A dynamo model for axisymmetric and non-axisymmetric solar magnetic fields

J. Jiang* and J. X. Wang*

National Astronomical Observatories, Chinese Academy of Sciences, Beijing 100012, China

ABSTRACT

More and more observations are showing a relatively weak, but persistent, non-axisymmetric magnetic field co-existing with the dominant axisymmetric field on the Sun. Its existence indicates that the non-axisymmetric magnetic field plays an important role in the origin of solar activity. A linear non-axisymmetric \( \alpha^2 - \Omega \) dynamo model is derived to explore the characteristics of the axisymmetric \((m = 0)\) and the first non-axisymmetric \((m = 1)\) modes and to provide a theoretical basis with which to explain the ‘active longitude’, ‘flip-flop’ and other non-axisymmetric phenomena. The model consists of an updated solar internal differential rotation, a turbulent diffusivity varying with depth, and an \( \alpha \)-effect working at the tachocline in a rotating spherical system. The difference between the \( \alpha^2 - \Omega \) and the \( \alpha - \Omega \) models and the conditions that favour the non-axisymmetric modes under solar-like parameters are also presented.

Key words: magnetic fields – MHD – Sun: activity – Sun: magnetic fields.

1 INTRODUCTION

The distribution of the magnetic field emerging on the solar surface provides clues to the mechanism of the field generation. One striking feature of this distribution is the clustering of active regions, which is commonly termed ‘active longitude’ (Bai 1987; Benevolenskaya et al. 1999; De Toma, White & Harvey 2000). Signatures of possible longitudinal inhomogeneities have also been reported in the distributions of the solar wind and the interplanetary magnetic field (Neugebauer et al. 2000). The ‘flip-flop’ phenomenon, i.e. two persistent active longitudes separated by \( 180^\circ \), has also been identified on the Sun (Berdyugina & Usonkin 2003). These observations indicate the involvement of a large-scale non-axisymmetric magnetic field in the formation and evolution of the dominant types of axisymmetric solar activity. It is therefore useful to establish a non-axisymmetric dynamo model to explain these non-axisymmetric solar magnetic fields.

The pioneering theoretical works on non-axisymmetric activity adopted two main approaches. One is that the generation sources are non-axisymmetric, and the non-axisymmetric magnetic field is produced accordingly. Bigazzi & Ruzmaikin (2004) and Moss, Piskunov & Sokoloff (2002), for example, adopted the non-axisymmetric distribution of the \( \alpha \)-effect. The other considers axisymmetric sources of generation that excite a non-axisymmetric field. The numerical results of Chan et al. (2004), for example, support this possibility.

Earlier studies (Stix 1971; Ivanova & Ruzmaikin 1985) considered a linear non-axisymmetric solar dynamo with decoupled axisymmetric and non-axisymmetric modes. These earlier studies, however, were unable to include the correct distribution of solar differential rotation, which was unknown at that time. Recently, there have been some studies on non-linear non-axisymmetric dynamo models. Moss (1999) obtained stable solutions that possessed a small non-axisymmetric field component co-existing with the dominant axisymmetric part using the updated solar rotation profile. Bigazzi & Ruzmaikin (2004) studied the generation of non-axisymmetric fields and their coupling with the axisymmetric solar magnetic field. Bassom et al. (2005) used an asymptotic WKBJ method to investigate a linear \( \alpha^2 - \Omega \) model with the aim of isolating the basic physical effects leading to the preferential excitation of non-axisymmetric solar and stellar magnetic structure. Is it possible, however, to model a non-axisymmetric solar dynamo, using the updated generation sources, that describes the non-axisymmetric phenomena? What are the differences, such as configuration and cycle, etc., between the axisymmetric and non-axisymmetric modes? When is the non-axisymmetric mode preferred? The main objectives of this paper are to answer these questions.

Using the axisymmetric sources of generation we develop a new high-precision non-axisymmetric code based on the spectral method, beginning with the linear non-axisymmetric mean-field dynamo equations. The mathematical formulations are presented in Section 2. The difference between the \( \alpha^2 - \Omega \) and the \( \alpha - \Omega \) models are demonstrated in Section 3. In Section 4, the condition for excitation of the dominant axisymmetric mode and the condition that favours the non-axisymmetric mode are discussed. The axisymmetric mode \( m = 0 \) and the first non-axisymmetric mode \( m = 1 \) are discussed in Sections 5 and 6, respectively.

