The dynamical stability of a Kuiper Belt-like region

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ABSTRACT

The dynamics of the Kuiper Belt region between 33 and 63 au is investigated just taking into account the gravitational influence of Neptune. Indeed the aim is to analyse the information which can be drawn from the actual exoplanetary systems, where typically physical and orbital data of just one or two planets are available. Under this perspective we start our investigation using the simplest three-body model (with Sun and Neptune as primaries), adding at a later stage the eccentricity of Neptune and the inclinations of the orbital planes to evaluate their effects on the Kuiper Belt dynamics. Afterwards we remove the assumption that the orbit of Neptune is Keplerian by adding the effect of Uranus through the Lagrange–Laplace solution or through a suitable resonant normal form. Finally, different values of the mass ratios of the primary to the host star are considered in order to perform a preliminary analysis of the behaviour of exoplanetary systems. In all cases, the stability is investigated by means of classical tools borrowed from dynamical system theory, like Poincaré mappings and Lyapunov exponents.

Key words: celestial mechanics – Kuiper Belt.

1 INTRODUCTION

Celestial mechanics and planetary sciences considerably advanced during the last decade of the 20th century, thanks to two fundamental astronomical discoveries: the existence of the Kuiper Belt and the detection of extrasolar planetary systems. Since their first observations the two populations continuously grow in number. An interesting question concerns the possibility of the existence of a Kuiper Belt-like region within exoplanetary systems. In order to answer this question, it is necessary to have a clear picture of the dynamical aspects of the Kuiper population. The results available in the literature are very wide and they cover several topics, from the stability analysis to the investigation of mean motion resonances. Among the others we mention the study of the main secular resonances performed by Knežević et al. (1991), the numerical exploration of Duncan, Levison & Budd (1995) with particular attention to the 3/4 and 2/3 mean motion resonances, the analytical and numerical approaches of Morbidelli, Brown & Levison (2003) to determine the primordial sculpting of the Kuiper Belt. The major mean motion resonances have also been studied in the paper of Malhotra (1996) from the 5/6 (about 34 au) up to the 1/3 resonance (about 63 au). Nesvorný & Roig (2000, 2001) performed a systematic study on the dynamics of the 2/3, 1/2, 3/4 and other weaker resonances. At present the Kuiper population is divided into two groups: classical (mainly residing between 42 and 48 au) and scattered objects (characterized by large eccentricity and inclination). A special role is performed by the plutinos, which form a crowded group in the 2/3 resonance with Neptune around 39 au (Chiang & Jordan 2002). Also the 1/2 resonance draws the attention of many researchers, due to the fact that it marks the edge of the classical Kuiper Belt around 50 au (see e.g. Hahn & Malhotra 2005). Within the classical population an interesting subdivision as dynamically cold and hot objects has been introduced (see Morbidelli et al. 2003).

The aim of this paper is to study the dynamical behaviour in the parameter space corresponding to the Kuiper Belt. To this end we implement standard techniques of dynamical systems, for example, the determination of families of periodic orbits and their linear stability (Hadjidemetriou 2006), the computation of the Poincaré mappings, the determination of the Lyapunov exponents (Benettin et al. 1980; Froeschlé 1984; Wolf et al. 1985). A particularly useful tool is represented by the construction of maps of dynamical stability, based on the computation of the (maximum) Lyapunov characteristic number (LCN), which depicts the distribution of order and chaos in particular planar sections of the phase space. The results on the Kuiper Belt of the Solar System will provide indications about the possible existence of a similar structure associated to exoplanetary systems. However, we need to keep in mind that the astronomical observations concerning exoplanetary systems often provide data about a very limited number of planets around the same star: typically just one planet, sometimes two bodies, very few cases with three or four planets. It is therefore essential to analyse how strong is the isolated influence of Neptune on the dynamical shape of the Kuiper Belt. To this end, we start by exploring the basic model, that is, the planar, circular, restricted three-body problem (hereafter

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RTBP). Afterwards we continue by evaluating the effect of the eccentricity of the primary as well as of the relative inclination by considering the planar-elliptic, the spatial-circular and the spatial-elliptic models. Finally, removing the assumption that the orbit of Neptune is Keplerian, we introduce two additional models where the trajectory of Neptune is perturbed by Uranus. In the first model we adopt the so-called Lagrange–Laplace solution (Morbidelli 2002), while in the second case we make use of a resonant normal form to evaluate the contribution of the 1:2 Uranus–Neptune mean motion resonance.

