, respectively, and $T(\cdot)$ denotes the transformation function.

Second, at the most probable point, $g(T(\mathbf{U}))$ could be approximated to a linear function as follows:

$$G(\mathbf{U}) = \beta + \boldsymbol{\alpha}\mathbf{U}^{\mathrm{T}} \tag{10}$$

Thirdly, with the new limit-state function $G(\mathbf{U})$ in Eq. (10), p_f is calculated by

$$p_f = \Pr\{G(\mathbf{U}) < 0\} = \Phi(-\beta) \tag{11}$$

3 System Reliability Prediction With Combined Statistics- and Physics-Based Methods

The objective of this study is to integrate statistics- and physics-based reliability methods so that the joint probability density function (PDF) of all the component states is available. As discussed previously, the two different types of components (failure modes) are defined as follows:

- Type I: Type I components have limit-state functions and their reliabilities can be estimated by physics-based reliability methods, such as FORM.
- Type II: Type II components do not have limit-state functions, and their reliabilities are estimated by statistics-based methods.

The main idea of this work is to construct limit-state functions for type II components using testing data. Then with all the available limit-state functions, the system reliability could be estimated.

3.1 Construct a Limit-State Function for a Type II Component. Assume that samples of a type II component are tested at a number of training points $\mathbf{x}_i$, $i = 1, 2, \ldots, m$. If the component is working at $\mathbf{x}_i$, the state is $y_i = +1$. If the component fails at $\mathbf{x}_i$, the state is $y_i = -1$. Then we have a dataset $(\mathbf{x}_i, y_i)$, $i = 1, 2, \ldots, m$. Through the X-to-U transformation, we have a new dataset $(\mathbf{u}_i, y_i)$, $i = 1, 2, \ldots, m$, where

$$u_{ij} = \Phi^{-1}(F_j(x_{ij})) \tag{12}$$

in which subscript j indicates the jth component of the ith sample point.

With sufficient number of experiments, the probability of failure of the component p_f can also be estimated with a statistics-based reliability method. We hence assume that the component reliability is available.

In this study, we use SVM to construct the limit-state function in the form of $G^{\mathrm{II}}(\mathbf{U}) = \beta + \boldsymbol{\alpha}\mathbf{U}^T$, in which β is known and is given by $\beta = -\Phi^{-1}(p_f)$. Now the task becomes to find a unit vector $\boldsymbol{\alpha}$ that defines the hyperplane $G^{\mathrm{II}}(\mathbf{U})$. This can be done with the following two steps:

Step 1: Assume the hyperplane function for dataset $(\mathbf{u}_i, y_i)$ obtained from SVM method is

$$H(\mathbf{U}) = \beta + \boldsymbol{\omega}\mathbf{U}^{\mathrm{T}} \tag{13}$$

in which

$$\boldsymbol{\omega} = \sum_{i=1}^{k} \lambda_i y_i \mathbf{u}_i \tag{14}$$

where λ_i is given by

$$\begin{cases} \max_{\lambda} L = \sum_{i=1}^{k}\lambda_i - \frac{1}{2}\sum_{i,j=1}^{k}\lambda_i\lambda_j y_i y_j \mathbf{u}_i\mathbf{u}_j^{\mathrm{T}} \\ \text{s.t. } \sum_{i=1}^{k} y_i\lambda_i = 0, \lambda_i \ge 0 \end{cases} \tag{15}$$

Step 2: Replace $\boldsymbol{\alpha}$ by $(\boldsymbol{\omega}/\|\boldsymbol{\omega}\|)$, then $G^{\mathrm{II}}(\mathbf{U})$ is rewritten as

$$G^{\mathrm{II}}(\mathbf{U}) = \beta + \frac{\boldsymbol{\omega}}{\|\boldsymbol{\omega}\|}\mathbf{U}^{\mathrm{T}} \tag{16}$$

3.2 System Reliability Analysis. We now discuss the system reliability analysis using the proposed integrated statistics- and physics-based reliability method. Assume there are m components (failure modes) and $m \ge 2$. The limit-state functions $g_i^{\mathrm{I}}(\cdot)$, $(i = 1, 2, \ldots, m_1)$ of m_1 type I components are available. For the other $m_2 (m_2 = m - m_1)$ type II components, their observations or training points $(\mathbf{x}, y)$ are available.

