The Symmetric-Range Accuracy under a One-Way Random Model with Balanced or Unbalanced Data

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The symmetric-range accuracy \( A \) of a sampler is defined as the fractional range, symmetric about the true concentration, that includes a specified proportion of sampler measurements. In this article, we give an explicit expression for \( A \) assuming that the sampler measurements follow a one-way random model so as to capture different components of variability, for example, variabilities among and within different laboratories or variabilities among and within exposed workers. We derive an upper confidence limit for \( A \) based on the concept of a ‘generalized confidence interval’. A convenient approximation is also provided for computing the upper confidence limit. Both balanced and unbalanced data situations are investigated. Monte Carlo evaluation indicates that the proposed upper confidence limit is satisfactory even for small samples. The statistical procedures are illustrated using an example.

Keywords: coverage probability; generalized confidence interval; generalized pivotal quantity; non-central chi-square distribution; upper confidence limit

INTRODUCTION

For quantifying the measurement accuracy of exposure data, the development of accuracy criteria is important. The National Institute of Occupational Safety and Health (NIOSH) accuracy criterion is based on the symmetric-range accuracy \( A \), and the NIOSH accuracy requirement states that a 95\% upper confidence limit for \( A \) does not exceed 0.25 (see Bartley et al., 2003; Bartley, 2001, 2008; Bartley and Lidén, 2008). By definition, the symmetric-range accuracy \( A \) is defined as the fractional range, symmetric about the true concentration \( C \), within which 100(1 − \( \alpha \)) \% of sampler measurements are to be found. Though the NIOSH accuracy criterion applies strictly to intra-laboratory variations, a similar criterion, for example, at the 0.50 level, could be adopted and applied also for controlling inter-laboratory variations. In Bartley (2001) and Bartley et al. (2003), an approximation is developed for \( A \), and an approximate 95\% upper confidence limit is derived, assuming a normal distribution for the original measurements. In a recent article, Krishnamoorthy and Mathew (2009) have derived an exact expression for \( A \). Also, an accurate upper confidence limit on \( A \) may be obtained by exploiting the generalized confidence interval idea. The work of Krishnamoorthy and Mathew (2009) is also in the setup of exposure measurements that follow a normal distribution. It appears that the symmetric-range accuracy and its confidence limit calculation have not been addressed in the context of other models; for example, in the context of a model that involves random effects.

The one-way random model for the log-transformed exposure data is widely used to capture variability among workers when repeated measurements are made on the same worker over time.

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(see Symanski et al., 2006 for a review). The one-way random model has also found applications in the evaluation of sampling/analytical methods where inter-sampler variation is significant. For example, Bartley et al. (1994) discuss the 10-mm nylon cyclone whose variation in inner diameter leads to inaccuracy in estimating respirable dust concentrations; a one-way random model is used to capture these variations. The use of a one-way random model results in a situation where a combined uncertainty has to be computed by adding the two variance components (corresponding to variability among and within laboratories). The calculation of such a combined uncertainty is pointed out in Bartley et al. (2003). The same situation arises in inter-laboratory evaluation of methodologies to determine workplace contaminations; the two variance components will now represent the variability between and within laboratories. The goal of this investigation is to derive an exact expression for the symmetric-range accuracy in this scenario and to compute an accurate upper confidence limit. We have once again used the generalized confidence interval idea in order to derive the required upper confidence limit. We have addressed both balanced and unbalanced data situations. Monte Carlo results are reported to judge the accuracy of the proposed upper confidence limits. We have also illustrated our methodology in both balanced and unbalanced data situations using data from an inter-laboratory evaluation of a methodology to determine trace beryllium in air filter samples.

