Virialization of cosmological structures in models with time-varying equation of state

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ABSTRACT

We study the virialization of the cosmic structures in the framework of flat cosmological models where the dark energy component plays an important role in the global dynamics of the Universe. In particular, our analysis focuses on the study of the spherical matter perturbations, as the latter decouple from the background expansion, start to ‘turn around’ and finally collapse. We generalize this procedure, taking into account models with an equation of state which vary with time, and provide a complete formulation of the cluster virialization attempting to address the non-linear regime of structure formation. In particular, assuming that clusters have collapsed prior to the epoch of $z_f \simeq 1.4$, in which the most distant cluster has been found, we show that the behaviour of the spherical collapse model depends on the functional form of the equation of state.

Key words: galaxies: formation – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

It is well known that the available high-quality cosmological data (Type Ia supernovae, CMB, etc.) are well fitted by an emerging ‘standard model’, which contains cold dark matter to explain clustering and an extra component with negative pressure, the dark energy, to explain the observed accelerated cosmic expansion. During the last decades there have been many theoretical speculations regarding the nature of the above exotic dark energy. Most of the authors claim that a scalar field which rolls down the potential $V(\phi)$ (Ratra & Peebles 1988; Caldwell, Dave & Steinhardt 1998; Peebles & Ratra 2003) could resembles that of the dark energy. The cosmic equation of state is given by $p/Q = w_\phi$, where $p_\phi$ is the pressure and $\rho_\phi$ is the corresponding density of the dark energy. Owing to the fact that $w < -1/3$ is related to the potential of the dark energy field, having an indication about its value may help us to understand the nature of the dark energy.

However, a serious issue here is how the large-scale structures, and in particular galaxy clusters, form. The cluster distribution basically traces scales that have not yet undergone the non-linear phase of gravitational clustering, thus simplifying their connections to the initial conditions of cosmic structure formation. The so-called spherical collapse model, which has a long history in cosmology, is a simple but still fundamental tool to describe the growth of systems in the Universe via gravitational instability (Gunn & Gott 1972). In the last decade many authors have been involved in this kind of study and have found that the main parameters of the spherical collapse model, such as the ratio between the final and the turn-around radius (hereafter collapse factor), is affected by the dark energy (Lahav et al. 1991; Wang & Steinhardt 1998; Iliev & Shapiro 2001; Basilakos 2003; Battye & Weller 2003; Weinberg & Kamionkowski 2003; Mota & van de Bruck 2004; Horellou & Berge 2005; Maor & Lahav 2005; Percival 2005; Zeng & Gao 2005; Nunes & Mota 2006; Wang 2006).

The aim of this work is along the same lines, attempting to investigate the cluster formation processes by generalizing the non-linear spherical model for a family of cosmological models with an equation-of-state parameter being a function of time, $w = w(t)$. This can help us to understand better the theoretical expectations of negative-pressure models as well as the variants from the Quintessence ($w = w_0 = \text{constant with } -1 \leq w_0 < -1/3$) and Phantom ($w = w_0 = \text{constant with } w_0 < -1$) cases, respectively.

The structure of the paper is as follows. The basic elements of the cosmological equations are presented in Section 2. Sections 3 and 4 outline the spherical collapse analysis in models where $w$ is a function of time, and finally we draw our conclusions in Section 5.