For a type I component, the limit-state functions in the U-space can be written as

$$g_i^{\mathrm{I}}(\mathbf{X}) \xrightarrow{\mathbf{X}\to T(\mathbf{U})\&\text{ FORM}} G_i^{\mathrm{I}}(\mathbf{U}) = \beta_i^{\mathrm{I}} + \boldsymbol{\alpha}_i^{\mathrm{I}}\mathbf{U}^{\mathrm{T}}\ (i = 1, 2, \ldots, m_1) \tag{17}$$

For type II components, the limit-state functions constructed by SVM can also be written as

$$\begin{aligned} g_j^{\mathrm{II}}(\mathbf{X}) &\xrightarrow{\mathbf{X}\to T(\mathbf{U})\&\text{ SVM}} G_j^{\mathrm{II}}(\mathbf{U}) = \beta_j^{\mathrm{II}} + \frac{\boldsymbol{\omega}_j}{\|\boldsymbol{\omega}_j\|}\mathbf{U}^{\mathrm{T}} \\ &= \beta_j^{\mathrm{II}} + \boldsymbol{\alpha}_j^{\mathrm{II}}\mathbf{U}^{\mathrm{T}}\ (j = m_1+1,\ m_1+2, \ldots, m) \end{aligned} \tag{18}$$

The system reliability is then given by

$$R_s = \Pr\left(\bigcap_{i=1}^{m_1} -G_i^{\mathrm{I}}(\mathbf{U}) < 0 \cap \bigcap_{j=m_1+1}^{m} -G_j^{\mathrm{II}}(\mathbf{U}) < 0\right) = \int_{\Omega} \phi_{\mathbf{G}}(\mathbf{v})d\mathbf{v} \tag{19}$$

where $\phi_{\mathbf{G}}(\cdot)$ is the joint PDF of the states of the m components, and Ω is the system safe region defined by

$$\begin{aligned} \Omega = \{\mathbf{U} | -G_i^{\mathrm{I}}(\mathbf{U}) < 0, -G_j^{\mathrm{II}}(\mathbf{U}) < 0\, (i = 1, 2, \ldots, m_1; \\ j = m_1+1,\ m_1+2, \ldots, m)\} \end{aligned} \tag{20}$$

Thus, $\phi_{\mathbf{G}}(\cdot)$ is actually the joint PDF of a multivariate normal distribution determined by the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. $\boldsymbol{\mu}$ is given by

$$\boldsymbol{\mu} = (-\beta_1^{\mathrm{I}} - \beta_2^{\mathrm{I}}, \ldots, -\beta_{m_1}^{\mathrm{I}}, -\beta_{m_1+1}^{\mathrm{II}}, -\beta_{m_1+2}^{\mathrm{II}}, \ldots, -\beta_m^{\mathrm{II}}) \tag{21}$$

in which $\beta_i^{\mathrm{I}}\ (i = 1, 2, \ldots, m_1)$ is obtained by FORM, and $\beta_j^{\mathrm{II}}\ (j = m_1+1,\ m_1+2, \ldots, m)$ is computed by $\beta_j^{\mathrm{II}} = -\Phi^{-1}(p_{fj})$. And $\boldsymbol{\Sigma}$ is given by

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1m} \\ \rho_{21} & 1 & & \rho_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{m1} & \rho_{m2} & \cdots & 1 \end{bmatrix}_{m\times m} \tag{22}$$

in which ρ_{ij} is the correlation coefficient between the ith and jth components and is calculated by

$$\rho_{ij} = \rho_{ji} = \begin{cases} \boldsymbol{\alpha}_i^{\mathrm{I}}(\boldsymbol{\alpha}_j^{\mathrm{I}})^{\mathrm{T}}, & i < j \le m_1 \\ \boldsymbol{\alpha}_i^{\mathrm{I}}(\boldsymbol{\alpha}_j^{\mathrm{II}})^{\mathrm{T}}, & i \le m_1 < j \\ \boldsymbol{\alpha}_i^{\mathrm{II}}(\boldsymbol{\alpha}_j^{\mathrm{II}})^{\mathrm{T}}, & m_1 < i < j \end{cases} \tag{23}$$

According to Eq. (18), we have $\boldsymbol{\alpha}_j^{\mathrm{II}} = (\boldsymbol{\omega}_j/\|\boldsymbol{\omega}_j\|)\ (j = m_1+1,\ m_1+2, \ldots, m)$. To verify the direction of $\boldsymbol{\alpha}_j^{\mathrm{II}}$, we first substitute $\boldsymbol{\alpha}_j^{\mathrm{II}} = (\boldsymbol{\omega}_j/\|\boldsymbol{\omega}_j\|)$ into Eq. (18). Since $g(\mathbf{X}) < 0$ or $G(\mathbf{U} < 0)$ means a failed state, Eq. (18) should be negative at any sample point with $y = -1$. Otherwise, we change the direction of $\boldsymbol{\alpha}_j^{\mathrm{II}}$ by reversing the signs of all the components in it. The details of doing this are shown in Example 1 in Sec. 4.

Now, we obtain $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, and the expression of $\phi_{\mathbf{G}}(\mathbf{v})$ is given by [16]

$$\phi_{\mathbf{G}}(\mathbf{v}) = \frac{1}{\sqrt{(2\pi)^n|\boldsymbol{\Sigma}|}}\exp\left(-\frac{1}{2}(\mathbf{v}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{v}-\boldsymbol{\mu})\right) \tag{24}$$

Then R_s can be easily evaluated by integrating $\phi_{\mathbf{G}}(\mathbf{v})$ in the safe region Ω, and the probability of system failure is $p_{fs} = 1 - R_s$.