*E-mail: jiangjie@ourstar.bao.ac.cn (JJ); wangjx@bao.ac.cn (JXW)
2 MATHEMATICAL FORMULATIONS

2.1 The basic equations

The starting point of our model is the mean-field dynamo equation, which governs the evolution of the large-scale magnetic field $\mathbf{B}$ in response to the flow field $\mathbf{U}$, the $\alpha$-effect, and the magnetic diffusivity $\eta$:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B} + \alpha \mathbf{B} - \eta \nabla \times \mathbf{B}). \quad (1)$$

Because the turbulent diffusivity is much larger than the molecular diffusivity, we ignore the molecular diffusivity in $\eta$. For the flow field, only the (differential) rotation $\Omega$ is considered, for simplicity. Because the magnetic field is divergence-free, we expand $\mathbf{B}$ in terms of two scalar functions $h$ and $g$, which represent the poloidal and toroidal potentials, respectively, in the spherical polar coordinates $(r, \theta, \phi)$ (see Chandrasekhar 1961; Moffatt 1978):

$$\mathbf{B} = \nabla \times (r \phi h(r, \theta, \phi, t) + \nabla \times r g(r, \theta, \phi, t)). \quad (2)$$

When $\alpha = \alpha(r, \theta, \phi, t)$, $\Omega = \Omega(r, \theta, \phi, t) = \eta(r)$, substituting equation (2) into equation (1) means that the governing equation reduces to

$$\partial_t L^2 h = R_u V^h_u + \eta \nabla^2 L^2 h + R_\Omega V^h_{2\Omega} + R_u V^h_{\alpha N}, \quad (3)$$

$$\partial_t L^2 g = R_u V^g_u + R_\Omega V^g_{2\Omega} + R_u V^g_{\alpha N} + \eta \nabla^2 L^2 g + \frac{\partial h}{\partial r} \frac{\partial g}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right] \frac{\partial g}{\partial \theta} + \frac{1}{r^2} L^2 g. \quad (4)$$

where

$$L^2 = -\frac{1}{\sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin \theta} \nabla^2 = -\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \right).$$

$V^h_{2\Omega}$ and $V^g_{\alpha N}$ are the terms that contain the azimuthal component $\partial / \partial \phi$, and $V_u$, $V^h_{\alpha N}$ and $V^g_{\alpha N}$ can be obtained from (see Appendix A)

$$r \cdot \nabla \times (r \phi \mathbf{B}) = V^h + V^h_{\alpha N}, \quad (5)$$

$$r \cdot \nabla \times (\mathbf{U} \times \mathbf{B}) = V^h_{\Omega}, \quad (6)$$

$$r \cdot \nabla \times [\nabla \times (r \phi \mathbf{B})] = V^g + V^g_{\alpha N}, \quad (7)$$

$$r \cdot \nabla \times [\nabla \times (\mathbf{U} \times \mathbf{B})] = V^g_{\Omega} + V^g_{\alpha N}. \quad (8)$$

Equations (3) and (4) are cast in non-dimensional form by expressing all lengths in units of the solar radius $R_\odot$ and time in units of the magnetic diffusion time $R^2_\odot / \eta_0$. This leads to the appearance of two dimensionless numbers:

$$R_u = \alpha_0 R_\odot / \eta_0, \quad (9)$$

$$R_\Omega = \Omega_0 R^2_\odot / \eta_0, \quad (10)$$

where $\alpha_0$ and $\eta_0$ are reference values for the $\alpha$-effect and the diffusivity in the convective zone (CZ), respectively. $\Omega_0$ is the characteristic value of the differential rotation. The quantities $R_u$ and $R_\Omega$ are dynamo numbers measuring the relative importance of inductive versus diffusive effects. $R_\Omega$ is discussed further in Subsection 2.2.

2.2 Internal rotation $\Omega(r, \theta)$

Based on the helioseismic inversion (Schou et al. 1998; Charbonneau et al. 1999), there are two strong radial shear regions inside the Sun. One is located in the tachocline, and the other is in the subphotospheric layer. For the sake of simplicity in computational solutions, we neglect the shear at the subsurface and assume that the dynamo works in the tachocline. The following expression for the solar interior rotation is adopted:

$$\Omega(r, \theta) = \Omega_0 + \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{2r - r_c}{d} \right) \right] \Omega_0, \quad (11)$$

where $\Omega_0 = \Omega_{0b} + \alpha_2 \cos^2 \theta + \alpha_1 \cos^4 \theta$ is the surface latitudinal rotation and $\theta$ is the colatitude. The parametric values are set as $r_c = 0.7 R_\odot$, $d = 0.05 R_\odot$, $\Omega_{0b} / 2\pi = 430.0$ nHz, $\Omega_{0b} / 2\pi = 455.8$ nHz, $\alpha_2 / 2\pi = -51.2$ nHz, $\alpha_1 / 2\pi = -84.0$ nHz. Fig. 1 shows the radial distribution of $\Omega(r, \theta)$ at various latitudes. It reveals that $\Omega$ depends weakly on depth in the bulk of the CZ. In the tachocline, however, the rotation rate changes from almost uniform in the radiative interior to depth-dependent in the CZ. Within the tachocline, rotation increases with distance from the core at low latitudes, while it decreases at high latitudes. At intermediate latitudes (near $35^\circ$, dashed line in Fig. 1), rotation is almost independent of depth.

