Cosmological parameters and cosmic topology

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ABSTRACT

Geometry constrains but does not dictate the topology of the three-dimensional space. In a locally spatially homogeneous and isotropic universe, however, the topology of its spatial section dictates its geometry. We show that, besides determining the geometry, the knowledge of the spatial topology through the circles-in-the-sky offers an effective way of setting constraints on the density parameters associated with dark matter ($\Omega_m$) and dark energy ($\Omega_\Lambda$). By assuming the Poincaré dodecahedral space as the circles-in-the-sky detectable topology of the spatial sections of the Universe, we re-analyse the constraints on the density parametric plane $\Omega_m - \Omega_\Lambda$ from the current Type Ia supernova plus X-ray gas mass fraction data, and show that a circles-in-the-sky detection of the dodecahedral space topology gives rise to strong and complementary constraints on the region of the density parameter plane currently allowed by these observational data sets.

Key words: cosmology: theory – dark matter – distance scale.

1 INTRODUCTION

The standard approach to cosmological modelling commences with the assumption that our three-dimensional space is homogeneous and isotropic at large scales. The most general space–time metric consistent with the existence of a cosmic time $t$ and the principle of spatial homogeneity and isotropy is

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + f^2(\chi)(d\phi^2 + \sin^2 \theta d\phi^2)],$$

where $f(\chi) = (\chi, \sin \chi, \sinh \chi)$ depending on the sign of the constant spatial curvature ($k = 0, 1, -1$), and $a(t)$ is the cosmological scalefactor. The metric (1) only express the above assumptions, it does not specify the underlying space–time manifold $M_4$ nor the corresponding spatial sections $M$. In this geometrical approach to model the physical world, the dynamics of the Universe is clearly provided by a metrical theory of gravitation as, for example, General Relativity.

The Friedmann equation $k = H_0^2 a(t)^2 (\Omega_{\text{tot}} - 1)$ makes apparent that the curvature (or the corresponding geometry) of the spatial section $M$ of the Universe is an observable in that for $\Omega_{\text{tot}} > 1$ the spatial section is positively curved, for $\Omega_{\text{tot}} = 1$ it is flat ($k = 0$), while for $\Omega_{\text{tot}} < 1$ it is negatively curved. Thus, a chief point in the search for the (spatial) geometry of universe is to constrain the density $\Omega_{\text{tot}}$ from observations. In the context of the current standard cosmological model, that is, the $\Lambda$ cold dark matter ($\Lambda$CDM) scenario, this amounts to determining regions in the density parametric plane $\Omega_m - \Omega_\Lambda$, which consistently account for the observations, and from which one expects to infer the spatial geometry of the Universe.

In practice, to observationally probe the spatial geometry of the Universe, by using, for example, a single observational data set as the current Type Ia supernova (SN Ia) observations, various degeneracies arise in the parametric plane $\Omega_m - \Omega_\Lambda$ (Riess et al. 2004). These degeneracies are mitigated by either imposing reasonable priors, or combining the data with complementary observations, or both. An example in this regard is the combination of the current SN Ia compilation (Riess et al. 2004) with the Sloan Digital Sky Survey (SDSS) galaxy power spectrum (Tegmark et al. 2004) and the cosmic microwave background (CMB) measurements by the Wilkinson Microwave Anisotropy Probe (WMAP) (Bennett et al. 2003; Spergel et al. 2003), which has remarkably narrowed the bounds on the cosmological density parameters $\Omega_m$ and $\Omega_\Lambda$. Besides being sensitive to different combinations of the density parameters, these measurements probe the geometry of the Universe at considerably different redshifts (typically $z < 2$ for SNe Ia or $z \sim 1100$ for CMB).