The proposed method provides a new way to approximate linear limit-state functions for components with only estimated probabilities of failure and limited reliability data. The dependency between components is automatically accommodated in the system covariance matrix. Also, it is computationally efficient due to the linear form of all the limit-state functions.

First-order reliability method is used in this feasibility study, but it is not a necessity of the proposed method. Other reliability methods, such as SORM and SVM-based methods, can also be used. One can choose a method if it satisfies the following two requirements: the method can produce the probability of failure p_f so that the reliability index is obtained by $\beta = -\Phi^{-1}(p_f)$, and a directional vector $\boldsymbol{\alpha}$ is available or can be derived. Both SORM and SVM-based methods satisfy the above requirements.

4 Examples

In this section, three examples are presented. To validate the proposed method, we at first assume that the true limit-state functions of type II components exist, but unknown to system designers. Using these true limit-state functions, we can obtain the true system reliability from a physics-based reliability method. To mimic the actual physical testing for type II components, we perform computer-based testing (random sampling) by calling the true limit-state functions and then apply the proposed method.

4.1 Example 1—A Mathematical Problem. A system consists of two physical components, and each has one failure mode. There are two basic random variables denoted by $\mathbf{X}=(X_1, X_2)$, where $X_1 \sim N(10, 0.8)$, and $X_2 \sim N(30, 1.5)$. The first component is a type I component with the limit-state function given by

$$g_1^{\mathrm{I}}(\mathbf{X}) = -152 + 8.6X_1 + 3.4X_2 \tag{25}$$

The limit-state function of the second component is unknown, and the component probability of failure $p_{f2} = 2.5625 \times 10^{-6}$ is estimated by a statistics-based approach. Thus, this component is a type II component, and a number of experiments are performed to estimate its reliability. To simulate the experiments, we assume the true limit-state function is given by

$$g_2^{\mathrm{II\,(true)}}(\mathbf{X}) = -198 + 5.4X_1 + 6.4X_2 \tag{26}$$

The corresponding true limit-state function in the U-space is

$$G_2^{\mathrm{II\,(true)}}(\mathbf{U}) = 4.5596 + 0.4104U_1 + 0.9119U_2 \tag{27}$$

Twenty-six samples of $\mathbf{X}$ are generated and the corresponding values of y are computed using Eq. (26). The simulated results are given in Table 1. Training points are given in both X- and U-spaces, and p_{f2} is also obtained based on Eq. (26).

For the first component, FORM is used directly. The new limit-state function is given by

$$G_1^{\mathrm{I}}(\mathbf{U}) = \beta_1^{\mathrm{I}} + \boldsymbol{\alpha}_1^{\mathrm{I}}\mathbf{U}^{\mathrm{T}} = 4.2036 + 0.8034U_1 + 0.5955U_2 \tag{28}$$

For the second component, $\beta_2^{\mathrm{II}} = -\Phi^{-1}(p_{f2}) = 4.5596$. Using the SVM method, an optimal hyperplane is obtained as shown in Fig. 2. The weight vector $\boldsymbol{\omega} = (0.2689, 0.5695)$ is acquired. Then the unit vector $\boldsymbol{\alpha}_2^{\mathrm{II}}$ is calculated by $\boldsymbol{\alpha}_2^{\mathrm{II}} = (\boldsymbol{\omega}_2/\|\boldsymbol{\omega}_2\|) = (0.4270, 0.9042)$, thereby building the approximated limit-state function of the second component

$$G_2^{\mathrm{II}}(\mathbf{U}) = \beta_2^{\mathrm{II}} + \boldsymbol{\alpha}_2^{\mathrm{II}}\mathbf{U}^{\mathrm{T}} = 4.5596 + 0.4270U_1 + 0.9042U_2 \tag{29}$$

To verify the direction of $\boldsymbol{\alpha}_2^{\mathrm{II}}$, we first arbitrarily pick one training point, for instance, $\mathbf{u}_1 = (-2.9932, -3.7953)$, where a failure $(y=-1)$ occurs. Then we plug $\mathbf{u}_1$ into Eq. (29) and obtain $G_2^{\mathrm{II}}(\mathbf{U}) = -0.11 < 0$, which indicates a failure. Thus, the failed state is consistent with the label of $\mathbf{u}_1$, that is $y_1 = -1$, which means $\boldsymbol{\alpha}_2^{\mathrm{II}}$ has a correct direction.