We set our model on a rotating spherical system of the inner core with the rotation velocity $\Omega_c$. Thus the differential rotation in the rotating frame is

$$\Omega(r, \theta) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{2r - r_c}{d} \right) \right] (2\pi \times 25.8) \times (1.00 - 1.98 \cos^2 \theta - 3.26 \cos^4 \theta) \text{ (nHz).} \quad (12)$$

The differential rotation of the surface at the equator is $(2\pi \times 25.8)$ nHz, and we regard this as the characteristic value of the differential rotation $\Omega_c$ in equation (10). Hence, the value of $R_\Omega$ adopted is $(8 \times 10^{10} / \eta_0) \text{ m}^2 \text{s}^{-1}$, which is dependent only on the reference value of the diffusivity $\eta_0$.

2.3 The diffusivity profile $\eta(r)$

We use the analytical expression of Dikpati & Charbonneau (1999) for the diffusivity profile:

$$\eta(r) = \eta_i + \frac{\eta_0}{2} \left[ 1 + \text{erf} \left( \frac{r - r_c}{d} \right) \right] \quad (13)$$

(see Fig. 2, dashed line). The diffusivity $\eta_i$ in the CZ is dominated by the turbulence. In a stably stratified core, the diffusivity $\eta_c$ is much lower because of the much lower levels of turbulence. In what
follows, we take $\eta_l/\eta_o = 0.01$. The transition from high to low diffusivity occurs near the tachocline, which is coincident with the rotational shear layer. Here, $\eta_o$ cannot be determined from observations and is widely known to fall in the range from $2 \times 10^{10}$ to $2 \times 10^{12}$ cm$^2$ s$^{-1}$.

2.4 The $\alpha$-effect $\alpha(r, \theta)$

The $\alpha$-effect cannot currently be determined from observations. The dominant physical mechanisms responsible for it can be categorized as the following three types. (1) The $\alpha$-effect works at the surface and it is produced by the decaying of active regions (Babcock 1961; Leighton 1969). (2) The $\alpha$-effect is directly related to turbulent convective motions (Parker 1955). It exists throughout the whole CZ and changes sign near the bottom of the CZ (Krivodubskii 1998; Kuzanyan et al. 2003). (3) The $\alpha$-effect works at the tachocline induced by the hydrodynamical shear instabilities (Dikpati et al. 2001) or MHD instabilities (Thelen 2000). It is possible that all of these mechanisms operate simultaneously inside the Sun. Here we only consider the $\alpha$-effect associated with the tachocline, which has the following expression:

$$\alpha(r, \theta) = \alpha \left[ \frac{1}{2} \right] \left[ 1 + \text{erf} \left( \frac{r - r_1}{d} \right) \right] \cos \theta,$$

where $r_1 = 0.675 R_\odot$, $r_2 = 0.725 R_\odot$. The solid line in Fig. 2 shows the variation of $\alpha(r, \theta)$ with $r$, which occurs mainly in the tachocline. The commonly used angular dependence $\cos \theta$ is adopted, which is the simplest way to guarantee antisymmetry across the equator. Moreover, we do not consider $\alpha$-quenching, because only linear solutions are sought.

2.5 The numerical scheme

Because the governing equations (3) and (4) are two coupled, linear, homogeneous equations in $h$ and $g$, with the given boundary conditions we can look for eigensolutions of the form

$$[h(r, \theta, \phi, t), g(r, \theta, \phi, t)] = [h(r, \theta, \phi), g(r, \theta, \phi)] e^{\sigma t},$$

where $\sigma$ is the eigenvalue and can be written as $\sigma = \sigma + i \omega$. Only the solution that neither grows nor decays ($\sigma \simeq 0$), i.e. the onset of dynamo action, is considered. The corresponding $R_\alpha$ is the critical $R_\alpha$. The solution with the lowest dynamo number is the easiest to excite and is the most stable. In what immediately follows, we solve the dynamo equations numerically using the spectral (Chebyshev-r) method (see Jiang & Wang 2006 for details).