However, the spatial geometry constrains but does not dictate the topology of the three-space $M$, and although the spatial section $M$ is usually taken to be one of the simply-connected spaces, namely Euclidean $\mathbb{R}^3$, spherical $S^3$, or hyperbolic $\mathbb{H}^3$, it is a mathematical result that the great majority of locally homogeneous and isotropic three–spaces $M$ are multiply connected quotient manifolds of the form $\mathbb{R}^3/\Gamma$, $S^3/\Gamma$, or $\mathbb{H}^3/\Gamma$, where $\Gamma$ is a fixed-point free group of isometries of the corresponding covering space. Thus, for example, for the Euclidean geometry ($k = 0$) besides $\mathbb{R}^3$ there are six classes of topologically distinct compact orientable spaces $M$ that can be endowed with this geometry, while for both the spherical ($k = 1$) and hyperbolic ($k = -1$) geometries there are an infinite...
number of topologically inequivalent manifolds with non-trivial topology that can be endowed with each one of these geometries.

Now since the spatial geometry is an observable that constrains but does not determine the topology of the three-space \( M \), two pertinent questions at this point are whether the topology may be an observable and, if so, to what extent it can be used to remove or at least reduce the degeneracies in the density parameter plane \( \Omega_m - \Omega_\Lambda \), which arise from statistical analyses with data from current observations.

The main aim of this paper, which extends and complements our previous work (Rebouças et al. 2006), is to address these questions in the context of the \( \Lambda \)CDM scenario, by focusing our attention on the finite and positively curved Poincaré space model (Luminet et al. 2003) that accounts for the suppression of power at large scales observed by WMAP (Bennett et al. 2003; Spergel et al. 2003), and also fits the WMAP temperature two-point correlation function (Aurich, Lustig & Steiner 2005a,b; Gundermann 2005). In other words, we will show how to use the Poincaré dodecahedral space (PDS) as a circles-in-the-sky observable topology of the Universe\(^2\) to reduce the inherent degeneracies in the density parameters \( \Omega_m \) and \( \Omega_\Lambda \), that arise from the so-called gold set of 157 SNe Ia, as compiled by Riess et al. (2004), along with the latest Chandra measurements of the X-ray gas mass fraction in 26 galaxy clusters, as provided by Allen et al. (2004).

2 COSMIC TOPOLOGY

In a number of recent papers, different strategies and methods to probe a non-trivial topology of the spatial section of the Universe have been discussed (see e.g. Lachieze-Rey & Luminet 1996; Roukema & Edge 1997; Falgudes & Gausmann 1999; Uzan, Lachieze & Luminet 1999; Gomero et al. 2002; Hajian & Souradeep 2005; Hajian, Souradeep & Cornish 2005; and the review articles, Lachieze-Rey & Luminet 1995; Starkman 1998; Levin 2002; Rebouças & Gomero 2004). An immediate observational consequence of a detectable non-trivial topology\(^3\) of the three-space \( M \) is that the sky will show multiple (topological) images of either cosmic structures observed by \( \Omega_m - \Omega_\Lambda \) or may not be detected, see Gomero et al. (2001a,b), Weeks et al. (2003) and Weeks (2003).

\(^2\)Here, in line with the usage in the literature, by topology of the Universe, we mean the topology of the space-like section \( M \).

\(^3\)For a detailed discussion on the extent to which a non-trivial topology may or may not be detected, see Gomero et al. (2001a,b), Weeks et al. (2003) and Weeks (2003).


2.1 Poincaré dodecahedral space model

The PDS \( D \) is a manifold of the form \( S^3 / \Gamma \) in which \( \Gamma = I^* \) is the binary icosahedral group of the order of 120. It is represented by a regular spherical dodecahedron (12 pentagonal faces) along with several further features of the model have been carefully considered. As a consequence, to reduce the inherent degeneracies in the density parameters, which arise from statistical analyses with data from current observations.

Regarding the question as to whether the topology can be used as an observable to reduce degeneracies in the cosmological density parameters, we first note that the topology of a locally homogeneous and isotropic three-manifold determines the sign of its curvature (see e.g. Bernshtein & Shvartsman 1980) and, therefore, the topology of the spatial section \( M \) of the Universe dictates its geometry. As a consequence, the detection of a generic cosmic topology alone would give rise to a strong degeneracy on both density parameters, since it only determines whether the density parameters take values in the regions below, above, or on the flat line \( \Omega_m = \Omega_\Lambda = 1 \).