With the obtained limit-state functions in Eqs. (28) and (29), we can easily estimate the system reliability by

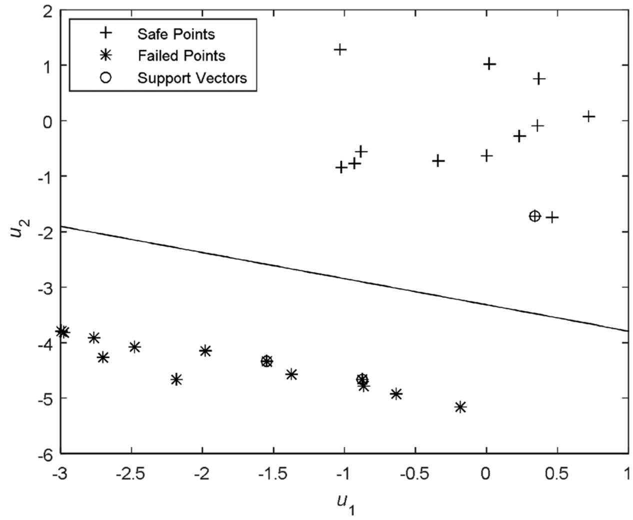

Fig. 2 Classification of training points using SVM

$$R_s = \Pr(-G_1^{\mathrm{I}}(\mathbf{U}) < 0 \cap -G_2^{\mathrm{II}}(\mathbf{U}) < 0) = \int_\Omega \phi_{\mathbf{G}}(\mathbf{v})d\mathbf{v} \tag{30}$$

The mean vector and covariance matrix of $\phi_{\mathbf{G}}(\mathbf{v})$ are, respectively, given by

$$\boldsymbol{\mu} = (-\beta_1^{\mathrm{I}}, -\beta_2^{\mathrm{II}}) = (-4.2036, -4.5596) \tag{31}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{21} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \boldsymbol{\alpha}_1^{\mathrm{I}}(\boldsymbol{\alpha}_2^{\mathrm{II}})^{\mathrm{T}} \\ \boldsymbol{\alpha}_1^{\mathrm{I}}(\boldsymbol{\alpha}_2^{\mathrm{II}})^{\mathrm{T}} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.8815 \\ 0.8815 & 1 \end{bmatrix} \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (30), we obtain $p_{fs} = 1 - R_s = 1.4441 \times 10^{-5}$.

To validate the result, we use $g_1^{\mathrm{I}}(\mathbf{X})$ and $g_2^{\mathrm{II\,(true)}}(\mathbf{X})$ to calculate the system reliability based on FORM. The result is 1.4523×10^{-5} and is regarded as the true probability of system failure. For comparison, we also compute the system reliability using the independence assumption method. All the results are listed in Table 2. The independence assumption method produces an error of 8.1%. The large error comes from neglecting the strong correlation indicated by $\rho_{12} = 0.8815$. The proposed method has an error of only 0.56%, which shows much higher accuracy.

4.2 Example 2—A Cantilever Beam Problem With Multiple Failure Modes. As shown in Fig. 3, a cantilever beam is subjected to moments M_1 and M_2; forces Q_1 and Q_2; and distributed loads denoted by (q_{L1}, q_{R1}) and (q_{L2}, q_{R2}). Among these external loads, M_1, M_2, and Q_1 are random variables. The other random variables are the dimensions of a_1, a_2, and b_1; the yield strength S_a; and the allowable shear stress τ_a. Thus, there are totally eight basic random variables, as listed in Table 3, in which N means Normal Distribution and Log$N$ means Lognormal Distribution. For each distribution, the first parameter is the mean value and the second one is the standard deviation. The deterministic parameters are listed in Table 4.

Table 1 Training points

	x		u		y
1	7.6054	24.3070	−2.9932	−3.7953	−1
2	9.4886	22.5981	−0.6392	−4.9346	−1
3	10.0151	31.5254	0.0189	1.0169	+1
⋮	⋮	⋮	⋮	⋮	⋮
25	8.7630	23.4822	−1.5462	−4.3452	−1
26	8.2552	22.9919	−2.1810	−4.6721	−1

Table 2 Results of system reliability from different methods

	Proposed method	Independence assumption method	True value
p_{fs}	1.4441×10^{-5}	1.5699×10^{-5}	1.4523×10^{-5}
Error (%)	0.56	8.1	—