3 THE $\alpha^2$-$\Omega$ DYNAMO MODEL VERSUS THE $\alpha$-$\Omega$ MODEL

For the mean-field dynamo theory, the poloidal field is created from the toroidal field by the $\alpha$-effect, and the toroidal field from the poloidal field in two ways, namely from differential rotation (the $\Omega$-effect) and the $\alpha$-effect. The model including all these ingredients is termed the $\alpha^2$-$\Omega$ model. When $R_\Omega > R_\alpha^2$, the $\alpha$-effect as the toroidal source can be ignored and the $\alpha$-$\Omega$ model is always adopted.
(Zeldovich et al. 1983). Is the simple $\alpha-\Omega$ model acceptable for the Sun? What quantitative conditions does it need to satisfy? What are the differences between the two models for solar-like parameters?

We first discuss the above questions based on the axisymmetric mode $A_0$, and increase the range of $\eta_0$ from $8 \times 10^{10}$ to $8 \times 10^{12}$ cm$^2$ s$^{-1}$. Thus $R_\Omega$ ranges from $10^2$ to $10^4$. For the $\alpha$–$\Omega$ model, the condition for the generation of an undamped magnetic field is determined only by $D = R_{\alpha} R_\Omega$ (Ivanov & Ruzmaikin 1985). Accordingly, we obtain the straight (solid) line in Fig. 3 with logarithmic abscissa, and $D = R_{\alpha} R_\Omega$ is about 3840. For the $\alpha^2$–$\Omega$ model, however, the situation is more complicated because of the generation of the toroidal field by the $\alpha$-effect (the dashed line in Fig. 3). Comparing the two lines of Fig. 3, it can be seen that, for the $\alpha^2$–$\Omega$ model, the dynamo action is increased in comparison with the $\alpha$–$\Omega$ model by the reduction of the critical $R_{\alpha}$ when $R_\Omega$ is small ($<3 \times 10^3$). With increasing $R_\Omega$, the difference in the corresponding critical $R_{\alpha}$ between the two models decreases. When $R_\Omega = 3 \times 10^3$, the agreement between the two models reaches the level of 0.3 per cent.

Because the absolute scale for the strength of the magnetic field cannot be given by linear eigenvalue calculations, we define the ratio of the magnetic energy between the toroidal and poloidal components as (Charbonneau & MacGregor 2001)

$$\Theta = \frac{\int B_T^2 \, dV}{\int B_P^2 \, dV},$$

(22)

where

$$B_T = -\frac{g}{\partial \theta} \hat{e}_\theta,$$

and

$$B_P = \frac{L^2 h}{r} \hat{e}_r + \left( \frac{1}{\partial h/\partial \theta} + \frac{\partial^2 h}{\partial \theta^2} \right) \hat{e}_\theta$$

for the axisymmetric model. Fig. 4 gives the energy ratios between the toroidal and poloidal fields at the onset state for the two models for different values of $R_\Omega$. When $R_\Omega$ is less than $3 \times 10^3$, the $\alpha$–$\Omega$ model has smaller $E_T/E_P$ than does the $\alpha^2$–$\Omega$ model. The larger the value of $R_\Omega$, the less difference there is between the two models. The agreement between the two models reaches 0.46 per cent when $R_\Omega = 3 \times 10^3$. When $R_\Omega > 3 \times 10^3$, the energy ratios are in close agreement with each other. Thus we can replace the $\alpha^2$–$\Omega$ model by the $\alpha$–$\Omega$ model, and the corresponding turbulent diffusivity $\eta_0$ should be less than $2.67 \times 10^{11}$ cm$^2$ s$^{-1}$.

![Figure 3](https://academic.oup.com/mnras/article-abstract/377/2/711/1037053/Downloaded-from-academic.oup.com.common/article-pdf/1377/11037053/07 April 2019)

Figure 3. A critical $R_\Omega - R_\Omega$ plot for the $\alpha$–$\Omega$ (solid line) and the $\alpha^2$–$\Omega$ (dashed line) models for the axisymmetric mode $A_0$. When $R_\Omega < 3 \times 10^3$, the $\alpha$–$\Omega$ model has a larger critical $R_{\alpha}$ than the $\alpha^2$–$\Omega$ model. When $R_\Omega > 3 \times 10^3$, the two models have the nearly same critical $R_{\alpha}$. For the $\alpha$–$\Omega$ model, $R_{\alpha} R_\Omega$ is about 3840.

![Figure 4](https://academic.oup.com/mnras/article-abstract/377/2/711/1037053/Downloaded-from-academic.oup.com.common/article-pdf/1377/11037053/07 April 2019)

Figure 4. The ratio of the magnetic energy between the toroidal and poloidal components vs. $R_\Omega$ for the $\alpha$–$\Omega$ (solid line) and the $\alpha^2$–$\Omega$ (dashed line) models for the mode $A_0$.