2 THE BASIC COSMOLOGICAL EQUATIONS

For homogeneous and isotropic cosmologies, driven by non-relativistic matter and an exotic fluid with an equation of state $p/Q = w(\alpha)$ with $w_\alpha < 0$, the Einstein field equations can be given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_\alpha + p_\alpha) - \frac{k}{a^2} \quad \text{(1)}$$
with \( k = -1, 0 \) or 1 for an open, flat or closed universe, respectively, and
\[
\ddot{a}/a = -4\pi G \left\{ \left[ w(a) + \frac{1}{3} \right] \rho_0 + \frac{1}{3} \rho_m \right\}
\]
where \( a(t) \) is the scalefactor, \( \rho_m = \rho_{\text{mat}} a^{-3} \) is the background matter density and \( \rho_0 = \rho_{\text{tot}} f(a) \) is the dark energy density, with
\[
f(a) = \exp \left\{ 3 \int_a^1 \left[ 1 + w(u) \right] du \right\}
\]
Thus, the scalefactor evolves according to Friedmann equation:
\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3} \rho_0 + \frac{1}{3} \rho_m(1 + w(a)) = \frac{1}{3} \Omega_0 + \frac{1}{3} \Omega_m(1 + w(0))
\]
where the Hubble parameter is written as \( H(a) = H_0 E(a) = H_0 \) and \( H_0 = H(a=0) \) is the Hubble constant with
\[
E(a) = \frac{\Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_r f(a)}{3}
\]
while \( \Omega_m = 8\pi G \rho_{\text{mat}}/3H_0^2 \) (density parameter), \( \Omega_k = -k/H_0^2 \) (curvature parameter) and \( \Omega_r = 8\pi G \rho_{\text{rad}}/3H_0^2 \) (dark energy parameter) at the present time with \( \Omega_m + \Omega_k + \Omega_r = 1 \).

In addition, \( \Omega_m(a) \) and \( \Omega_r(a) \) could also evolve with the scalefactor as
\[
\Omega_m(a) = \frac{\Omega_m a^{-3}}{E^2(a)} \quad \text{and} \quad \Omega_r(a) = \frac{\Omega_r f(a)}{E^2(a)}.
\]
Note that in this paper we consider a spatially flat \((k = 0)\) low-\(\Omega_m\) \((\Omega_m = 0.3)\) cosmology with \( H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1} \) which is in agreement with the cosmological parameters found from the recent observations (see Friedman et al. 2001; Spergel et al. 2003; Tegmark et al. 2004; Basilakos & Plionis 2005; Spergel et al. 2006, and references therein). Finally, in order to address the negative-pressure term it is essential to define the functional form of the equation parameter \( w = w(a) \) (see Section 4).

### 3 VIRIALIZATION IN THE SPHERICAL MODEL

The spherical collapse model is still a powerful tool, despite its simplicity, for understanding how a small spherical patch of homogeneous overdensity forms a bound system via gravitation instability. Technically speaking, the basic cosmological equations, mentioned before, are valid either for the entire Universe or for homogeneous spherical perturbations [by replacing the scalefactor with radius \( R(t) \)]
\[
\frac{\ddot{R}}{R} = -4\pi G \left\{ \left[ w(R) + \frac{1}{3} \right] \rho_k + \frac{1}{3} \rho_m \right\}
\]
where \( \rho_m \) and \( \rho_k \) are the time-varying matter and dark energy density, respectively (for spherical perturbations).

We study the cluster virialization in models with dark energy, generalizing the notations of Lahav et al. (1991), Wang & Steinhardt (1998), Maimini et al. (2003), Basilakos (2003), Lokas (2001), Mota & van de Bruck (2004), Horellou & Berge (2005), Zeng & Gao (2005), Mao & Lahav (2005), Bartelmam, Doran & Wetterich (2006), Nunes & Mota (2006) and Wang (2006) in order to take into account models with a time-varying equation of state. Thus, in this section, we review only some basic concepts of the problem. Within the framework of the spherical collapse model we assume a spherical mass overdensity shell, utilizing both the virial theorem \( T_v = -\frac{1}{2} U_{\text{tot}} + U_{\text{tot}} \) and the energy conservation \( T_{v,i} + U_{\text{tot},i} = U_{\text{tot},i} + U_{\text{tot},i} \), where \( T_v \) is the kinetic energy, \( U_{\text{tot}} = -3GM^2/5R \) is the potential energy and \( U_{\text{tot}} \) is the total energy associated with the dark energy for the spherical overdensity. In particular, the potential energy induced by the dark energy component (see Horellou & Berge 2005, and references therein) is
\[
U_{\text{tot}} = (1 + 3w)\rho_0 \frac{4\pi GM}{10} R^2.
\]
Using the above formulation we can obtain a cubic equation which relates the ratio between the final (virial) \( R_f \) and the turn-around outer radius \( R_o \):
\[
2n\lambda^3 - (2 + n)\lambda + 1 = 0,
\]
where \( \lambda = R_o/R_f \) is the collapse factor,
\[
r_1 = -(3w + 1)\frac{\Omega_m f(a)}{\Omega_0} \frac{\rho_{\text{tot}}}{\rho_m} \frac{\rho_{\text{tot}}}{\rho_m} = \left( R_o/R_f \right) ^{-3}
\]
and
\[
r_2 = -(3w + 1)\frac{\Omega_m f(a)}{\Omega_0} \frac{\rho_{\text{tot}}}{\rho_m} \frac{\rho_{\text{tot}}}{\rho_m} = \left( R_o/R_f \right) ^{-3}
\]
with
\[
\xi \equiv \frac{\rho_{\text{tot}}}{\rho_{\text{tot}}} = \left( \frac{R_o}{R_f} \right) ^{-3}
\]
(the definition of the \( \alpha_i \) and \( \alpha_m \) factors are presented bellow).