**SYMMETRIC-RANGE ACCURACY**

Let $X_{ij}$ denote the $j$th sampler measurement on the $i$th worker or from the $i$th laboratory, assumed to follow the one-way random model given by

$$X_{ij} = \mu + \tau_i + e_{ij}, j = 1, 2, ..., n_i, i = 1, 2, ..., k, \quad (1)$$

where $\mu = E(X_{ij})$ is the overall mean of the measurements, and the $\tau_i$’s and $e_{ij}$’s are independent random variables with $\tau_i \sim N(0, \sigma^2_{\tau})$ and $e_{ij} \sim N(0, \sigma^2_{e})$. Thus, $X_{ij} \sim N(\mu, \sigma^2_{\tau} + \sigma^2_{e})$. Here, $k$ denotes the number of workers or number of laboratories from where sampler measurements are obtained, and $n_i$ denotes the number of sampler measurements on the $i$th worker or from the $i$th laboratory. We note that the $X_{ij}$ represent the original measurements. Furthermore, the random variable $\tau_i$ represents an effect due to the $i$th laboratory or the $i$th worker.

If $C$ denotes the true concentration and $X$ follows the model in equation (1), then the symmetric-range accuracy $A$ satisfies

$$P((1-A)C < X < (1+A)C) = 1 - \alpha, \quad (2)$$

where $0 < \alpha < 1$, and for NIOSH applications, $\alpha = 0.05$. Here, we have followed the notations and definitions in Bartley et al. (2003), Bartley (2001) and Krishnamoorthy and Mathew (2009). As $X \sim N(\mu, \sigma^2_{\tau} + \sigma^2_{e})$, we can write

$$P((1-A)C < X < (1+A)C) = P$$

$$\left(\frac{(1-A)C - \mu}{\sqrt{\sigma^2_{\tau} + \sigma^2_{e}}} < \frac{X - \mu}{\sqrt{\sigma^2_{\tau} + \sigma^2_{e}}} < \frac{(1+A)C - \mu}{\sqrt{\sigma^2_{\tau} + \sigma^2_{e}}}\right)$$

$$= P \left(\frac{(1-A)C - \mu}{\sqrt{\sigma^2_{\tau} + \sigma^2_{e}}} < Z_0 < \frac{(1+A)C - \mu}{\sqrt{\sigma^2_{\tau} + \sigma^2_{e}}}\right), \quad (3)$$

where $Z_0 = (X - \mu) / \sqrt{\sigma^2_{\tau} + \sigma^2_{e}}$ follows a standard normal distribution. Let

$$b = \frac{C - \mu}{\sqrt{\sigma^2_{\tau} + \sigma^2_{e}}}. \quad (4)$$

Then, equation (3) can be expressed as

$$P \left(|Z_0 - b| < AC / (\sqrt{\sigma^2_{\tau} + \sigma^2_{e}})\right),$$

or equivalently

$$P ((Z_0 - b)^2 < A^2C^2 / (\sigma^2_{\tau} + \sigma^2_{e})).$$

Thus, the symmetric-range accuracy $A$ should satisfy

$$P \left((Z_0 - b)^2 < A^2C^2 / (\sigma^2_{\tau} + \sigma^2_{e})\right) = 1 - \alpha. \quad (5)$$

Note that $(Z_0 - b)^2 \sim \chi^2_{1}\left(\mu_{(b^2)}\right)$, a non-central chi-square distribution with one degree of freedom and a non-centrality parameter $b^2$. Hence, $A^2C^2 / (\sigma^2_{\tau} + \sigma^2_{e}) \sim \chi^2_{1\alpha}\left(\mu_{(b^2)}\right)$, the 100$(1-\alpha)$ percentile of the non-central chi-square distribution. Thus,

$$A = \frac{\sqrt{\sigma^2_{\tau} + \sigma^2_{e}}}{C} \sqrt{\chi^2_{1\alpha}\left(\mu_{(b^2)}\right)}, \quad (5)$$

where $b$ is defined in equation (4). Thus, equation (5) gives an explicit expression for $A$, similar to what is obtained in Krishnamoorthy and Mathew

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(2009). Such an explicit expression facilitates the computation of an upper confidence limit for $A$, as noted below.