### Table 1.

<table>
<thead>
<tr>
<th>$R_\Omega$</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
<th>6000</th>
<th>7000</th>
<th>8000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1.02</td>
<td>1.04</td>
<td>1.06</td>
<td>1.08</td>
<td>1.10</td>
<td>1.12</td>
<td>1.14</td>
<td>1.16</td>
<td>1.18</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>1.04</td>
<td>1.08</td>
<td>1.12</td>
<td>1.16</td>
<td>1.20</td>
<td>1.24</td>
<td>1.28</td>
<td>1.32</td>
<td>1.36</td>
</tr>
<tr>
<td>$d$</td>
<td>1.23</td>
<td>1.27</td>
<td>1.31</td>
<td>1.35</td>
<td>1.39</td>
<td>1.43</td>
<td>1.47</td>
<td>1.51</td>
<td>1.55</td>
</tr>
</tbody>
</table>

Now we study the non-axisymmetric mode $S_1$. Table 1 gives the difference between the two models for various values of $R_\Omega$. When $R_\Omega > 4 \times 10^3$, the energy ratios are in close agreement with 0.46 per cent. When $R_\Omega$ is larger than 4000, $\eta_0$ will be less than $2 \times 10^{11}$ so that the $\alpha - \Omega$ model is equivalent to the $\alpha^2 - \Omega$ model. It is perfectly reasonable in this case to use the $\alpha$–$\Omega$ model to replace the $\alpha^2$–$\Omega$ model.

### 4 AXISYMMETRIC VERSUS NON-AXISYMMETRIC MODE

It is commonly held that strong differential rotation favours the axisymmetric mode. Without it, all single-main-sequence stars with outer CZs have non-axisymmetric field configurations (Rüdiger & Elstner 1994). It is the non-uniform rotation that produces the observed dominant oscillatory dipolar field on the Sun. However, is the differential rotation expressed by equation (12) sufficient to produce the dominant axisymmetric field? What condition does it need to satisfy to make the non-axisymmetric mode the preferred one?

With the given rotation profile (12) of the Sun, we vary $R_\Omega$ from 500 to 8000. Accordingly, $\eta_0$ varies from $1.0 \times 10^{11}$ to $1.6 \times 10^{12}$ cm$^2$ s$^{-1}$. Fig. 5 shows the critical $R_{\alpha}$ versus $R_\Omega$, for the $\alpha^2$–$\Omega$ model. When $R_\Omega > 3000$ it is the axisymmetric mode $A_0$ (solid line) that has the lower value of critical $R_{\alpha}$ and will be preferentially excited. For $R_\Omega < 3000$, on the other hand, it is the non-axisymmetric mode $S_1$ that has the lower critical $R_{\alpha}$ and will be the preferred mode. The smaller the value of $R_{\alpha}$, the easier it will be to excite $S_1$. According to Section 3, it is not at the $\alpha$–$\Omega$ limit ($R_\Omega < 3000$) that $S_1$ is the preferred mode. In other words, it is impossible to favour the non-axisymmetric mode at the $\alpha$–$\Omega$ limit. To obtain the preferred non-axisymmetric modes, the contribution of the $\alpha$-effect to the generation of the toroidal field cannot completely vanish (Rädler 1986a). As it deviates further from the limit, the role the $\alpha$-effect plays in producing the toroidal field becomes more important, and the non-axisymmetric mode becomes easier to excite. This is also
consistent with the analytical results of Bassom et al. (2005). These authors give the reason that the wind-up of the non-axisymmetric structures can be compensated by phase mixing which is inherent in the \( \alpha^2-\Omega \) dynamo.

Furthermore, rather than the differential rotation, it is \( R_\alpha \) that is the decisive parameter in deciding which kind of mode is preferred. We can also say that the turbulent diffusivity \( \eta_o \) is the key parameter because the differential rotation is determined by observations (Jiang & Wang 2007). For the \( \alpha^2-\Omega \) model, when \( R_\alpha < 3000 \), i.e. \( \eta_o > 2.67 \times 10^{11} \text{ cm}^2 \text{ s}^{-1} \), the non-axisymmetric mode will be preferred.

In the following two sections we consider the \( \alpha^2-\Omega \) model with \( \eta_o = 1.6 \times 10^{11} \text{ cm}^2 \text{ s}^{-1} \) and \( R_\alpha = 5000 \) so that the axisymmetric mode will be preferred.