However, we would like to point out that there is some confusion in the literature regarding equation (9) which is based on energy conservation. Indeed recently it has been shown (Maor & Lahav 2005) that the above formulation is problematic due to the fact that the total energy of the bound system is not conserved in dark energy models with \( w \neq -1 \). In order to avoid these systematic effects in this work we utilize two different assumptions. First of all, we assume that when the matter epoch just begins, the dark energy moves synchronously with ordinary matter on both the Hubble scale and the galaxy cluster scale; the so-called clustered dark energy scenario (see Zeng & Gao 2005, and references therein). It is interesting to note that the synchronous dark energy evolution (hereafter clustered) was designed to conserve energy. In particular, Maor & Lahav (2005) address the issue of the clustered dark energy model based on the following assumptions: (i) clustered Quintessence considering that the whole system virializes (matter and dark energy) and (ii) the dark energy remains clustered but now only the matter virializes (for more details see their Section 4). Note that in this work we are utilizing the second possibility. In that case the equation which defines the collapse factor becomes
\[
(1 + q)\lambda - \frac{q}{2} (1 - 3w)\lambda^{-3w} = \frac{1}{2},
\]
where the factor \( q \) is given by
\[
q = \frac{\rho_{\text{tot}}}{\rho_{\text{tot}}} = \frac{\rho_{\text{tot}}}{\rho_{\text{tot}}} = \frac{\rho_{\text{tot}}}{\rho_{\text{tot}}} = \frac{v}{\xi}
\]
(the definition of the \( v \) parameter is presented in Section 3.1).

If we assume that the dark energy component in a galaxy cluster scale can be treated as being homogeneous then we get the following cubic equation defined by Maor & Lahav (2005)
\[
2\lambda^3 - (1 + q)\lambda + \frac{1}{2} = 0.
\]
However, it was pointed out recently (see Wang 2006) that if the value of \( q \) is less than 0.01 then the problem of energy conservation
does not really affect the virialization process and thus equation (9) is still a good approximation.

From now on, we will call $a_t$ the scalefactor of the Universe where the overdensity reaches its maximum expansion and $\alpha_t$ the scalefactor in which the sphere virializes, while $R_t$ and $R_\alpha$ are the corresponding radii of the spherical overdensity. Note that $\rho_{mc,t}$ is the matter density which contains the spherical overdensity at the turn-around time and $\rho_{mc,\alpha}$ is the background matter density at the same epoch. Therefore, for bound perturbations which do not expand forever, the time needed to recollapse is twice the turn-around time, $t_t = 2t_\alpha$. Taking this and equation (4) into account, it is easy to estimate the relation between the $\alpha_t$ and $\alpha_t$

$$\int_0^{t_t} \frac{d\alpha}{H(\alpha)/a} = 2 \int_0^{t_\alpha} \frac{d\alpha}{H(\alpha)/a}.$$  (16)