We are aware of the fact that these models are very limited, since they do not include the gravitational influence of all giant planets of the Solar System. Nevertheless a comparison with the existing literature, where more complete models are adopted, shows a good agreement with our spatial, inclined RTBP as well as with the non-Keplerian models. Indeed, one of the first numerical investigations of the stability of the Kuiper Belt has been performed by Torbett (1989), who integrated test particles under the influence of the four giant planets, though it was assumed that they were moving on Keplerian orbits. Later, we mention the numerical investigations including the effective dynamics of the giant planets as performed by Duncan et al. (1995), Kuchner, Brown & Homan (2002) and Jones, Williams & Melita (2005), where a high number of samples have been integrated over billion years. Our results about the shape of the stability region agree with those found in the above-mentioned papers, except for a very limited region which deserves more careful investigations, that is, the gap corresponding to [40, 43] au (extensively studied in Duncan et al. 1995; Jones et al. 2005).

Remarkable investigations of stable regions of habitable zones in exoplanetary systems and of test particles in specific extrasolar multiple planetary systems have been performed in Rivera & Haghighipour (2007) and Sándor et al. (2007).

This paper is organized as follows. In Sections 2 and 3 we briefly describe, respectively, the dynamical models and the numerical tools. In Section 4 we present families of periodic orbits and we show their correspondence with the phase-space topology. The results in the framework of the RTBPs (circular, elliptic, spatial) are presented in Section 5, while the non-Keplerian models are discussed in Section 6. A preliminary investigation of Kuiper Belt-like regions in exoplanetary systems is performed in Section 7, while conclusions are presented in Section 8.

2 MODELS

We briefly introduce the models that we are going to use throughout this work. We start by considering some models in the framework of the RTBP; first we assume the basic planar, circular three-body problem; then we proceed to consider the elliptic model and finally the spatial problem. The procedure of increasing the complexity of the models allows to evaluate the effect of the eccentricity and inclination on the overall dynamics. Finally, we extend the model to include the influence of Uranus on the orbit of Neptune.

2.1 Restricted three-body problems

For completeness we review the equations of motion of the most general form of the RTBP, referring for complete details to classical textbooks, like, for example, Szebehely (1967).

We consider the motion of a body $p$ of infinitesimal mass moving in the gravitational field of two primaries, the Sun (S) and Neptune (N). In normalized units $m_S = 1 - \mu$ and $m_N = \mu = 5.178 \times 10^{-5}$ are the masses of the primaries ($0 < \mu < 1$), which are supposed to move in a Keplerian orbit around their common barycentre $O$. We normalize the units of length and time so that the semimajor axis of the relative orbit of N around S and the gravitational constant are set to unity. We consider a barycentric reference frame $Oxyz$, which rotates with variable angular velocity, so that the angle of rotation is equal, at any instant, to the true anomaly. Let $(\rho, \theta)$ be the polar coordinates of N in the inertial barycentric reference frame $O\xi\eta\zeta$ (i.e. $\xi_N = (1 - \mu)\rho \cos \theta, \eta_N = (1 - \mu)\rho \sin \theta, \zeta_N = 0$). The equations of motion are of the form

\[
\dot{x} = \dot{\theta} y + \dot{\theta} y + (y + \dot{\theta} x) \dot{\theta} - \frac{1 - \mu}{r_1^3} (x + \mu \rho - \frac{\mu}{r_2^3} [x - (1 - \mu)\rho])
\]

\[
\dot{y} = -\dot{\theta} x - \dot{\theta} x - (x - \dot{\theta} y) \dot{\theta} - \frac{1 - \mu}{r_1^3} y - \frac{\mu}{r_2^3} y
\]

\[
\dot{z} = -\frac{1 - \mu}{r_1^3} z - \frac{\mu}{r_2^3} z,
\]

where

\[
r_1 = \sqrt{(x + \mu \rho)^2 + y^2 + z^2},
\]

\[
r_2 = \sqrt{[x - (1 - \mu)\rho]^2 + y^2 + z^2}.
\]

The circular case corresponds to $\rho = 1, \dot{\theta} = 1(\dot{\theta} = 0)$ and there exists the Jacobi integral:

\[
h = \frac{1}{2} [x^2 + y^2 + z^2 - (x^2 + y^2)] - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}.
\]

For the elliptic case notice that $\dot{\theta} = \sqrt{1 - e_N^2} / \rho^2$, where we assume $e_N = 0.009$. The planar case is simply obtained by setting $z = 0$ and $\dot{z} = 0$.