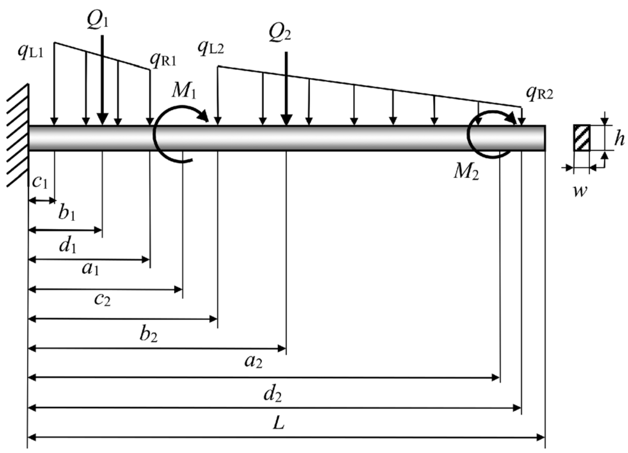

Fig. 3 A cantilever beam system

Table 3 Basic random variables

	Random variables	Distribution
X_1	$M_1(\mathrm{N\cdot m})$	$N(50\times10^3, 2\times10^3)$
X_2	$M_2(\mathrm{N\cdot m})$	$N(30\times10^3, 2\times10^3)$
X_3	$a_1(\mathrm{m})$	$N(1.5, 0.005)$
X_4	$a_2(\mathrm{m})$	$N(4.5, 0.005)$
X_5	$Q_1(\mathrm{N})$	$\mathrm{Log}\,N(65\times10^3, 13\times10^3)$
X_6	$b_1(\mathrm{m})$	$N(0.7, 0.005)$
X_7	$S_a(\mathrm{Pa})$	$N(62.5\times10^6, 10^6)$
X_8	$\tau_a(\mathrm{Pa})$	$N(3.6\times10^6, 10^5)$

The cantilever beam has three failure modes. Thus, this problem involves a system reliability analysis, and each failure mode is considered as a component. The first failure mode is due to excessive normal stress, and its limit-state function is known and given by

$$g_1^{\mathrm{I}}(\mathbf{X}) = S_a - \frac{6M}{wh^2} \quad (33)$$

in which M is the bending moment at the root and is calculated by

$$M = \sum_{i=1}^{2} M_i + \sum_{i=1}^{2} Q_i b_i + \sum_{i=1}^{2} q_{Li}(d_i - c_i)(d_i + c_i)/2 + \sum_{i=1}^{2} [(q_{Ri} - q_{Li})(d_i - c_i)/2][c_i + 2(d_i - c_i)/3] \quad (34)$$

Thus, this failure mode is treated as a type I component.

Table 4 Deterministic parameters

	Parameters	Values
1	$Q_2(\mathrm{N})$	30×10^3
2	$b_2(\mathrm{m})$	2.5
3	$q_{L1}(\mathrm{N/m})$	30×10^3
4	$q_{L2}(\mathrm{N/m})$	20×10^3
5	$c_1(\mathrm{m})$	0.25
6	$c_2(\mathrm{m})$	1.75
7	$q_{R1}(\mathrm{N/m})$	20×10^3
8	$q_{R2}(\mathrm{N/m})$	10^3
9	$d_1(\mathrm{m})$	1.25
10	$d_2(\mathrm{m})$	4.75
11	$L\,(\mathrm{m})$	5.1
12	$w\,(\mathrm{m})$	0.204
13	$h\,(\mathrm{m})$	0.403

The second failure mode is caused by the excessive shear stress with a known limit-state function given by

$$g_2^{\mathrm{I}}(\mathbf{X}) = \tau_a - \tau_{\max} \quad (35)$$

in which τ_a is the allowable shear stress, and $\tau_{\max}$ is the maximal shear stress computed by

$$\tau_{\max} = \frac{3}{2wh}\left(\sum_{i=1}^{2} Q_i + \sum_{i=1}^{2} q_{Li}(d_i - c_i) + \sum_{i=1}^{2} \frac{(q_{Ri} - q_{Li})(d_i - c_i)}{2}\right) \quad (36)$$

Thus, this failure mode is also treated as a type I component.

The third failure mode is caused by excessive deflection, and its limit-state function is not available. It is therefore treated as a type II component. Then reliability testing is performed to estimate the probability of failure and the result is p_{f3}. To simulate physical experiments, we generate training points by simulation. We assume that the true limit-state function for (FM3) is given by

$$g_3^{\mathrm{II\,(true)}}(\mathbf{X}) = v_a - v_{\max} \quad (37)$$

in which $v_a = 8.4$ mm is the allowable deflection, and $v_{\max}$ is the maximal tip deflection given by

$$v_{\max} = \frac{1}{EI}\left[\frac{ML^2}{2} + \frac{BL^3}{2} + \sum_{i=1}^{2} \frac{M_i(L-a_i)^2}{2} - \sum_{i=1}^{2} \frac{Q_i(L-b_i)^3}{6}\right] + \frac{1}{EI}\left[-\sum_{i=1}^{2} \frac{q_{Li}(L-c_i)^4}{24} - \sum_{i=1}^{2} \frac{(q_{Ri}-q_{Li})(L-c_i)^5}{120(d_i - c_i)} + \sum_{i=1}^{2} \frac{q_{Ri}(L-d_i)^4}{24}\right] + \frac{1}{EI}\sum_{i=1}^{2} \frac{(q_{Ri}-q_{Li})(L-d_i)^5}{120(d_i - c_i)} \quad (38)$$

in which B is the reaction force at the fixed end. The Young's modulus is $E = 200\times10^9$ Pa and the moment of inertia is $I = (wh^3/12)$. We then generate 24 samples of $\mathbf{X}$ and obtain samples of y using Eq. (37). Also, $p_{f3} = 4.1309\times10^{-4}$ is given, which is obtained based on Eq. (37).