In order to reduce the parameter space of the overall problem we adopt further observational constraints. Indeed, $\alpha_t$ can be defined from the literature assuming that clusters have collapsed prior to the epoch of $z_t \simeq 1.4$ in which the most distant cluster has been found (Mullis et al. 2005; Stanford et al. 2006). Therefore, considering $\alpha_t \simeq 0.41$ and utilizing equation (16), it is routine to obtain the scalefactor at the turn-around time.$^1$ In the case of a $\Lambda$ cosmology we get an analytical solution:

$$\sin h^{-1} \left( a_t^{3/2} \right) = 2 \sin h^{-1} \left( a_t^{3/2} \right),$$  (17)

where $v_t = (1 - \Omega_m)/\Omega_m$. For the general problem we have to solve equation (16) numerically. The ratio between the scalefactors converges to the Einstein–de Sitter value ($\alpha_t/\alpha_t = 2^{2/3}$) at high redshifts.

### 3.1 The rescaled equations

In this section we present some of the basic concepts of the spherical collapse model. In particular, we assume a spherical overdensity which contains a dark energy component which behaves either as clustered or homogeneous. For a flat cosmology ($k = 0$), using the basic differential equations (see equations 1 and 7) and performing the following transformations

$$x = \frac{\alpha}{\alpha_t} \quad \text{and} \quad y = \frac{R}{R_t},$$  (18)

the evolution of the scalefactors of the background and of the perturbation are governed, respectively, by the following two equations:

$$\dot{x}^2 = H^2_t \Omega_{mc,\alpha} \left[ \Omega_m(x,x) \right]^{-1}$$  (19)

and

$$\dot{y} = - \frac{H^2_t \Omega_{mc,\alpha}}{2} \left[ \frac{\zeta}{y^2} + v y f(x, y) \right].$$  (20)

Note that in order to obtain the above set of equations we have used the following relations:

$$\rho_{mc} = \rho_{mc,\alpha} \left( \frac{R}{R_t} \right)^3 = \frac{\xi \rho_{mc,\alpha}}{y^3}$$  (21)

and

$$J(x, y) = \begin{cases} [1 + 3w(R(y))] f(R(x)) & \text{for clustered DE} \\ [1 + 3w(x)] f(x) & \text{homogeneous DE} \end{cases}$$  (22)

$^1$The epoch of the turn around is roughly $z_t \simeq 2.5$.

## 3.2 The general solution for the clustered dark energy scenario

In this section we present our analytical solution of the $\zeta$ parameter by integrating the above system of differential equations (equations 19 and 20), using at the same time the boundary conditions of (d $y/d x)_{y=0} = 0$ and $y = 1$. In the case of homogeneous dark energy the above system is solved only numerically. The novelty of our approach here is that for the clustered dark energy case the system can be solved analytically including models, where the dark energy parameter $w$ is a function of the cosmic time. Indeed, due to the fact that now the second part of equation (20) is a function only of $y$ we can easily perform the integration

$$\int_0^1 \left[ \frac{y}{\zeta + vy P(y) - (\zeta + P(1)v)y} \right]^{1/2} dy = \int_0^1 \frac{1}{\left[ x \Omega_m(x) \right]^{1/2}} dx.$$  (29)

### 4 SPECIFIC DARK ENERGY MODELS

Knowing the functional form of the function $f(x)$, the collapse scalefactor (in our case $\alpha_t \simeq 0.41$), the turn-around scalefactor $\alpha_t$ (from

$$v = \frac{\rho_{mc,\alpha}}{\rho_{mc,\alpha}} = \frac{1 - \Omega_{mc,\alpha}}{\Omega_{mc,\alpha}}.$$  (23)

The function $R(y)$ is given by the combination of equations (12) and (18): $R(y) = R_1 y = \zeta^{-1/3} \alpha_t y$. Finally, $\Omega_m(x)$ is given by

$$\Omega_m(x) = \frac{1}{1 + vx^3 f(x)}.$$  (24)

where $C$ is the integration constant and

$$G(y) = \frac{v}{f(\alpha_t)} \int y [1 + 3w(R(y))] f(R(y)) dy.$$  (25)

Considering now that the functional form of $f(R(y))$ is exponential (see Appendix B) we have the following useful formula:

$$\int y [1 + 3w(R(y))] f(R(y)) dy = -2 \int y f(R(y)) dy - \int y^2 \frac{d f(R(y))}{dy} dy = -y^2 f(R(y)).$$  (26)