2.2 The secular planetary solution

The motion of a Kuiper Belt object under the influence of Neptune and of the Sun is now investigated by removing the assumption that the orbit of Neptune is Keplerian. Indeed we suppose that Neptune’s trajectory is described by the so-called Lagrange–Laplace solution, which is based on a secular normal form dynamics (see Morbidelli 2002). The time dependence of the eccentricity, inclination, longitude of perihelion and longitude of the node is expressed in terms of circular functions with frequencies, phases and coefficients provided in Morbidelli (2002) (see tables 7.1, 7.2, 7.3 therein). We stress that in the present study we consider only the effect of Uranus on Neptune’s orbit.

2.3 The Uranus–Neptune resonant motion

We now give special relevance to the 1:2 resonance between Uranus and Neptune using a proper resonant normal form. More precisely, the trajectory of Neptune is obtained through the equations of motion of the planar RTBP Sun–Uranus–Neptune (assuming first that Uranus moves on a circular orbit and then on an elliptic one with eccentricity 0.0472). The corresponding perturbing function is expanded in Fourier series and only specific terms are retained, that is, those associated to the 1:2 orbital resonance between Uranus and Neptune. We are aware that the lack of this model relies on the fact that we do not consider the gravitational influence of Neptune on Uranus using a more appropriate planetary three-body model; however, we intend this model as a first approach to highlight the role of the 1:2 Uranus–Neptune mean motion resonance.

In practical computations we determine the Fourier expansion of the perturbing function up to the order 5 in the eccentricities and
up to the 12th order in the semimajor axes ratio and we retain only the combinations of the mean anomaly which correspond to the 1:2 resonance. Beside these terms we include in the normal form the secular part as well as the terms depending just on the argument of perihelion of Neptune. The integration of the resulting equations provides the time variation of the orbital elements of Neptune, which is then inserted into the equations of motion of the Kuiper Belt object in a heliocentric reference frame.

3 NUMERICAL METHODS

The stability analysis is performed through classical methods borrowed from dynamical system theory: the computation of periodic orbits, Poincaré maps and Lyapunov exponents.

Periodic orbits are exceptional solutions of the RTBP. Their positions and linear stability are strongly related with the topology of the phase space and of the underlying dynamics. Stable periodic orbits are surrounded by invariant tori which are associated with regular orbits and long-term stability. Instead, unstable periodic orbits are associated with the existence of weak or broad homoclinic chaos. The computation of periodic orbits is performed by applying a Newton–Raphson shooting algorithm (Press et al. 1992).

The Poincaré map (or surface of section) is a very efficient tool for understanding the topology and the underlying dynamics of systems with two degrees of freedom. In particular we consider the planar circular RTBP in a rotating frame and we define the Poincaré map as the set of points \( P = \{ (x, \dot{x}) | y = 0, \dot{y} < 0, h = \text{constant} \} \). Invariant tori are represented on the section by points which form an invariant curve, while scattered points denote a chaotic regime. However, for the extended models (elliptic or/and spatial RTBP) the Poincaré maps are described by higher dimensions: they are presented through projections and often they do not depict clearly the topology of the phase space. In these cases the computation of Lyapunov indicators turns out to be more efficient. We limit to the computation of the maximum LCN (denoted by \( L_{\text{max}} \)), which provides an efficient measure of the sensitive dependence to initial conditions: the highest is its value, the more chaotic is the trajectory. We perform two types of numerical simulations.