In what follows, we examine to what extent a combination of the circles-in-the-sky detection of a specific spatial topology, namely, the PDS topology (which accounts for some observed anomalies in WMAP CMB data) with the current SN Ia and galaxy cluster measurements may reduce the intrinsic density parameter degeneracies of the \( \Omega_m - \Omega_\Lambda \) plane.
3 OBSERVATIONS

3.1 Method

An important class of manifolds with a non-trivial topology is comprised by the globally homogeneous manifolds. These manifolds satisfy a topological principle of (global) homogeneity, in the sense that all points in $M$ are topologically equivalent. In particular, in these spaces the pairs of matching circles of the circles-in-the-sky method will be antipodal, as shown in Fig. 1.

The PDS $\mathcal{D}$ is globally homogeneous, and gives rise to six pairs of antipodal matched circles on the LSS, centred in a symmetrical pattern as the centres of the faces of the dodecahedron. Fig. 1 gives an illustration of two of these antipodal circles. Clearly, the distance between the centres of each pair of circles is twice the radius $r_{\text{antipodal}}$ of the sphere inscribable in $\mathcal{D}$. Now, a straightforward use of a Napier's rule to the right-angled spherical triangle shown in Fig. 1 furnishes a relation between the angular radius $\alpha$ and the angular sides $r_{\text{antipodal}}$ and radius $\chi_{\text{LSS}}$ of the LSS, namely

$$
\chi_{\text{LSS}} = \frac{\tan^{-1} \left( \frac{\tan r_{\text{antipodal}}}{\cos \alpha} \right)}{2} \cos \alpha,
$$

where $r_{\text{antipodal}}$ is a topological invariant, equal to $\pi/10$ for the dodecahedral topology, and the distance $\chi_{\text{LSS}}$ to the origin in units of the curvature radius, $a_0 = a(t_0) = (H_0 \sqrt{1 - \Omega_{\text{m}}})^{-1}$, is given by

$$
\chi_{\text{LSS}} = \frac{d_{\text{LSS}}}{a_0} = \sqrt{\Omega_{\Lambda}} \sqrt{\Omega_{\text{m}}} \int_{1+z}^{\infty} \frac{dx}{\sqrt{\Omega_{\text{m}} x^2 + \Omega_{\text{m}} + \Omega_{\Lambda}}},
$$

where $d_{\text{LSS}}$ is the radius of the LSS. $x = 1 + z$ is an integration variable, $\Omega_{\text{m}} = 1 - \Omega_{\Lambda}$, and $z_{\text{LSS}} = 1089$ (Spergel et al. 2003).

Equations (2) and (3) give the relations between the angular radius $\alpha$ and the cosmological density parameters $\Omega_{\Lambda}$ and $\Omega_{\text{m}}$, and thus can be used to set bounds on these parameters. To quantify this, we proceed in the following way. First, we assume the angular radius $\alpha = 50^{\circ}$, as estimated by Aurich et al. (2005a). Second, since the measurements of the radius $\alpha$ unavoidably involve observational uncertainties we take, in order to obtain very conservative results, $\delta\alpha \approx 6^{\circ}$, which is the scale below which the circles are blurred (Aurich et al. 2005a).

3.2 Statistical analysis

In order to study the effect of the PDS topology on the parametric space $\Omega_{\text{m}} - \Omega_{\Lambda}$, we use the most recent compilation of SN Ia data, the so-called gold sample of 157 SNe Ia, recently published by Riess et al. (2004) along with the latest Chandra measurements of the X-ray gas mass fraction in 26 X-ray luminous, dynamically relaxed galaxy clusters spanning the redshift range $0.07 < z < 0.9$, as provided by Allen et al. (2004). We emphasize that this particular combination of observational data covers complementary aspects of the $\Omega_{\text{m}} - \Omega_{\Lambda}$ plane, in that while X-ray measurements are very effective to place limits on the clustered matter ($\Omega_{\text{m}}$) the new SN Ia sample tightly constrains the unclustered component ($\Omega_{\Lambda}$).