Therefore, the basic differential equation for the evolution of the overdensity perturbations under the framework of the boundary conditions [described before $C = \zeta + P(1)v$] takes the form

$$\frac{dy}{dx} = \frac{\zeta y^{-1} + v P(y) - \zeta - P(1)v}{[x \Omega_m(x)]^{-1}},$$  (27)

with

$$P(y) = \frac{y^2 f(R(y))}{f(\alpha_t)}.$$  (28)

Finally, below we present the general integral equation which governs the behaviour of the density contrast $\zeta$ at the turn-around epoch, for generic flat cosmological models, as a function of the equation of state (which also depends on time) and the perturbation collapse time

$$\int_0^1 \left[ \frac{y}{\zeta + vy P(y) - (\zeta + P(1)v)y} \right]^{1/2} dy = \int_0^1 \frac{1}{\left[ x \Omega_m(x) \right]^{1/2}} dx.$$  (29)

equation 16) and assuming a low matter density flat cosmological model with \(\Omega_m = 1 - \Omega_Q = 0.3\), we can obtain the parameter \(\zeta\) solving numerically the equation (29). Note that in this work, we deal with four different type of equations of state, which we present in the coming sections.

### 4.1 Quintessence – Phantom models

In this case the equation of state is constant (see for a review, Caldwell 2002; Peebles & Ratra 2003; Corasaniti et al. 2004). If \(-1 \leq w_0 < -1/3\) we have the so-called Quintessence models while for \(w_0 < -1\) we get the so-called Phantom cosmologies (hereafter QP models). Despite the fact that for these models the cluster virialization has been investigated thoroughly in several papers, we have decided to re-estimate it in order to understand the variants from the \(w = w(t)\) case. In particular, for \(w(x) = w_0\) we get \(f(x) = \Omega_m^{-3(1 + w_0)}\) and the overall problem (equation 29) reduces to the integral equation found by Zeng & Gao (2005). In particular, the basic factors become

\[
P(y) = \zeta^{1 + w_0}y^{-3w_0 - 1}\quad \text{(30)}
\]

and

\[
\Omega_m(x) = \frac{1}{1 + vx^{-3w_0}}.\quad \text{(31)}
\]

In Fig. 1 we present the \(\zeta\) solution (top panel) for the clustered and homogeneous dark energy (solid and dashed lines, respectively) as a function of the equation-of-state parameter, and it is obvious that the two models give almost the same results. Also Fig. 1 (bottom panel) shows the dependence of the density contrast at virialization on \(w\):

\[
\Delta_{\text{vir}} = \frac{\rho_{\text{vir},t}}{\rho_{\text{vir}}(x)} = \frac{\zeta}{\lambda^3} \left(\frac{\alpha_l}{\alpha_i}\right)^3,\quad \text{(32)}
\]

where \(\rho_{\text{vir},t}\) is the matter density in the virialized structure and \(\rho_{\text{vir}}(x)\) is the background matter density at the same epoch. The factor \(\Delta_{\text{vir}}\) is a key parameter in this kind of study because we can compare the predictions provided by the spherical collapse model with observations. In this framework, it is also interesting to note that the collapse factor is inbetween \(0.48 \leq \lambda \leq 0.50\) while for the homogeneous dark energy we get \(0.47 \leq \lambda \leq 0.50\). Both cases tend to the standard value 0.5 in an Einstein–de Sitter universe in agreement with previous studies (Maor & Lahav 2005; Wang 2006). This is to be expected simply because at large redshifts matter dominates over the dark energy in the Universe, which means that the parameter \(q\) has small values (see the insert of Fig. 1). Note that \(\Delta_{\text{vir}}\) models can be described by Quintessence models with \(w_0\) strictly equal to \(-1\), and thus equation (29) is written as

\[
\int_0^1 \left[\frac{y}{\zeta + vy^3 - (\zeta + vy)^y}\right]^{1/2} dy = \frac{\ln(\sqrt{1+v^2} + \sqrt{v^2})}{\sqrt{v^2}}.\quad \text{(33)}
\]