**Interval estimation with balanced data**

Consider the model in equation (1) with balanced data, i.e. the $n_j$’s are equal with $n$ denoting their common value. Thus, we have sampler measurements $X_{ij}$’s following the model

$$X_{ij} = \mu + \tau_i + \epsilon_{ij}, \quad j = 1,2,\ldots,n, \quad i = 1,2,\ldots,k. \quad (6)$$

Define $\bar{X}_i = \frac{1}{n} \sum_{j=1}^n X_{ij}$ and $\bar{X}_. = \frac{1}{k} \sum_{i=1}^k \sum_{j=1}^n X_{ij}$. Recall that the $\tau_i$’s are random variables representing effects due to the different workers or laboratories. Now let $SS_t$ denote the between worker or between laboratory sum of squares, and $SS_e$ denote the within worker or within laboratory sum of squares. These are given by

$$SS_t = \frac{k}{n} \sum_{i=1}^n \left( \bar{X}_i - \bar{X}. \right)^2$$

and

$$SS_e = \sum_{i=1}^k \sum_{j=1}^n \left( X_{ij} - \bar{X}_i \right)^2.$$  

See Montgomery (2012, Chapter 3). It is known that $\bar{X}_i$, $SS_t$, and $SS_e$ are independently distributed with

$$Z = \sqrt{k} \frac{\left( \bar{X}_i - \mu \right)}{\sqrt{n} \sigma^2 + \sigma^2_e} \sim N(0,1),$$

$$SS_t = \frac{n}{n^2} \chi^2_{k-1}, \quad \text{and} \quad SS_e = \frac{n}{n^2} \chi^2_{k(n-1)}.$$  

where $\chi^2_r$ denotes the central chi-square distribution with $r$ degrees of freedom (df).

A 100$\gamma$% upper confidence limit for $A$ will now be computed using the generalized confidence interval idea, as done in Krishnamoorthy and Mathew (2009). The first step toward this computation is the derivation of a generalized pivotal quantity (GPQ) for $A$. As $A$ is a function of $\mu$ and $\sigma^2_t + \sigma^2_e$, a GPQ for $A$ can be obtained by substituting the GPQs of these parameters in the expression for $A$ in equation (5). In order to obtain GPQs for the parameters, let $\bar{X}_i$, $ss_t$, and $ss_e$ denote the observed values of the random variables $\bar{X}_i$, $SS_t$, and $SS_e$, respectively. The observed values $\bar{X}_i$, $ss_t$, and $ss_e$ are numbers computed from a given set of data and are to be treated as fixed. A GPQ is a function of the random variables $\bar{X}_i$, $SS_t$, and $SS_e$, and their observed values $\bar{X}_i$, $ss_t$, and $ss_e$, and is required to satisfy two conditions: (i) given the observed quantities $\bar{X}_i$, $ss_t$, and $ss_e$, the distribution of the GPQ should be free of unknown parameters, and (ii) in the definition of the GPQ, if the random variables $\bar{X}_i$, $SS_t$, and $SS_e$ are replaced by the corresponding observed values, the GPQ will simplify to the parameter of interest (the parameter of interest being the symmetric-range accuracy $A$). To find a GPQ for $\sigma^2_t + \sigma^2_e$, let

$$\sigma^2_t = n \sigma^2_t + \sigma^2_e$$

and $\sigma^2_e = \sigma^2_e$ so that

$$\sigma^2_t + \sigma^2_e = \frac{1}{n} \left[ \sigma^2_t + (n-1) \sigma^2_e \right]. \quad (8)$$

A GPQ for $\sigma^2_t$ is given by

$$G_{\sigma^2_t} = \frac{\sigma^2_t}{SS_t} = \frac{ss_t}{\chi^2_{k-1}},$$

and a GPQ for $\sigma^2_e$ is given by

$$G_{\sigma^2_e} = \frac{\sigma^2_e}{SS_e} = \frac{ss_e}{\chi^2_{k(n-1)}}.$$  

Substituting these GPQs in equation (8), we obtain a GPQ for $\sigma^2_t + \sigma^2_e$ as

$$G_{\sigma^2_t + \sigma^2_e} = \frac{1}{n} \left[ \frac{ss_t}{\chi^2_{k-1}} + (n-1) \frac{ss_e}{\chi^2_{k(n-1)}} \right]. \quad (9)$$