(i) We define a grid of 500 initial conditions by varying the semimajor axis and keeping constant the remaining initial values. For each initial condition on the grid we compute \( L_{\text{max}} \) for an integration time equal to 300,000 revolutions of Neptune in the circular model and 30,000 revolutions in all other cases. Then we plot the size of the maximum LCN versus the semimajor axis.

(ii) We create 2D grids (of size 200 × 50) of initial conditions in the plane given by the semimajor axis versus the eccentricity. Then we create a plot (called map of dynamical stability) which provides the value of \( \log(L_{\text{max}}) \) according to a colour scale: from black (indicating low LCNs or stable motion) to white (indicating high LCNs or chaotic motions). The value of \( L_{\text{max}} \) was computed over 50,000 revolutions of Neptune.

The numerical integration was performed using a Burlisch–Stoer algorithm with variable step-size (for the computation of Poincaré sections and of the maps of dynamical stability), or using the fourth-order symplectic integrator developed by Yoshida (1990) (to compute the maximum LCN and to check the validity of the integrations when using the non-symplectic methods), or implementing a fourth-order Runge–Kutta method (when the 1:2 Uranus–Neptune resonant motion is taken into account).

4 PERIODIC ORBITS AND PHASE-SPACE TOPOLOGY

The study of periodic orbits starts from the unperturbed RTBP, that is, when the mass of the planet is set to zero (\( \mu = 0 \)). The analysis of such model pertains to the two-body problem and one can easily obtain circular periodic orbits in the rotating frame. These trajectories form a family of periodic orbits (called the ‘circular family’) by changing smoothly their radius or, equivalently, the semimajor axis. As a consequence, the ratio \( n/n' \) of the mean motion of the small body and the planet varies along the family. The orbits with rational ratio \( n/n' = p/q \), where \( p, q \) are prime integers, are the resonant periodic orbits.

4.1 The planar circular RTBP

For \( \mu \neq 0 \) the circular family continues to exist and it consists of almost circular orbits. An exception occurs at the resonances of the form \( p/(p+1) \), where the family breaks and gaps are formed; then we obtain a family of periodic orbits along which the ratio \( n/n' \) varies. The resonant circular orbits are bifurcation points for two families of periodic orbits, denoted as \( I \) and \( II \). In the case of \( p/(p+1) \)-resonances the families \( I \) and \( II \) continue smoothly from the circular family (for details see Bruno 1994; Hénon 1997; Hadjidemetriou 2006). Along these families the eccentricity increases while the ratio \( n/n' \) is almost constant; for each resonance \( p/q \) there exist two families of resonant periodic orbits: the family \( I_{p/q} \), where the small body starts from perihelion (\( M = 0^\circ \)), and the family \( II_{p/q} \) where the small body starts from aphelion (\( M = 180^\circ \)).

As a particular example, the 2/5 resonance is presented in Fig. 1 (a large number of resonances in the Kuiper Belt is studied in Voyatzis & Kotoulas 2005), Fig. 1(a) shows the characteristic curve of the 2/5 periodic orbit in the plane \((x_0, h)\), where \( x_0 = x(0) \) and \( h \) is the Jacobi integral; the two branches correspond to families \( I \) and \( II \), which bifurcate from the circular family \( C \). Unstable periodic orbits characterize the beginning of family \( I \); at \( h \approx -1.49 \) (i.e. eccentricity \( e \approx 0.5 \)) there appears a collision orbit, after which the family continues with stable periodic orbits at high eccentricities, which may cross Neptune’s orbit; nevertheless, collisions are avoided because of the phase protection mechanism due to the resonance. The family \( II \) consists only of stable periodic orbits and collisions are avoided (at least for low and moderate eccentricity values). In Fig. 1(b) the same families are presented in the plane \((a, e)\) of semimajor axis and eccentricity, where we can observe that they extend close to the line \( a = 1.84 \) where the 2/5 resonance of the unperturbed system is located.

Four Poincaré sections at different \( h \)-levels are presented in Fig. 2. For \( h = -1.60 \) (Fig. 2a) the families \( I \) and \( II \) of the 2/5 resonant periodic orbit correspond to hyperbolic and elliptic fixed points, respectively, of period 3. The elliptic fixed points are surrounded by islands, which correspond to librations of the resonant angle \( \sigma = -2\lambda' + S_3 - 3\omega_0 \), where \( \lambda', \lambda \) are the mean longitudes of Neptune and of the small body, respectively. Chaos occupies tiny regions around the hyperbolic fixed points. New resonances appear as \( h \) increases (see e.g. Fig. 2b).