In order to explore further the virialized structures we investigate the connection between the infalling velocity at the virialized epoch and the equation of state. It is well known from the literature that the non-linear infalling velocity field is described by the following expression (see Lilje & Efstathiou 1989; Croft, Dalton & Efstathiou 1999):

\[
V^{\text{inf}}(r) = -\frac{1}{3} H(x)\Omega_m^0(x) r \frac{\delta(r)}{[1 + \delta(r)]^{3/2}},\quad \text{(34)}
\]

where \(\delta(r)\) is the fluctuation field. It should be noted that in this work we have used the generic expression for the growth index \(\Omega_m^0\) defined by Wang & Steinhardt (1998) with

\[
\beta = \frac{3}{5} - w_0/(1 - w_0) + \frac{3(1 - w_0)(1 - 3w_0/2)}{125(1 - 6w_0/5)^4}(1 - \Omega_m).\quad \text{(35)}
\]

Due to interplay between the infalling velocity and the virial radius, in Fig. 2 we present the corresponding ratio between them.

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**Figure 1.** The upper panel shows the \(\zeta\) parameter as a function of \(w_0\). Note that the solid line presents the clustered dark energy scenario while the dashed line corresponds to the homogeneous dark energy. The lower panel shows the corresponding density contrast at virialization, for a flat cosmology, \(\Omega_m = 1 - \Omega_Q = 0.3\) with a constant equation of state. Finally, the indices \(P\) and \(Q\) represent Phantom and Quintessence, respectively.

**Figure 2.** The non-linear infalling velocity in the virialized regime as a function of the equation-of-state parameter.
4.2 Models with time variation \( w \)

In the last few years there have been many theoretical speculations regarding the nature of the exotic ‘dark energy’. Various candidates have been proposed in the literature, among which was a dynamical scalar field acting as vacuum energy (Ozer & Taha 1987). Under this framework, high-energy field theories generically show that the equation of state of such a dark energy is a function of the cosmic time. If this is the case then the basic equations regarding the clustering in models with a time-varying equation of state become more complicated than in models with constant \( w \). Note that in the literature, due to the absence of a physically well-motivated fundamental theory, there are plenty of dark energy models that can fit current observational data (for a review see Liberato & Rosenfeld 2006).

In this work for example, we use several parameters regarding the dark energy component in order to evaluate the integrals of equations (3) and (29). In particular, we consider a simple expression for the equation of state:

\[
    w(\alpha) = w_0 + w_1 X(\alpha)
\]

which is available in the literature (see e.g. Goliath et al. 2001; Linder 2003; Cepa 2004; Liberato & Rosenfeld 2006, and references therein). The function \( X(\alpha) \) satisfies the following constraint at the present epoch: \( X(1) = 0 \), and thus, \( w_0 = -1/3 \). Here we sample the parameters as follows: \( w_0 \in [-1.05, -0.35] \) and \( w_1 \in [0.1, 1.5] \), in steps of 0.1. To this end, we need to emphasize that our approach is powerful, in the sense that for different parametric forms of \( w(\alpha) \) the system of equations (19) and (20) can be solved analytically in the framework of a clustered dark energy. Finally, in this work we assume that the parameters \((w_0, w_1)\) and the functional form of the equation-of-state parameter remain the same either for the entire Universe or for the spherical perturbations.

4.2.1 The \( w(\alpha) = w_0 + w_1 (1 - \alpha) \) model

In this case (see Chevalier & Polarski 2001; Linder 2003; Cepa 2004), where \( w_0 \) and \( w_1 \) are constants, using equations (3), (24) and (28) we derive the following basic functions of the differential equation (29):

\[
    f(\alpha) = \alpha^{-3(1+w_0+w_1)} e^{3b_1(\alpha-1)},
\]

\[
    \Omega_{m0}(x) = \frac{1}{1 + x^{1-3b_0-3w_0+w_1} e^{b_1(1-\alpha)}},
\]

and

\[
    P(y) = x^{1+w_0+w_1} e^{b_1(1-\alpha)} y^{3w_0+w_1+1}.
\]

For the homogeneous dark energy scenario we find that the collapse factor lies in the range \( 0.47 \leq \lambda \leq 0.50 \) while for the clustered case we get \( 0.30 \leq \lambda \leq 0.50 \). In particular, Fig. 3 shows the behaviour of the collapse factor for possible pairs of \((w_0, w_1)\) starting from \( w_0 = -0.35 \) (solid line), \( w_0 = -0.45 \) (short-dashed line), \( w_0 = -0.55 \) (dashed line) and \( w_0 = -0.65 \) (dot-dashed line).