Using the proposed SVM method, eight support vectors are obtained, and the weight vector is $\boldsymbol{\omega}_3 = (0.137, 0.560, -0.065, -0.037, 0.483, -0.059, -0.013, 0.012)$. Then the approximated limit-state function for FM3 is given by

$$G_3^{\mathrm{II}}(\mathbf{U}) = \beta_3^{\mathrm{II}} + \boldsymbol{\alpha}_3^{\mathrm{II}}\mathbf{U}^{\mathrm{T}} \quad (39)$$

in which $\beta_3^{\mathrm{II}} = -\Phi^{-1}(p_{f3}) = 3.3439$ and $\boldsymbol{\alpha}_3^{\mathrm{II}}$ is the unit vector calculated by $\boldsymbol{\alpha}_3^{\mathrm{II}} = (\boldsymbol{\omega}_3/\|\boldsymbol{\omega}_3\|) = (0.181, 0.738, -0.085, -0.049, 0.637, -0.077, -0.017, 0.016)$.

The first two failure modes are type I components, and FORM produces the following results:

$$G_i^{\mathrm{I}}(\mathbf{U}) = \beta_i^{\mathrm{I}} + \boldsymbol{\alpha}_i^{\mathrm{I}}\mathbf{U}^{\mathrm{T}}\ (i = 1, 2) \quad (40)$$

in which

$$\beta_1^{\mathrm{I}} = 2.9806,\ \boldsymbol{\alpha}_1^{\mathrm{I}} = (-0.121, -0.121, 0, 0, -0.926, -0.033, 0.335, 0)$$
$$\beta_2^{\mathrm{I}} = 2.7065,\ \boldsymbol{\alpha}_2^{\mathrm{I}} = (0, 0, 0, 0, -0.968, 0, 0, 0.250) \quad (41)$$

Thus, $G_i^{\mathrm{I}}(\mathbf{U})\,(i = 1, 2)$ and $G_3^{\mathrm{II}}(\mathbf{U})$ follow a multivariate normal distribution with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ given by

$$\boldsymbol{\mu} = (-2.9806, -2.7065, -3.3439) \quad (42)$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.897 & 0.705 \\ 0.897 & 1 & 0.613 \\ 0.705 & 0.613 & 1 \end{bmatrix} \quad (43)$$

Table 5 Results of system reliability

	Proposed method	Independence assumption method	True value
p_{fs}	4.1528×10^{-3}	5.2442×10^{-3}	4.1582×10^{-3}
Error (%)	0.13	26.12	—

where ρ_{12}, ρ_{13}, and ρ_{23} are the correlation coefficients between $G_1^{\mathrm{I}}(\mathbf{U})$ and $G_2^{\mathrm{I}}(\mathbf{U})$; $G_1^{\mathrm{I}}(\mathbf{U})$ and $G_3^{\mathrm{II}}(\mathbf{U})$; $G_2^{\mathrm{I}}(\mathbf{U})$ and $G_3^{\mathrm{II}}(\mathbf{U})$, respectively. Then, the system reliability is evaluated and is given by $p_{fs} = 4.1528 \times 10^{-3}$.

With all the given limit-state functions $g_1^{\mathrm{I}}(\mathbf{X})$, $g_2^{\mathrm{I}}(\mathbf{X})$, and $g_3^{\mathrm{II\,(true)}}(\mathbf{X})$, the true system reliability could be obtained using FORM and is assumed as a benchmark for comparison. Likewise, we also estimate the probability of system failure using the independence assumption method. The results are shown in Table 5, which indicates that the proposed method is more accurate than the independence assumption method.

4.3 Example 3—A System With Multiple Components.

Figure 4 shows a crank-slider system consisting of four physical components. An external moment is applied to joint *A*, which drives beam AB rotating around *A*. For this example, we only focus on the system reliability when $\theta_2 = \pi/2$.

Physical component 1 is beam AB with a length of l_1, and the cross section is defined by the width b_1 and height h_1. Beam AB has one failure mode (FM1) due to excessive normal stress, and the limit-state function is known and is given by

$$g_1^{\mathrm{I}}(\mathbf{X}) = S_{a1} - S_1 \tag{44}$$

in which S_{a1} is the allowable normal stress, and $S_1 = (M(h_1/2)/b_1h_1^3/12)$ is the maximal normal stress developed in the beam.

Physical component 2 is beam BC with a length of l_2, the cross section is defined by the width b_2 and height h_2. The single failure mode (FM2) for beam BC is caused by buckling with a known limit-state function given by

$$g_2^{\mathrm{I}}(\mathbf{X}) = P_{\mathrm{cr}} - F_{\mathrm{BC}} \tag{45}$$

in which $P_{\mathrm{cr}} = (\pi^2E_2I_2/(Kl_2)^2)$ is the critical force for buckling, $I_2 = (b_2h_2^3/12)$, and $F_{\mathrm{BC}} = M/l_1$ is the force developed in the beam.