Break-up of tori and large-scale chaos is generated for larger values of \( h \) (see Figs 2c and d). We remark that for high energy levels the orbits of family \( I \) become stable and form stability islands; for example, in Fig. 2(d) the 2/5 resonance gives rise to two chains of islands denoted as 2/5' and 2/5''.
4.2 The planar elliptic RTBP

When the circular RTBP is perturbed by introducing the eccentric motion of the planet ($e_N \neq 0$), the system becomes non-autonomous. In normalized units, the dynamical system has period $2\pi$; therefore the associated periodic orbits should retain the same period or a multiple of it, namely, $T = 2k\pi, k = 1, 2, \ldots$ (Hadjidemetriou 2006). In conclusion, the periodic orbits of families I and II with period $T = 2\pi$ are bifurcation points for families of resonant periodic orbits in the elliptic model. Such families are obtained by parametric continuation, varying the eccentricity $e_N$ of the planet. The families of periodic orbits generated by the $2/5$ resonance are presented in Fig. 3. There are two bifurcation points belonging to the family II and to the family C of circular orbits (for a minor body eccentricity $e \approx 0$), each one giving rise to two families of periodic orbits, where the planet is initially at perihelion (family $E_p$) or at aphelion (family $E_a$) (see also Voyatzis & Kotoulas 2005). For a particular value of $e_N$ (say $e_N = 0.009$), the associated periodic orbits are isolated and they correspond to specific eccentricity values of the small body. Since $e_N$ is typically much less than 1, the periodic orbits of the elliptic model are obtained for initial conditions close to those of the bifurcation points. The fact that the periodic orbits are isolated shows that the introduction of the planet’s eccentricity affects critically the dynamics only in the neighbourhood of the bifurcation points.

4.3 The spatial circular RTBP

Non-planar periodic orbits and their families arise in the spatial circular problem as parametric continuation (e.g. by varying the inclination) from planar periodic orbits, which are critical with respect to vertical perturbations (Hénon 1973). From these bifurcation points we obtain families of resonant periodic orbits, which are either symmetric to the $xz$ plane (Type $F$) or symmetric to the $x$-axis.
The spatial circular model at the 2/5 resonance, where $e_N$ is the eccentricity of the perturbing body, while $e$ is the eccentricity of the small body. Solid and dotted lines denote stability and instability, respectively.

Figure 3. Families of periodic orbits of the elliptic problem corresponding to the 2/5 resonance; $e_N$ is the eccentricity of the perturbing body, while $e$ is the eccentricity of the small body. Solid and dotted lines denote stability and instability, respectively.

(Type G). A study of 3D periodic orbits in the Kuiper Belt dynamics (Kotoulas & Voyatzis 2005) shows the existence of stable periodic orbits up to relatively large inclination values. In the neighbourhood of these orbits the motion is regular and the argument of perihelion librates. For the 2/5 resonance the families of spatial periodic orbits are shown in Fig. 4; they are presented in the plane of initial eccentricity and inclination of the small body. The families bifurcate from four critical orbits of the family $I$ and from the family $C$ of circular planar orbits. We can distinguish four families of periodic orbits. The characteristic curves $G_1$, $G_2$ start and end at a bifurcation point of the planar problem. The family $F_1$ ends at a bifurcation point corresponding to retrograde motion ($i = 180^\circ$) of the planar system. The family $F_2$ consists of stable periodic orbits up to $e \approx 0.85$. For larger eccentricity values the orbits become unstable and the family terminates at a collision orbit with the Sun ($e = 1$).