From the figure, it becomes evident that for large values of \( w_1 \) the collapse factor is significant less than 0.5 which means that under the framework of the present equation of state, a candidate structure (large-scale overdensity) is located in a large density regime \( (\Delta_{\text{vir}} > 400) \) and at the virialized epoch \( [r = R_1, \alpha = \alpha_1 \text{ and } \delta(R_1) \simeq \Delta_{\text{vir}}] \), as a function of the equation-of-state parameter. Therefore, it becomes evident that for models with \( w_0 < -1 \), the so-called Phantom models (see Caldwell 2002), the infalling velocity is unaffected by the equation of state. When we use models with \( w > -1 \) the infalling velocity is a decreasing function of \( w \) (the difference is \( \sim 37 \) per cent). The latter is to be expected because of the \( \Delta_{\text{vir}} \) behaviour (see Fig. 1).

4.2.2 The \( w(\alpha) = w_0 + w_1 (\alpha^{-1} - 1) \) model

Using now the equation of state derived by Goliath et al. (2001), the formulas described before become

\[
    f(\alpha) = \alpha^{-3(1+w_0-w_1)} e^{3b_1(\alpha-1)},
\]

\[
    \Omega_{m0}(x) = \frac{1}{1 + x^{1-3w_0-w_1} e^{b_1(1-\alpha)}},
\]

and

\[
    P(y) = x^{1+w_0-w_1} e^{b_1(1-\alpha)} y^{3w_0+w_1+1}.
\]

Following the same paradigm as in the previous case, in Fig. 4 we present for the clustered case the distribution of \( \lambda \). It is obvious that this equation of state also affects the virialization of the large-scale structures in the same way as before (see Section 4.2.1), but the corresponding \( w_1 \) parameter takes much lower values. For \( w_1 < 0.15 \) the collapse factor tends to 0.5. Note that for the homogeneous dark energy the collapse factor is in the range \( 0.45 \leq \lambda \leq 0.49 \).

4.2.3 The \( w(\alpha) = w_0 + w_1 (1 - \alpha \alpha) \) model

In this case (see Liberato & Rosenfeld 2006) we get

\[
    f(\alpha) = \alpha^{-3(1+w_0)} e^{b_1(\alpha-1)^2},
\]

\[
    \Omega_{m0}(x) = \frac{1}{1 + x^{1-3w_0} e^{b_1(1-\alpha)^2}},
\]

and

\[
    P(y) = x^{1+w_0} e^{b_1(1-\alpha)^2} y^{3w_0+1}.
\]
Thus produce more bound systems (0.30 ⩽ \( R / t \)) where the collapse factor is between 0.45 and 0.50. This result is in agreement with those derived in the QP models, despite the fact that the two dark energy models have completely different equations of state. Finally, considering now the homogeneous dark energy scenario, the collapse factor is almost in the same interval, that is, 0.45 ⩽ \( \lambda \) ⩽ 0.48.