A GPQ for $\mu$ is given by

$$G_{\mu} = \bar{X}_. - \frac{\sqrt{k} \left( \bar{X}_i - \mu \right)}{\sqrt{SS_t}} \sqrt{\frac{ss_t}{kn}} \sqrt{\chi^2_{k-1}} \sqrt{\frac{ss_t}{kn}},$$

where $Z$ is defined in equation (7). Finally, a GPQ for $A$ can be obtained by replacing the parameters in equation (5) by their GPQs and is given by

$$G_A = \frac{\sqrt{G_{\sigma^2_t + \sigma^2_e}}}{C - \sqrt{G_{\mu}} \left( G_{\sigma^2_e} \right)},$$

with $G_{\sigma^2_e} = \left( \frac{C - G_{\mu}}{G_{\sigma^2_t + \sigma^2_e}} \right)^2$. \quad (10)

**An approximation**

In order to numerically obtain a percentile of $G_A$ that will provide the required upper confidence
limit, simulation must be carried out a large number of times. Furthermore, the non-central critical value $X_{1:1-\alpha}^2(G_b^2)$ must be evaluated for each simulation run and this makes the calculation somewhat time-consuming. The following approximation can be used to avoid the calculation of $X_{1:1-\alpha}^2(G_b^2)$ for each simulated $G_b^2$:

$$X_{m,p}^2(\delta) = (m + \delta) \left[ z_p \left( \frac{2}{9} \frac{m + 2\delta}{(m + \delta)^2} - \frac{2}{9} \frac{m + 2\delta}{(m + \delta)^2} + 1 \right) \right]$$

$$= X_{m,p}^2(\delta),$$

where $z_p$ is the $100p$ percentile of a standard normal distribution. The above approximation with $m = 1$, $p = 1-\alpha$, and $\delta = G_b^2$ can be used to evaluate $X_{1:1-\alpha}^2(G_b^2)$ in equation (10) as

$$X_{1:1-\alpha}^2(G_b^2) = (1 + G_b^2) \left[ z_{1-\alpha} \left( \frac{2}{9} \frac{1 + 2G_b^2}{(1 + G_b^2)^2} - \frac{2}{9} \frac{1 + 2G_b^2}{(1 + G_b^2)^2} + 1 \right) \right]$$

$$= X_{1:1-\alpha}^2(G_b^2), \text{ say.} \quad (11)$$

Simulation studies in the sequel indicate that this approximation is very satisfactory and provide results very close to those based on equation (10). Note that, for given $(\bar{x}_e, ss_x, ss_e)$, the distribution of $G_b$ does not depend on any unknown parameters, and so the percentiles of $G_b$ can be estimated by Monte Carlo simulation as shown in Algorithm 1. The $100\gamma$ percentile of $G_A$ is a $100\gamma\%$ upper confidence limit for $A$.

**Algorithm 1**

1. For given sampler measurements following the model in equation (1), compute $\bar{x}_e$, $ss_x$, and $ss_e$.
2. Generate random variates $Z \sim N(0,1)$, $X_{k-1}^2$, and $X_{k(n-1)}^2$.
3. Compute $G_{\mu} = \bar{x}_e - \frac{Z}{\sqrt{\frac{ss_x}{k-1}}} \sqrt{\frac{ss_e}{kk}}$, $G_{\sigma^2 + \sigma_e^2} = \frac{1}{n} \left[ \frac{ss_x}{k-1} + \frac{ss_e}{k(k-1)} \right]$, and

$$G_{b^2} = \frac{(C-G_{\mu})^2}{\sigma^2 + \sigma_e^2}.$$  

4. Compute

$$G_A = \frac{G_{\sigma^2 + \sigma_e^2}}{\sigma^2 + \sigma_e^2} \sqrt{X_{1:1-\alpha}^2(G_b^2)}$$

$$G_A = \frac{G_{\sigma^2 + \sigma_e^2}}{\sigma^2 + \sigma_e^2} \sqrt{X_{1:1-\alpha}^2(G_b^2)}, \text{ where } X_{1:1-\alpha}^2 \text{ is given in equation (11).}$$

5. Repeat Steps 2–4 a large number of times, say, $M$.

The $100\gamma$ percentile of $G_A$’s generated above is a $100\gamma\%$ upper confidence limit for $A$.