Physical component 3 is the shaft DE with a length of l_3 and a diameter of d_4. The shaft has two failure modes (FM3 and FM4) caused by excessive deflection and excessive normal stress, respectively. The corresponding limit-state functions are known and given by

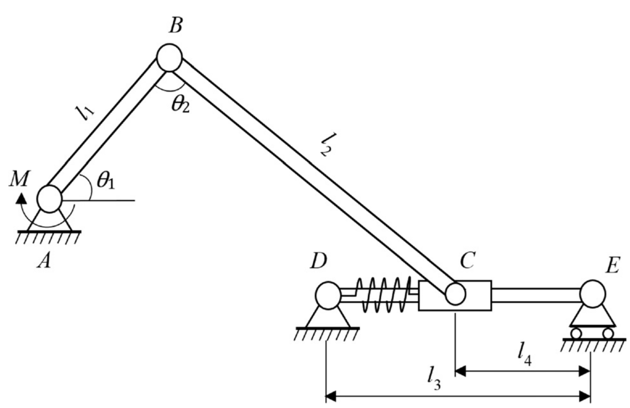

Fig. 4 A crank-slider system

Table 6 Random variables

	Random variables	Distribution
X_1	$M_1(\mathrm{N\cdot m})$	$N(350, 65)$
X_2	$l_1(\mathrm{m})$	$N(0.3, 10^{-4})$
X_3	$l_2(\mathrm{m})$	$N(0.9, 10^{-3})$
X_4	$b_1(\mathrm{m})$	$N(0.022, 5 \times 10^{-4})$
X_5	$h_1(\mathrm{m})$	$N(0.019, 5 \times 10^{-4})$
X_6	$b_2(\mathrm{m})$	$N(0.015, 5 \times 10^{-4})$
X_7	$h_2(\mathrm{m})$	$N(0.009, 5 \times 10^{-4})$
X_8	$d_4(\mathrm{m})$	$N(0.0228, 10^{-4})$

$$\begin{cases} g_3^{\mathrm{I}}(\mathbf{X}) = \delta_{a3} - \delta_3 \\ g_4^{\mathrm{I}}(\mathbf{X}) = S_{a4} - S_4 \end{cases} \tag{46}$$

in which δ_{a3} is the allowable deflection, and δ_3 is the maximal deflection given by

$$\delta_3 = \frac{F_{\mathrm{BC}}\sin(\pi/2 - \theta_1)l_4\left(l_3^2 - l_4^2\right)^{3/2}}{9\sqrt{3}\,l_4E_4(\pi/4)(d_4/2)^4} \tag{47}$$

where E_4 is the Young's modulus of shaft DE. S_{a4} is the allowable normal stress, and S_4 is the maximal normal stress developed in the shaft and is calculated by

$$S_4 = \frac{M_{\max}c}{I_4} = \frac{F_{\mathrm{BC}}\sin(\pi/2 - \theta_1)(d_4/2)}{(\pi/4)(d_4/2)^4} \tag{48}$$

Physical component 4 is the spring CD with one failure mode (FM5) due to excessive shear stress applied to the spring coils. The limit-state function is unknown while the probability of failure is given by $p_{f3} = 1.04 \times 10^{-3}$. Likewise, to simulate the testing, we assume the true limit-state function as

$$g_5^{\mathrm{II\,(true)}}(\mathbf{X}) = \tau_{a5} - \tau_5 \tag{49}$$

in which $\tau_{a5} \sim N(100 \times 10^6)$ Pa is the allowable shear stress of the spring coil, and τ_5 is the developed maximal shear stress and calculated by

$$\tau_5 = \frac{F_{\mathrm{BC}}\cos(\pi/2 - \theta_1)D}{\pi d^3}\left(\frac{4D - d}{4D - 4d} + \frac{0.615d}{D}\right) \tag{50}$$

in which $D \sim N(34.7 \times 10^{-3},\ 10^{-4})$m is the outer diameter of the spring, and $d = 29.5 \times 10^{-3}$ m is the spring inner diameter. We then generate 30 training points of $\mathbf{X}$ and obtain samples of y based on Eq. (49) and the distributions of M_1, l_1, D, and τ_{a5}.

All the random variables known by the system designers are listed in Table 6 and the known deterministic parameters are listed in Table 7. Since D and τ_{a5} are only known by the spring supplier, they are not listed in Table 6. They are denoted as X_9 and X_{10}. Thus, there are actually ten basic random variables in the system. For FM5, the training points are provided in the form of (X_1, X_2, X_9, X_{10}).

Table 7 Deterministic parameters

No.	Deterministic parameters	Values
1	$E_2(\mathrm{Pa})$	200×10^9
2	$E_4(\mathrm{Pa})$	200×10^9
3	K	1
4	$l_3(\mathrm{m})$	0.96
5	$l_4(\mathrm{m})$	0.31
6	$S_{a1}(\mathrm{Pa})$	400×10^6
7	$S_{a4}(\mathrm{Pa})$	460×10^6
8	$\delta_{a4}(\mathrm{m})$	0.0032

Table 8 Results of system reliability

	Proposed method	Independence assumption method	True value
p_{fs}	0.0133	0.0238	0.0142
Error (%)	6.49	67.74	—

As discussed in Sec. 4, the five FMs in the system are treated as five components at the system level. The first four FMs with known limit-state functions $g_i^{\mathrm{I}}(\mathbf{X})\ (i=1,2,3,4)$ belong to type I components, and FM5 is a type II component since its limit-state function $g_5^{\mathrm{II}}(\mathbf{X})$ is not available.