5 CONCLUSIONS

The launch of the recent observational data has brought great progress in understanding the physics of gravitational collapse as well as the mechanisms that have given rise to the observed large-scale structure of the Universe. In this work, assuming that the dark energy moves synchronously with ordinary matter on both the Hubble scale and the galaxy cluster scale (cluster dark energy), we have treated analytically the non-linear spherical collapse scenario considering different models with negative pressure which also contains a time-varying equation of state. We verify that having an overdense region of matter, which will create a cluster of galaxies prior to the epoch of \( z \approx 1.4 \), dark energy affects the virialization of the large-scale structures in the following manner: in flat cosmological models that contain either (i) a constant equation state or (ii) an equation of state with \( w = w_0 + w_1(1 - \alpha \lambda) \), the virialization radius divided by the turn-around radius is in between 0.47 ⩽ \( R / t \) ⩽ 0.50 and the density contrast at virialization lies in the interval 171 ⩽ \( \Delta_{\text{vis}} \) ⩽ 240. It is interesting to mention that the same behaviour is also found for the homogeneous dark energy. Finally, we find that the equations of state \( w = w_0 + w_2(1 - \alpha \lambda) \) and \( w = w_0 + w_2(\alpha^{-1} - 1) \) correspond to relatively larger \( \Delta_{\text{vis}} \) values, and thus produce more bound systems (0.30 ⩽ \( R / t \) ⩽ 0.50) with respect to the other two equations of state mentioned before. However, for the homogeneous dark energy we found structures where the collapse factor is in between 0.45 ⩽ \( R / t \) ⩽ 0.49.

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APPENDIX A

Without wanting to appear too pedagogical, we remind the reader of some basic elements of algebra. Given a cubic equation: \( \lambda + a_1 \lambda^2 + a_2 \lambda + a_3 = 0 \), let \( D \) be the discriminant:

\[
D = a_1^2 a_2^2 - 4a_1^3 a_3 - 4a_1^2 a_2 - 27a_1^2 + 18a_2 a_3
\]

(A1)

and

\[
x_1 = -a_1 + \sqrt[3]{\frac{2}{3}D} \quad \text{and} \quad x_2 = -a_1 - \sqrt[3]{\frac{2}{3}D}.
\]

If \( D > 0 \), all roots are real (irreducible case). In that case \( \lambda_1, \lambda_2, \lambda_3 \) can be written as

\[
\lambda_\mu = -\frac{a_1}{3} + \frac{2^{1/3}}{3} \cos \left( \frac{\theta}{3} \right), \quad \mu = 1, 2, 3
\]

(A2)

where \( r = \sqrt{x_1^2 + x_2^2} \) and \( \theta = \cos^{-1}(x_1/r) \).

In this study, we derive analytically the exact solution of the basic cubic equation (equation 15), having polynomial parameters \( a_1 = 0, a_2 = -(1 + q)/2q \) and \( a_3 = 1/4q \). Then the discriminant of equation (15) is

\[
D(q) = \frac{8(1 + q)^3 - 27q^2}{16q^3}.
\]
Of course, in order to obtain physically acceptable results we need to take \( q > 0 \) which gives \( D(q) > 0 \). Therefore, all roots of the cubic equation are real (irreducible case) but one of them, \( \lambda_3 \), corresponds to expanding shells. It is obvious that for \( q \to 0 \) the above solution tends to the Einstein–de Sitter case \( (\lambda_3 \to 0, 50) \), as it should.

**APPENDIX B**

We present here some more details regarding the integration of equation (26). Considering a spherical overdensity with radius \( R \), the functional form of equation (3) becomes

\[
 f(R) = \exp \left\{ \int_R^1 \left[ \frac{1 + w(u)}{u} \right] du \right\}. \tag{B1}
\]

Now taking into account that \( R(y) = R_y y^{-1/3} \alpha_y \) (described well in Section 3.1) and differentiating the above integral we have the following useful formula:

\[
 \frac{df(R(y))}{dy} = \frac{df}{dR} \frac{dR}{dy} = -3y^{-1}[1 + w(R(y))]f(R(y)). \tag{B2}
\]

Using now the integral of equation (26),

\[
 \int y[1 + 3w(R(y))]f(R(y)) \, dy = -2 \int yf(R(y)) \, dy + 3 \int [1 + w(R(y))]yf(R(y)) \, dy \tag{B3}
\]

and taking into account equation (B2) we can easily solve the integral

\[
 \int y[1 + 3w(R(y))]f(R(y)) \, dy = -2 \int yf(R(y)) \, dy - \int y^2 \frac{df(R(y))}{dy} \, dy = -y^2 f(R(y)). \tag{B4}
\]

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