5 THE AXISYMMETRIC MODE

The left part of Table 2 gives the truncation levels in the calculation of the axisymmetric mode \( A_0 \) and the corresponding critical \( R_\alpha \) and frequency \( \omega \). \( N \) is the number of radial harmonics expanded in terms of the Chebyshev functions, and \( L \) is the number of harmonics in the Legendre functions (see equations 16 and 17). When \( N = 38 \) and \( L = 50 \) there is convergence. The critical \( R_\alpha \) is 0.765, and the frequency \( \omega \) is \( \pm 410.37 \). With the dimensionless time \( t = R_\odot^2/\eta_o \), we obtain a period of about 15 yr, which is shorter than the 22-yr dynamo wave propagating equatorwards or polewards. Although some high-frequency dynamo waves have been identified as propagating polewards on the Sun (Makarov 1989; Obridko et al. 2006), the poleward solution obtained in our method cannot be used to explain these observations. Hence, the poleward solution is meaningless and should be neglected. Fig. 6(b) is a time–latitude diagram of the radial field at the surface. The phase shift between the two components is approximately \( \pi/2 \), which is consistent with observations (Sheeley 1991).

Fig. 7 shows the evolution of the toroidal field in a meridional plane \( \phi = 0 \) at intervals of 1/6th of the solar cycle period. The dot-dashed line is 0.7 \( R_\odot \). The magnetic field is strongest in the region where the radial shear of differential rotation is strong, i.e. the high

\[
N \quad L \quad R_\alpha \quad \omega \quad N \quad L \quad R_\alpha \quad \omega
\]

| 34 | 46 | 0.762 | \pm 408.97 | 24 | 66 | 1.201 | \pm 0.794 |
| 36 | 46 | 0.767 | \pm 412.06 | 26 | 66 | 1.198 | \pm 0.919 |
| 38 | 46 | 0.764 | \pm 409.90 | 28 | 66 | 1.198 | \pm 0.967 |
| 38 | 48 | 0.765 | \pm 410.39 | 28 | 68 | 1.203 | \pm 0.946 |
| 38 | 50 | 0.765 | \pm 410.37 | 28 | 70 | 1.203 | \pm 1.022 |

Figure 5. A critical \( R_\alpha – R_\Omega \) plot for the \( \alpha^2-\Omega \) model for the modes \( A_0 \) (solid line) and \( S_1 \) (dashed line). When \( R_\Omega \leq 3 \times 10^3 \), the mode \( S_1 \) has the smaller \( R_\alpha \). In contrast, for \( R_\Omega > 3 \times 10^3 \) the mode \( A_0 \) has the smaller \( R_\alpha \).

Figure 6. Butterfly diagrams of (a) the toroidal field \( B_\phi \) for the mode \( A_0 \) at the depth \( r = 0.7 R_\odot \), and (b) \( B_t \) at the surface \( r = R_\odot \). Solid (dashed) contours correspond to a positive (negative) magnetic field. The diffusion time \( R_\odot/\eta_o \) is taken as the time unit.

and it should be equatorwards, because the product of the \( \alpha \)-effect and radial gradient of differential rotation is negative (Parker 1955; Yoshimura 1975). The solution with \( \omega = -410.37 \) corresponds to the dynamo wave propagating polewards. Although some high-frequency dynamo waves have been identified as propagating polewards on the Sun (Makarov 1989; Obridko et al. 2006), the poleward solution obtained in our method cannot be used to explain these observations. Hence, the poleward solution is meaningless and should be neglected. Fig. 6(b) is a time–latitude diagram of the radial field at the surface. The phase shift between the two components is approximately \( \pi/2 \), which is consistent with observations (Sheeley 1991).

Fig. 7 shows the evolution of the toroidal field in a meridional plane \( \phi = 0 \) at intervals of 1/6th of the solar cycle period. The dot-dashed line is 0.7 \( R_\odot \). The magnetic field is strongest in the region where the radial shear of differential rotation is strong, i.e. the high
latitude of the tachocline. This follows the general rule that differential rotation tends to destroy any deviation from axisymmetry, and that the toroidal field is preferentially produced in the strong radial shear region (Moffatt 1978; Bigazzi & Ruzmaikin 2004).

It seems that the location of the toroidal field produced in the model is higher than that of observations. Furthermore, when the toroidal flux rope rises through the CZ to emerge, it will have some deflection to the pole owing to the effect of the Coriolis force (Cagnoli, Moreno-Insertis & Schüssler 1995). In the model, however, we omit an important ingredient, namely the meridional circulation. If the meridional circulation is considered, the strong field produced within the tachocline at high latitudes will be carried to low latitudes. The toroidal flux entering the CZ will become buoyantly unstable and emerge to form active regions at low latitudes (Nandy & Choudhuri 2002).