For type I components, $g_i^{\mathrm{I}}(\mathbf{X})$ could be approximated by FORM as

$$G_i^{\mathrm{I}}(\mathbf{U}) = \beta_i^{\mathrm{I}} + \boldsymbol{\alpha}_i^{\mathrm{I}}\mathbf{U}^T\ (i=1,2,\ldots,4) \tag{51}$$

in which

$$\begin{aligned}
&\beta_1^{\mathrm{I}}=2.5099,\ \boldsymbol{\alpha}_1^{\mathrm{I}}=(-0.91,0,0,0.16,0.38,0,0,0,0,0)\\
&\beta_2^{\mathrm{I}}=2.6609,\ \boldsymbol{\alpha}_2^{\mathrm{I}}=(-0.60,1.4\times10^{-3},-0.02,0,0,0.14,0.79,0,0,0)\\
&\beta_3^{\mathrm{I}}=2.5653,\ \boldsymbol{\alpha}_3^{\mathrm{I}}=(-0.99,2.6\times10^{-4},1.6\times10^{-2},0,0,0,0,0,0.14,0)\\
&\beta_4^{\mathrm{I}}=2.4145,\ \boldsymbol{\alpha}_4^{\mathrm{I}}=(-0.99,2.6\times10^{-4},1.5\times10^{-2},0,0,0,0,0,0.10,0)
\end{aligned} \tag{52}$$

For the type II components, using the proposed SVM method, the limit-state function is reconstructed as

$$G_5^{\mathrm{II}}(\mathbf{U}) = \beta_5^{\mathrm{II}} + \boldsymbol{\alpha}_5^{\mathrm{II}}\mathbf{U}^{\mathrm{T}} \tag{53}$$

in which $\beta_5^{\mathrm{II}} = -\Phi^{-1}(p_{f5}) = 3.0785$ and $\boldsymbol{\alpha}_5^{\mathrm{II}} = (0.911,-0.144,0,0,0,0,0,0,0.320,-0.217)$.

Then, $G_i^{\mathrm{I}}(\mathbf{U})\ (i=1,2,\ldots,4)$ and $G_5^{\mathrm{II}}(\mathbf{U})$ follow a multivariate normal distribution with the mean vector and covariance matrix given by

$$\boldsymbol{\mu} = (-2.5099, -2.6099, -2.5653, -2.4145, -3.0785) \tag{54}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.546 & 0.903 & 0.907 & 0.831 \\ 0.546 & 1 & 0.593 & 0.595 & 0.546 \\ 0.903 & 0.593 & 1 & 0.999 & 0.902 \\ 0.907 & 0.595 & 0.999 & 1 & 0.906 \\ 0.831 & 0.546 & 0.902 & 0.906 & 1 \end{bmatrix} \tag{55}$$

Using Eq. (24), the estimated probability of system failure is $p_{fs} = 1 - R_s = 0.0133$.

With all the exactly known limit-state functions $g_i^{\mathrm{I}}(\mathbf{X})\ (i=1,2,3,4)$ and $g_5^{\mathrm{II\,(true)}}(\mathbf{X})$, the true system reliability could be directly acquired using FORM. The results from different methods are shown in Table 8. It is concluded that the proposed method performs much better than the independence assumption method.

5 Conclusions

This work verifies the feasibility of integrating statistics- and physics-based method for system reliability analysis. It is common that component reliability is estimated by a statistic-based method; with the increasing use of physics-based computational models, it is also possible that component reliability is estimated by a physics-based method. This study deals with the difficulty of obtaining the joint probability density when physics-based methods are used for some components (type I) and statistics-based methods are used for other components (type II).

The physics-based method employed in this study is FORM, which is directly used for type I components whose physics-based limit-state functions are available. For type II components whose physics-based limit-state functions are unknown, with a statistics-based method, reliability experiments are performed. Then their reliabilities are estimated. A supervised learning strategy through SVM is developed to create limit-state functions for type II components. The proposed method makes the limit-state functions of all the components available thereby leading to a multivariate normal probability density function, whose integration in the safe region then produces the system reliability.

This feasibility study makes a number of assumptions, such as the distributions of basic random variables for both types of components are known, the component reliability is calculated by FORM, the safety-failure boundary is linear with respect to basic variables in the standard normal space, and sample points from reliability testing in both safe and failure regions are available. If the data set has a nonlinear pattern, the proposed method can still accommodate such nonlinearity by introducing slack variables to the SVM model so that the linear assumption could be violated slightly. If the nonlinearity is high, SVM methodologies that produce nonlinear models should be employed.

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