**A special case**

If it is assumed that $\mu = C$ (i.e. the bias is negligible) so that $X_{\mu} \sim N(C, \sigma^2 + \sigma_e^2)$, then $b = 0$ (see equation (4)), and the accuracy range $A$ defined in equation (5) becomes $\frac{\sigma^2 + \sigma_e^2}{\sigma^2 + \sigma_e^2} \sqrt{X_{1:1-\alpha}^2}$, where $X_{1:1-\alpha}^2$ denotes the $100(1-\alpha)$th percentile of a central chi-square distribution with one degree of freedom. In this case, the GPQ for $A$ is given by

$$G_A = \frac{G_{\sigma^2 + \sigma_e^2}}{\sigma^2 + \sigma_e^2} \sqrt{X_{1:1-\alpha}^2}. \text{ Clearly, a } 100\gamma\% \text{ upper confidence limit for } A \text{ can now be obtained after numerically obtaining the } 100\gamma \text{ percentile of } G_{\sigma^2 + \sigma_e^2}. \text{ The estimation of this percentile based on Monte Carlo simulation should be clear from Algorithm 1.}$$

However, we note that in this special case as

$$A = \frac{\sqrt{\sigma^2 + \sigma_e^2}}{\sigma^2 + \sigma_e^2} \sqrt{X_{1:1-\alpha}^2},$$

what is required is a $100\gamma\%$ upper confidence limit for $\sigma^2 + \sigma_e^2$. An approximate upper confidence limit can be explicitly obtained using the Satterthwaite approximation (see Montgomery, 2012, Chapter 13). In order to see this, the chi-square distributions in equation (7) provide us with the unbiased estimators of $n\sigma^2 + \sigma_e^2$ and $\sigma_e^2$, say $n\hat{\sigma}^2 + \hat{\sigma}_e^2$ and $\hat{\sigma}_e^2$, given by $n\hat{\sigma}^2 + \hat{\sigma}_e^2 = SS_e / (k-1) = MS_e$ (say) and $\hat{\sigma}_e^2 = SS_e / [k(n-1)] = MS_e$ (say). From these, we immediately get

$$\hat{\sigma}^2 + \hat{\sigma}_e^2 = MS_e \frac{1}{n} + \frac{1}{n} \right) MS_e,$$

which is a linear combination of $MS_e$ and $MS_e$. According to the Satterthwaite approximation, we then have the distribution
symmetric-range accuracy

\[
\frac{\hat{\sigma}_e^2 + \hat{\sigma}_r^2}{\hat{\sigma}_r^2 + \hat{\sigma}_e^2} \approx \frac{\chi^2}{f},
\]

(12)

approximately, where the degrees of freedom \( f \) is given by

\[
f = \left[ \frac{MS_e}{n} + \left(1 - \frac{1}{n}\right) MS_e \right]^2 \left[ \frac{MS_e^2}{\sigma^2(k-1)} + \left(1 - \frac{1}{n}\right)^2 \frac{MS_e^2}{k(n-1)} \right].
\]

(13)

We note that the degrees of freedom is estimated using the data. A \( 100\gamma \% \) approximate upper confidence limit for \( \hat{\sigma}_e^2 + \hat{\sigma}_r^2 \) is now given by \( f(\hat{\sigma}_e^2, \hat{\sigma}_r^2) / \chi^2_{f, 1-\gamma} \). Consequently, a \( 100\gamma \% \) approximate upper confidence limit for the symmetric-range accuracy \( A \) is now given by

\[
\sqrt{f(\hat{\sigma}_e^2, \hat{\sigma}_r^2) / \chi^2_{f, 1-\gamma}} \sqrt{\chi^2_{1-\alpha}},
\]

where the degrees of freedom \( f \) is given in equation (13).