4.4 The spatial elliptic model

As in the planar elliptic case, the periodic orbits have period $T = 2k\pi$, $k = 1, 2, \ldots$. The corresponding families arise from parametric continuation (varying $e_N$) and bifurcate from the periodic orbits of the spatial circular problem with period $T = 2k\pi$. For the Kuiper Belt dynamics, where $e_N = 0.009$, such orbits are isolated and the computations show that they exist for inclinations $80^\circ < i < 104^\circ$ (Kotoulas & Voyatzis 2005). In the particular case of the 2/5 mean motion resonance, there is one bifurcation point in the family $F_1$ of the spatial circular model at $e = 0.123$ and $i = 93.7^\circ$. Therefore, we may claim that the periodic orbits of the spatial elliptic model do not affect in practice the dynamics of the objects in the Kuiper Belt.

5 DYNAMICAL STABILITY IN THE FRAMEWORK OF THE RTBP MODELS

The maximum LCN ($L_{\text{max}}$) and the maps of dynamical stability are computed for a large number of initial conditions in the framework of the RTBP models (see Section 2).

5.1 The planar circular model

We consider semimajor axes in the interval $[1.1, 2.1]$ in units of Neptune’s semimajor axis. We set the mean anomaly $M = 0^\circ$ and the longitude of perihelion $\omega = 0^\circ$ of the small body. For a fixed value of the orbital eccentricity, we vary the semimajor axis over a grid of 500 points equally spaced in the interval $[1.1, 2.1]$; the $L_{\text{max}}$ is computed over 300 000 Neptune’s revolutions. Some objects experience escape trajectories, which typically occur after close encounters; in particular, our integration fails due to a close encounter or due to an escape solution whenever the osculating eccentricity exceeds 0.97.

When an escape or a collision orbit occurs within the above integration interval, the numerical integration fails and we set $L_{\text{max}} = -0.1$. An example of the convergence is displayed in Fig. 5(a), while Figs 5(b)–(d) show the value of $L_{\text{max}}$ for eccentricities $e = 0.1$, $e = 0.2$, $e = 0.3$. Typically we have the following behaviour: for small values of the semimajor axis we observe many escape orbits and very few regular orbits (corresponding to $L_{\text{max}} \approx 0$); the interval where $L_{\text{max}} > 0$ denotes a region of chaotic motion, which extends up to a given value, while for larger values the dynamics is stable. Notice that the extension of the chaotic region grows as the eccentricity increases.

In order to get a global description of the stability character as the orbital elements vary we compute the maps of dynamical stability presented in Fig. 6. We integrated the equations of motion of the planar, circular RTBP over a grid of $200 \times 50$ orbits in the plane ($a, e$). The $L_{\text{max}}$ was computed over 50 000 revolutions of the period of Neptune’s orbit, which are typically sufficient to get a good approximation of the maximum LCN. The initial conditions are fixed to $\omega = 0^\circ$ and $M = 0^\circ$ in Fig. 6(a), while $\omega = 0^\circ$ and $M = 180^\circ$ in Fig. 6(b).

The relation $q = a(1 - e) = 1$ denotes that the orbits of the planet and the small body intersect; as a consequence we refer to $q = 1$ as the collision line. Below the collision line there is a wide region of ordered motions, which extend to higher eccentricity values as $a$ increases. In Fig. 6(a) the stable regions above the collision line are confined to narrow resonant zones, which correspond to motions near the resonant families $I$ of stable periodic orbits. In Fig. 6(b) ($M = 180^\circ$) the tongues of stability are not broken by the collision line and they extend along the family $I$ of resonant periodic orbits, where collisions are avoided due to the phase-protection mechanism given by the mean motion resonances, which avoid close encounters even when the orbits do intersect (Morbidelli 2002). An exception is provided by the 1:2 and 1:3 resonances, where stability is associated with asymmetric librations (Voyatzis, Kotoulas & Hadjidemetriou 2005).

5.2 The planar elliptic problem

The study of the planar, elliptic RTBP (with $e_N = 0.009$) shows an overall behaviour which we shortly describe as follows. For small
values of the semimajor axis the close proximity to Neptune is definitely responsible for the existence of escape trajectories. A region of chaotic motion bridges to a zone of dynamical stability as one gets far from Neptune; the extent of such region shrinks as the eccentricity is increased. Due to the low value of Neptune’s eccentricity, the maps of dynamical stability of the planar elliptic problem do not show relevant differences with respect to the circular case.