3.3 $f_{\text{gas}}$ versus redshift test

The X-ray gas mass fraction test $[f_{\text{gas}}(z)]$ was first introduced by Sasaki (1996) and further developed by Allen et al. (2002a,b) (see also Ettori, Tozzi & Rosati 2003; Lima, Cunha & Alcaniz 2003, 2004; Chen & Ratra 2004; Alcaniz & Zhu 2005; Zhu & Alcaniz 2005). This is based on the assumption that rich clusters of galaxies are large enough to provide a fair representation of the baryon and dark matter distributions in the Universe (Fukugita, Hogan & Peebles 1998). Following this assumption, the matter content of the Universe can be expressed as the ratio between the baryonic content and the gas mass fraction, that is, $\Omega_{\text{m}} \propto \Omega_{b}/f_{\text{gas}}$. Moreover, as shown by Sasaki (1996), since $f_{\text{gas}} \propto d_{\text{LSS}}^{\frac{3}{2}}$ the model function can be defined as (Allen et al. 2002a):

$$
f_{\text{gas}}^{\text{mod}}(z) = \frac{h_{\text{obs}}}{(1 + 0.19\sqrt{H}) \Omega_{\text{m}}} \left[ 2 h_{\text{mod}} d_{\text{LSS}}^{\frac{3}{2}}(z) \right]^{1.5},
$$

where the bias factor $b$ is a parameter motivated by gas dynamical simulations that takes into account the fact that the baryon fraction in clusters is slightly depressed with respect to the Universe as a whole (Eke, Navarro & Frenk 1998; Bialek et al. 2001), the term $0.19\sqrt{H}$ stands for the optically luminous galaxy mass in the cluster and the ratio between the angular diameter distances $d_{\text{LSS}}^{\text{mod}}(z)/d_{\Lambda M}^{\text{mod}}(z)$ accounts for deviations in the geometry of the Universe (here modelled by the $\Lambda$CDM model) from the default cosmology used in the observations, that is, the Einstein–de Sitter (E–dS) model (see Allen et al. 2004, for more observational details).

In order to perform the $f_{\text{gas}}$ test, three Gaussian priors are added to our analysis, namely, on the baryon density parameter, $\Omega_{b} h^{2} = 0.0224 \pm 0.0009$ (Spergel et al. 2003), on the Hubble parameter, $h = 0.72 \pm 0.08$ (Freedman et al. 2001), and on the bias factor, $b = 0.824 \pm 0.089$ (Eke et al. 1998; Bialek et al. 2001). Thus, the total minimization $\chi^2_{\text{fgas}}$ is written as

$$
\chi^2_{\text{fgas}} = \sum_{i=1}^{26} \left[ \frac{f_{\text{gas}}^{\text{mod}}(z_i) - f_{\text{gas},i}}{\sigma^2_{\text{fgas},i}} \right]^2 + \frac{\Omega_{\text{m}} h^2 - 0.0224}{0.0009}^2 + \left( \frac{h - 0.72}{0.08} \right)^2 + \left( \frac{b - 0.824}{0.089} \right)^2,
$$


$\chi^2_{\text{fgas}}$ is a statistic that quantifies the goodness of fit of the $f_{\text{gas}}$ model to the data.
where \( f_{\text{gas}}(z) \) is given by equation (4) and \( f_{\text{gas}}(z) \), are the observed values of the X-ray gas mass fraction with errors \( \sigma_{f_{\text{gas}}} \).