**INTERVAL ESTIMATION WITH UNBALANCED DATA**

We shall now consider the model in equation (1) with unbalanced data, i.e. not all the \( n_i \)'s are equal. Thus, we have sampler measurements \( X_{ij} \)'s following the model

\[
X_{ij} = \mu + \tau_i + e_{ij}, \quad j = 1,2,...,n_i, \quad i = 1,2,...,k,
\]

(14)

where the various quantities in the model are as before, satisfying the same distributional assumptions. Note that the number of observations is not the same on the different workers or from the different laboratories. In the case of unbalanced data, a simple approach to compute an upper confidence limit for the symmetric-range accuracy \( A \) is to reduce the problem to the case of balanced data using a fairly widely used approximation. In order to introduce the approximation, define

\[
h = \frac{1}{k} \sum_{i=1}^{k} n_i^{-1}, \quad \bar{X} = \frac{1}{k} \sum_{i=1}^{k} X_i, \quad \text{and} \quad SS\_\pi
\]

\[
\frac{k}{i=1} (X_i - \bar{X})^2.
\]

(15)

Then,

\[
\bar{X} - N\left(\mu, \frac{\sigma^2_r + h\sigma^2_e}{k} \right).
\]

By direct calculation, it can be verified that \( E(SS\_\pi) = (k-1)(\sigma^2_r + h\sigma^2_e) \). An approximate chi-square distribution associated with \( SS\_\pi \) states that

\[
\frac{SS\_\pi}{\sigma^2_r + h\sigma^2_e} - \chi^2_{k-1} \quad \text{approximately.}
\]

(16)

If we define \( SS\_e = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2 \), then \( SS\_e \) is the usual error sum of squares and

\[
\frac{SS\_e}{\sigma^2_e} - \chi^2_{N-k}, \quad \text{where} \quad N = \sum_{i=1}^{k} n_i.
\]

(17)

Using \( \bar{X}, SS\_\pi, \) and \( SS\_e \), we shall now imitate the derivation in the balanced case to obtain an upper confidence limit for \( A \). Let \( \bar{X}, ss\_\pi, \) and \( ss\_e \), respectively, denote the observed values of \( \bar{X}, SS\_\pi, \) and \( SS\_e \), respectively. To find a GPQ for \( \sigma^2_r + \sigma^2_e \), let \( \hat{\sigma}_1^2 = \sigma^2_r + h\sigma^2_e \) and \( \hat{\sigma}_2^2 = \sigma^2_e \) so that

\[
\sigma^2_r + \sigma^2_e = \hat{\sigma}_1^2 + (1-h)\sigma^2_e.
\]

(18)

A GPQ for \( \hat{\sigma}_1^2 \) is given by

\[
G_{\hat{\sigma}_1^2} = \frac{\hat{\sigma}_1^2}{SS\_\pi} = \frac{SS\_\pi}{\chi^2_{k-1}}.
\]

and a GPQ for \( \sigma^2_e \) is given by

\[
G_{\sigma^2_e} = \frac{\sigma^2_e}{SS\_e} = \frac{SS\_e}{\chi^2_{N-k}}.
\]

We note that the GPQ for \( \hat{\sigma}_1^2 \) is only approximate as the chi-square distribution associated with \( SS\_\pi \) is only approximate. Substituting these GPQs in equation (18), we obtain a GPQ for \( \sigma^2_r + \sigma^2_e \) as
\[ G_{\sigma^2 + \sigma^e} = \frac{SS_x}{k_{h-1}} + (1-h) \frac{SS_e}{\chi^{2}_{N-k}}. \]  

(19)