6 THE NON-AXISYMMETRIC MODE

The right part of Table 2 gives the truncation levels of the mode $S1$. When $N = 28$ and $L = 70$, convergence is obtained, which is slower than in the calculation for the axisymmetric mode. The critical $R_\alpha$ is $1.203$, much larger than that for the axisymmetric mode. Therefore, we should have included a more complicated field configuration on the main activity cycle (Berdyugina 2004). For a more realistic result a value that is 3–4 times shorter (about 3.7 yr for the Sun) than the full flip-flop cycle has the same length as the $\alpha$-effect working in the tachocline, where the diffusivity is less than in other regions (see Fig. 2). Fig. 9(a) is a butterfly diagram for the mode $S1$ at the depth $r = 0.75 R_\odot$ (solid line) and (b) butterfly diagram of the toroidal field at the depth $r = 0.75 R_\odot$ for the mode $S1$. The fields concentrate in regions of low diffusivity (below the dot–dashed line $0.7 R_\odot$ in (a)) and weak differential rotation (about the latitude $35^\circ$ in (b)). Solid (dashed) contours correspond to a positive (negative) magnetic field.

The frequency $\omega$ is $-1.022$. Hence the period of mode $S1$ is nearly 400 times longer than that of mode $A0$. Thus the non-axisymmetric mode $S1$ appears to be fairly steady or weakly oscillating compared with the axisymmetric mode $A0$ (Berdyugina 2004). The time variations of the mode $A0$ are periodic. By changing the sign of the mode $A0$, the predominant longitude jumps by about $180^\circ$, which is just the flip-flop phenomenon (for details see section 3 of Fluri & Berdyugina 2004). Based only on these two modes, however, the full flip-flop cycle has the same length as the $A0$ cycle rather than a value that is 3–4 times shorter (about 3.7 yr for the Sun) than the main activity cycle (Berdyugina 2004). For a more realistic result we should have included a more complicated field configuration on the Sun.

Both the non-axisymmetric and the axisymmetric magnetic fields are generated by the same axisymmetric sources. They evolve independently of each other in linear theory. In reality, however, some non-linearities, such as the non-axisymmetric $\alpha$-effect (Bigazzi & Ruzmaikin 2004), MHD instability (Dikpati & Gilman 2005), magnetic buoyancy (Chatterjee et al. 2004) and $\alpha$-quenching (Zhang et al. 2003; Moss 2005), cause the two modes to become coupled, producing the flip-flop cycle (Moss 2004).

Fig. 9(a) shows the contours of the toroidal magnetic field in the meridional plane. The field is mainly concentrated around $35^\circ$ latitude, where the radial shear is weak (see Fig. 1). This is consistent with the results of Bigazzi & Ruzmaikin (2004), namely that the non-axisymmetric field survives only in regions of weak differential rotation and low diffusivity.

7 DISCUSSION AND CONCLUSIONS

In this paper we have investigated the properties of the axisymmetric and the non-axisymmetric modes with a linear $\alpha^2-\Omega$ model in a rotating frame, trying to understand the active longitudes, flip-flops and other non-axisymmetric phenomena. The model consists of updated differential rotation, turbulent diffusivity varying with depth, and the $\alpha$-effect working in the tachocline. The differences between the $\alpha-\Omega$ and $\alpha^2-\Omega$ models and the conditions for excitation of the non-axisymmetric modes with solar-like parameters are presented. The Chebyshev–$\tau$ method is used to solve the problem numerically with high precision. We tested this code with the analytical solutions of a simple $\alpha^2$ model and compared our results with those of Stix (1976); they were in good agreement.

Although the conditions that favour non-axisymmetric modes have been investigated by several authors (Rädler 1986a,b; Bassom et al. 2005), we are the first to have used updated solar parameters and a numerical method to obtain them. We also point out the role of the turbulent diffusivity $\eta_\alpha$ in the model. Diffusivity is poorly known and cannot be directly deduced from observations. When it is lower than $2.0 \times 10^{11}$ cm$^2$ s$^{-1}$ and $R_\alpha$ is correspondingly greater than 4000, the $\alpha-\Omega$ model is at the limit of the $\alpha^2-\Omega$ model. Furthermore, based on the $\alpha^2-\Omega$ model, when $R_\alpha$ is lower than 3000, the non-axisymmetric mode $m = 1$ will be increasingly more easily
excited than the axisymmetric mode $m = 0$ when increasing the value of the diffusivity $\eta$, and decreasing the corresponding $R_2$.

The non-axisymmetric mode $S_1$ has a much longer period than the axisymmetric mode $A_0$. The co-existence of the oscillating mode $A_0$ and the nearly steady $S_1$ results in the flip-flop phenomenon. Because the differential rotation affects axisymmetric and non-axisymmetric magnetic fields in different ways, the two kinds of fields occur in different regions. The axisymmetric magnetic field is mainly concentrated near the high latitude of approximately 55°, where the radial shear of differential rotation is strong. The non-axisymmetric field, however, occurs near the intermediate latitude of 35° at the bottom of the tachocline, where the differential rotation is weak and the diffusivity is low.