### 3.4 Magnitude versus redshift test

Supernova observations provide the most-direct evidence for the current cosmic acceleration. To perform a statistical analysis with the current supernova data, we first define the predicted distance modulus for a supernova at redshift \( z_i \), given a set of parameters \( s \), that is,

\[
\mu_p(z | s) = m - M = 5 \log h + 25, \tag{6}
\]

where \( m \) and \( M \) are, respectively, the apparent and absolute magnitudes, the complete set of parameters \( s \equiv (h, \Omega_m, \Omega_\Lambda) \) and \( h \) stands for the luminosity distance (in units of megaparsecs).

The set of parameters \( s \) is estimated by using a \( \chi^2 \) statistics, with

\[
\chi^2_{\text{SNe}} = \sum_{i=1}^{157} \frac{(\mu_p(z | s) - \mu_p(z | s))^2}{\sigma_i^2}, \tag{7}
\]

where \( \mu_p(z | s) \) is given by equation (6), \( \mu_p(z | s) \) is the extinction-corrected distance modulus for a given SN Ia at \( z_i \), and \( \sigma_i \) is the uncertainty in the individual distance moduli, which includes uncertainties in galaxy redshift due to a peculiar velocity of 400 km s\(^{-1}\) [For more details on statistical analyses involving SN Ia observations, we refer the reader to Padmanabhan & Choudhury (2003), Zhu & Fujimoto (2003), Alcaniz (2004), Alcaniz & Pires (2004), Nesseris & Perivolaropoulos (2004), Choudhury & Padmanabhan (2005) and Shafieloo et al. (2006)].

### 3.5 Topological constraint

Similarly to the Gaussian priors added to the \( f_{\text{gas}} \) test, the PDS topology is included in our statistical analysis as a prior relative to the value of \( \chi_{\text{gas}}^2 \), which can be easily obtained from an elementary combination of equations (2) and (3). In other words, the contribution of the topology to our statistical analysis is a term of the form

\[
\chi^2_{\text{topology}} = \frac{(\chi_{\text{gas}} - \chi_{\text{gas}}^\text{obs})^2}{(\delta \chi_{\text{gas}})^2}, \tag{8}
\]

where \( \chi_{\text{gas}}^\text{obs} \) is given by equation (2) and the uncertainty \( \delta \chi_{\text{gas}} \) comes from the uncertainty \( \delta \alpha \) of the circles-in-the-sky. This means that the total \( \chi^2_{\text{total}} \) minimization function is given by

\[
\chi^2_{\text{total}} = \chi^2_{\text{gas}} + \chi^2_{\text{SNe}} + \chi^2_{\text{topology}}. \tag{9}
\]

### 4 RESULTS AND DISCUSSIONS

The left-hand panel in Fig. 2 shows the results of our statistical analysis. Confidence regions – 68.3 and 95.4 per cent confidence limits (CLs) – in the parametric space \( \Omega_m - \Omega_\Lambda \) are displayed for the particular combination of observational data described above. For the sake of comparison, we also show in the right-hand panel the \( \Omega_m - \Omega_\Lambda \) plane for the conventional SN Ia plus galaxy cluster analysis, that is, the one without the above cosmic topology assumption. By comparing both analyses, it is clear that our initial premiss that a circles-in-the-sky detection of a non-trivial space topology reduces considerably the parametric space region allowed by the current observational data, and also breaks some degeneracies arising from the current SN Ia and X-ray gas mass fraction measurements. The best-fitting parameters for this SN Ia plus X-ray plus topology analysis are \( \Omega_m = 0.30 \) and \( \Omega_\Lambda = 0.79 \) with the reduced \( \chi^2 \equiv \chi^2_{\text{min}} / \nu \approx 1.12 \) (\( \nu \) is defined as degree of freedom). Note that such a value of \( \chi^2 \) is slightly smaller than the one obtained by fitting the gold sample of SNe Ia to the flat \( \Lambda \)CDM (concordance) scenario, that is, \( \chi^2 \approx 1.14 \), and equal to the value found for the \( \Lambda \)CDM model with arbitrary curvature (Riess et al. 2004). At 95.4 per cent CL, we also