A GPQ for \( \mu \) is given by

\[ G_{\mu} = \bar{x} + \frac{\sqrt{k}(\bar{x} - \mu)}{\sqrt{SS_x}} \sqrt{\frac{SS_x}{k}} \]

\[ = \bar{x} + \frac{Z}{\sqrt{\chi^{2}_{k-1}}} \sqrt{\frac{SS_x}{k}}, \]  

(20)

where \( Z = \frac{\sqrt{k} (\bar{x} - \mu)}{\sqrt{\sigma^2 + \sigma^e}} \sim N(0,1). \) Finally, a GPQ for \( A \) can be obtained as in equation (10) by replacing the parameters in equation (5) by their GPQs:

\[ G_A = \sqrt{\frac{G_{\sigma^2 + \sigma^e}}{C}} \sqrt{\chi^{2}_{1,\alpha}(G_{b}^2)}, \]  

with \( G_{b}^2 = \frac{(C - G_{\mu})^2}{G_{\sigma^2 + \sigma^e}}. \)  

(21)

The percentiles of \( G_A \) can be estimated by Monte Carlo simulation using an algorithm similar to Algorithm 1. The 100 \( \gamma \) percentile of \( G_A \) is a 100 \( \gamma \) \% upper confidence limit for \( A \). It should be clear that the approximation based on equation (11) that was mentioned for the balanced case can also be used in the unbalanced case. Furthermore, similar to the balanced case, the results simplify in the special case of negligible bias with \( \mu = C \). The Satterthwaite approximation can also be used in this special case, as follows. Let \( MS_x = SS_x / (k-1) \) and \( MS_e = SS_e / [k(n-1)] \). In view of equations (16) and (17), an unbiased estimator of \( \sigma^2_x + \sigma^2_e \) is given by \( \hat{\sigma}^2 + \hat{\sigma}_e^2 = MS_x + (1-h)MS_e \). A 100 \( \gamma \) \% upper confidence limit for \( A \), based on the Satterthwaite approximation, is given by

\[ \frac{\sqrt{(\hat{\sigma}^2 + \hat{\sigma}_e^2) / \chi^{2}_{1,\alpha}}}{\chi^{2}_{1,\alpha}} \sqrt{\chi^{2}_{1,\alpha}}, \]  

where the degrees of freedom \( f \) is given by

\[ f = \frac{MS_x^2 + (1-h)MS_e^2}{MS_x^2 / k_{h-1} + (1-h)MS_e^2 / N_{k-1}}. \]

We shall conclude our analysis of unbalanced data by making several comments on the chi-square approximation given in equation (16). The approximation is due to Thomas and Hultquist (1978), and these authors have investigated its accuracy. They conclude that the chi-square approximation in equation (16) is accurate except for \( \sigma^2_x / \sigma^2_e < .25 \), and the data are very unbalanced. Although it is difficult to quantify the extent of the unbalancedness under which the chi-square approximation given in equation (16) becomes unsatisfactory, several authors have used the approximation and have also commented on its accuracy (see e.g. Sahai and Ojeda, 2005, Section 11.8). It should also be noted that in the place of \( SS_x \), there are other formulations of the sum of squares (see once again Sahai and Ojeda, 2005, Chapter 11). Furthermore, these alternative formulations have found applications in industrial hygiene (see Harper et al., 2012). However, for our purpose, \( SS_x \) is preferable due to the chi-square approximation associated with it.