Usoskin, Berdyugina & Poutanen (2005) calculated that the ratio between the non-axisymmetric strength and the axisymmetric strength is approximately 1:10 by analysing sunspot-group data for the past 120 yr. Based on the non-linear models, (Moss 1999, 2004) gave the energy ratios between the two kinds of modes, although the values do not match well with observations. Because our model is linear and the different modes are decoupled, it cannot provide the energy ratios between the two modes.

In this work, we assume that the strong radial shear exists only in the tachocline and omit the subsurface shear and other details of the distribution of differential rotation. Brandenburg (2005) argued for alternative ideas concerning the dynamo operating in the bulk of the CZ, or perhaps even in the subsurface shear layer. Moreover, we assume that it is only the $\alpha$-effect that operates in the tachocline, and ignore the other two possible mechanisms mentioned in Section 2.4. The two generation sources (the radial shear and the $\alpha$-effect) are still hot topics of debate. Because we do not aim to give a detailed description of the Sun, just emphasizing the basic characteristics of the axisymmetric and non-axisymmetric modes, it is feasible for us to take the two simple generation sources and set up a thin-layer dynamo model. Of course, richer and more realistic models will open up new avenues for understanding the solar magnetic field.

Furthermore, because we tried to expatiate on our objectives with the simple generation sources, the meridional circulation is not considered in the work. It plays an important role in the axisymmetric model (Nandy & Choudhuri 2002; Guerrero & Muñoz 2004), carrying the strong asymmetric toroidal field produced at high latitudes to the low ones and forming the active regions there with the effect of the magnetic buoyancy. According to the work of Bigazzi & Ruzmaikin (2004), however, it does not have much influence on the non-axisymmetric field.

In forthcoming studies we will include non-linearities and the meridional circulation and investigate their influence on the coupling of the different modes and on the non-axisymmetric dynamo.

ACKNOWLEDGMENTS

We thank the anonymous referee for valuable comments that helped to improve the paper. We extend our thanks to A. R. Choudhuri and K. M. Kuzanyan for reviewing the manuscript and making many helpful suggestions. JJ would like to thank X. H. Liao and K. K. Zhang for kind supervision of the work. This work was supported by the National Natural Science Foundation of China (10233050, 10573025, 10603008) and by the National Basic Research Program (G2006CB806303).

REFERENCES

Radler K. H., 1986a, ESA SP-251, Plasma Astrophysics. ESA Publications Division, Noordwijk, p. 569

Downloaded from https://academic.oup.com/mnras/article-abstract/377/2/711/1037053 by guest on 07 April 2019

APPENDIX A: THE FULL FORM OF THE GOVERNING EQUATIONS

The full forms of the governing equations of the toroidal field \( g \) and poloidal field \( h \) in (3) and (4) are as follows.

\[
\frac{\partial L^2 h}{\partial t} = R_\alpha \left[ \alpha L^2 g - \frac{\partial \alpha}{\partial \theta} \frac{\partial g}{\partial \theta} \right] + \eta \nabla^2 L^2 h - R_\Omega \left[ \Omega \frac{\partial L^2 h}{\partial \phi} \right] + R_\alpha \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left[ \frac{1}{r} \frac{\partial h}{\partial \phi} + \frac{\partial^2 h}{\partial r^2} \right], \tag{A1}
\]

\[
\frac{\partial L^2 g}{\partial r} = R_\alpha \left[ -\alpha \nabla^2 L^2 h + \frac{\partial \alpha}{\partial \theta} \frac{\partial \nabla^2 h}{\partial \theta} + \frac{\partial^2 \alpha}{\partial r \partial \theta} \frac{\partial \nabla^2 h}{\partial r \partial \theta} - \frac{\partial \alpha}{\partial r} \frac{\partial L^2 h}{\partial r} - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \alpha}{\partial \theta} \right) L^2 h - \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial h}{\partial \theta} \right) L^2 h - \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( \sin \theta \frac{\partial h}{\partial \theta} \right) \right]
+ R_\Omega \left[ \frac{1}{r \sin \theta} \frac{\partial}{\partial r} \left( \sin \theta \frac{\partial h}{\partial \theta} \right) \right] + \eta \nabla^2 L^2 g + \frac{\partial \eta}{\partial r} \frac{\partial L^2 g}{\partial r} + \frac{1}{r} \frac{\partial \eta}{\partial r} L^2 g + \frac{1}{r} \frac{\partial \eta}{\partial \phi} L^2 g
- R_\Omega L^2 \left( \Omega \frac{\partial g}{\partial \phi} \right), \tag{A2}
\]

where \( \alpha, \Omega \) and \( \eta \) are the expressions after non-dimensionalization.

This paper has been typeset from a \TeX/\LaTeX\ file prepared by the author.