![Figure 2](https://academic.oup.com/mnras/article-abstract/369/4/1693/1088885/1694)

**Figure 2.** The results of our statistical analyses. The panels show confidence regions (68.3 and 95.4 per cent CLs) in the \( \Omega_m - \Omega_\Lambda \) plane from the latest Chandra measurements of the X-ray gas mass fraction in 26 galaxy clusters \((0.07 < z < 0.9)\) plus determinations of the baryon density parameter, measurements of the Hubble parameter and the gold sample of 157 SNe Ia. In the left-hand panel, a specific circles-in-the-sky detection of the dodecahedral space topology with \( \alpha = 50^\circ \), \( \delta \alpha = 6^\circ \) is assumed. The best-fitting values for the matter and vacuum density parameters are, respectively, \( \Omega_m = 0.30 \pm 0.04 \) and \( \Omega_\Lambda = 0.79 \pm 0.03 \) at 95.4 per cent CL, which provides \( \Omega_m < 1.09 \pm 0.05 \). In the right-hand panel, the conventional SN Ia plus X-ray analysis is shown for comparison.
obtain $0.26 \leq \Omega_m \leq 0.34$ and $0.76 \leq \Omega_\Lambda \leq 0.82$ providing $\Omega_{\text{tot}} \simeq 1.09 \pm 0.05$, which is consistent at $2\sigma$ level with the value reported by the WMAP team, that is, $\Omega_{\text{tot}} = 1.02 \pm 0.02$ at $1\sigma$ (Spergel et al. 2003). Note that, the above best-fitting scenario is in full agreement with most of the current observational analyses (see e.g. Roos 2005, for an updated review) and corresponds to a current accelerated universe with $q_0 \simeq -0.64$ and a total expanding age of $9.67 \pm 1$ Gyr. Finally, we also note that for values of the angular radius in the range $0 \leq \alpha \leq 90^\circ$, equation (2) (e.g. $\Omega_m = 0.28$) gives the interval $1.01 \leq \Omega_{\text{tot}} \leq 1.3$, making clear that the very detection of first six pairs of circles predicted by the PDS topology would restrict the allowed range for the total density parameter. This sensitivity of $\alpha$ with the value of $\Omega_m$ was recently discussed by Roukema et al. (2004), and reinforced by Aurich et al. (2005a). Clearly, additional limits on the total density arise when one assumes a specific value of $\alpha$ and a related uncertainty $\delta\alpha$.

5 CONCLUDING REMARKS

By assuming the PDS as circles-in-the-sky detected topology of the spatial sections of the Universe, we have re-analysed the constraints on the parametric space $\Omega_m$-$\Omega_\Lambda$ from the current SN Ia and X-ray gas mass fraction data. As the main outcome of this analysis, we have shown that once detected the PDS topology along with the corresponding circles-in-the-sky, they give rise to very strong and complementary constraints on the region of density parameter plane currently allowed by the cosmological observations. We have also discussed how the degeneracies inherent to a joint analysis involving SN Ia observations and X-ray gas mass fraction measurements are drastically reduced by the assumption of a circles-in-the-sky detection of a PDS topology. According to equation (2), the main reason for this breakdown of the degeneracies is the sensitivity of the angular radius with the total density parameter. Finally, our results also indicate that cosmic topology may offer a fruitful strategy to constrain the density parameters associated with dark energy and dark matter.

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REFERENCES

Aurich R., Lastig S., Steiner F., 2005a, Class. Quantum Grav., 22, 2061
Aurich R., Lastig S., Steiner F., 2005b, Class. Quantum Grav., 22, 3443
Gomero G. I., Rebouças M. J., Tavakol R., 2001a, Class. Quantum Grav., 18, 4461
Gomero G. I., Rebouças M. J., Tavakol R., 2001b, Class. Quantum Grav., 18, L145
Starkman G. D., 1998, Class. Quantum Grav., 15, 2529

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