**SIMULATION STUDIES**

Because a generalized confidence limit is generally known to be only approximate, we note that all of the upper confidence limits derived above are only approximate. In order to assess the accuracy of the upper confidence limits, we performed a simulation study and estimated the coverage probabilities. The simulation was carried out as follows. We first generated 2500 sample statistics \( (\bar{x}, ss_x, ss_e). \) For each generated statistic, we used Algorithm 1 with \( M = 5000 \) to find generalized upper confidence limit for \( A \). The percentage of the 2500 upper confidence limits so obtained that are greater than \( A \) is a Monte Carlo estimate of the coverage probability. For the simulation, we chose \( 1 - \alpha = \gamma = 0.95 \), and \( (\mu, C) = (2, 2) \) and \( (2, 1.5). \) In other words, we are considering the cases \( \mu = C \) and \( \mu \neq C \). Our extensive preliminary simulation study indicates that the coverage probabilities are not much affected by the values of \( n \), and they do depend on the values of \( k \). So, simulation results are reported for different
values of $k$ for $n = 2, 3,$ and $4$. Furthermore, the generalized confidence limits $G_A$ based on Algorithm 1 turned out to be similar to the ones based on the approximation to $G_A$ given in Step 4, and so we used the approximate $G_A$ for our simulation studies. The simulated coverage probabilities are given in Table 1 for the balanced case. It is clear from Table 1 that for smaller $k$, the generalized upper confidence limits are slightly conservative, especially when $C = \mu$. However, confidence limits are less conservative when $C$ is different from $\mu$.

In the simulation study for the unbalanced case, we choose $k = 6$ and $7$, and the results are given in Table 2. The $n_i$'s used in the simulation are given at the end of Table 2. The reported coverage probabilities are close to the nominal level for all the cases considered. Overall, we see that our confidence limits perform satisfactorily in terms of coverage probability, even though they are slightly conservative for smaller values of $k$.

AN EXAMPLE

The example is on the determination of trace beryllium in air filter samples, and the purpose is an inter-laboratory evaluation of a standardized inductively coupled plasma mass spectrometry method (see Ashley et al., 2009). The relevant data are given in Table 3. We note that there are 20 laboratories, with three replicate observations from each laboratory, except two laboratories, which had only two replicate measurements. We first consider the balanced case after omitting the data for laboratories 13 and 15, which had only two replications each. Thus, $k = 18$ and $n = 3$.

Using the notations in the case of balanced data, the summary statistics are

\[
\bar{x} = 8.084, \quad ss_x = 81.298, \quad ss_e = 33.791.
\]

The 95% upper confidence limit based on Algorithm 1 with the exact $G_A$ in Step 4 is calculated as $0.5329$ and with the approximate $G_A$ is $0.5264$. As these upper confidence limits are not $<0.50$, the data do not indicate that the symmetric-range accuracy is $<0.50$. In other words, an accuracy requirement at the $0.50$ level would not be met in the example.

We shall now calculate 95% upper confidence limit for $A$ based on data from all 20 laboratories. Using the notations for the unbalanced case, the required summary statistics are calculated as

\[
\bar{x} = 8.065, \quad ss_x = 28.329, \quad ss_e = 34.794.
\]

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<tr>
<th>$\mu = C = 1$</th>
<th>$\mu = 1.5, C = 2$</th>
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<td>$(k, n)$</td>
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<tr>
<td>$A$</td>
<td>(6,2)</td>
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(a)–(e) represent the different choices of the $n_i$'s: (a): (3,2,4,5,3,2); (b): (2,2,1,6,12); (c): (4,3,9,2,1,1); (d): (2,2,1,3,3,3); (e): (2,2,3,2,4,2,12).
Using the expressions in equations (19) and (20), we find

\[ G_{\sigma_T^2 + \sigma_c^2} = \frac{28.329}{X_{19}^2} \left( 1 - 0.35 \right) \frac{34.794}{X_{38}^2}, \]

and

\[ G_{\mu} = 8.065 + \frac{Z}{\sqrt{X_{19}^2}} \sqrt{\frac{28.329}{20}}. \]

Using the above expressions in \( G_A \) given in equation (21) and Monte Carlo simulation of 100 000 runs, we estimated the 95th percentiles of \( G_A \) as .5186. Thus, the 95% upper confidence limit for \( A \) based on all data in Table 3 is .5186. This upper confidence limit is close to the one based only on the balanced data, and so, we arrive at the same earlier conclusion that an inter-laboratory requirement at the 0.50 level would not be met in this example.

An R function routine that calculates upper confidence limits for the symmetric-range accuracy \( A \) is posted at www.ucl.berkeley.edu/~kksk4695
REFERENCES