5.3 The spatial circular case

The map of dynamical stability associated to the spatial circular case (Fig. 7) shows the existence of a wide stable region below the collision line $q = 1$. Above such curve, only some thin zones exist for high eccentricity values, which typically correspond to first-order mean motion resonances. We noticed that in the circular case the dynamics is not much affected by the inclination. The most remarkable difference is observed for the 1:2 resonant zone above
the collision line, which almost disappears after the introduction of the inclination.

5.4 The spatial elliptic problem

In the framework of the inclined, elliptic problem we set $e_N = 0.009$ and we consider the following values for the initial inclination: $5^\circ$, $10^\circ$, $20^\circ$, and $30^\circ$. The computation of $L_{\text{max}}$ in the interval $a \in [1.1, 2.1]$ and for eccentricity $e = 0.4$ and initial conditions $\tilde{\omega} = M = 0$ is presented in Fig. 8. The top panel corresponding to $i = 5^\circ$ indicates a dynamical behaviour similar to that described for the circular problem. However, an important change in the dynamics is obtained when the inclination increases, as we observe the formation of a broader region of stable motions for small values of the semimajor axis.

This situation is confirmed also by the maps of dynamical stability presented in Fig. 9 for the inclinations $5^\circ$, $10^\circ$, $20^\circ$, and $30^\circ$. The most

Figure 8. Maximum LCN versus the semimajor axis for the spatial, elliptic RTBP. The orbital elements are: $e = 0.4$, $\omega = 0^\circ$, $M = 0^\circ$ for $i = 5^\circ$, $10^\circ$, $20^\circ$, $30^\circ$ from top to bottom.

Figure 9. Maps of dynamical stability for the 3D elliptic case: (a) $i = 5^\circ$, (b) $i = 10^\circ$, (c) $i = 20^\circ$ and (d) $i = 30^\circ$. In all cases we have $\tilde{\omega} = M = 0^\circ$. The dotted curve corresponds to the collision line $q = 1$. The colour scale is the same as in Fig. 6.
significant stability region is confined below the collision line. For \( i = 5^\circ \) narrow zones of stability are observed at high eccentricities, which typically correspond to mean motion resonances. These zones widen as the inclination increases, thus showing that for moderate inclinations (say 20°–30°) a new upper broad region of stability appears.

6 EFFECT OF THE NON-KEPLERIAN NEPTUNE’S MOTION

6.1 Neptune’s Lagrange–Laplace solution

Following Section 2.2 we assume that the trajectory of Neptune is described by the Lagrange–Laplace solution, according to which

\[
\begin{align*}
\omega & = a, \\
\nu & = e, \\
\tau & = M.
\end{align*}
\]

Similarly to the previous sections we construct a map of dynamical stability for semimajor axes in the interval \([1.1, 2.1]\) and eccentricity within \([0, 0.5]\) (Fig. 10). The initial values of the phases are \( \nu = 0^\circ, M = 0^\circ \). The inspection of Fig. 10 (upper panel) shows the existence of a large stability region similar to that obtained for the RTBPs. However, the resonant stability zones above the collision line are destroyed by the secular perturbations. In contrast to the restricted three-body models, we observed that the results do not change significantly when the initial mean anomaly is set to \( M = 180^\circ \). This fact can be attributed to the variation of Neptune’s orbital elements, which cause a mixing of the phases for sufficiently large eccentricities; in other words, no matter the initial position of the small body (at pericentre or at apocentre) the mean motion resonances cannot prevent collisions with Neptune and therefore they do not guarantee long-term stability.

The stable region is broken by a narrow unstable zone [extending from \((a, e) \approx (2.1, 0.5)\) to \((a, e) \approx (1.3, 0.05)\)], whose origin can be blamed to the presence of the \( v_{18} \) resonance (see also Morbidelli, Thomas & Moons 1995) which excites the inclination (see Fig. 10, middle panel). The \( v_{18} \) resonance excites also the eccentricity, as it is shown in Fig. 10 (bottom panel) where the eccentricity increases gradually and, after 8 Myr, oscillates around the value \( e = 0.25 \). The middle and bottom panels show also the evolution of the inclination and eccentricity, respectively, of two additional orbits (indicated as orbits 1 and 2), which start outside the domain of \( v_{18} \). For these orbits no excitation of the eccentricity or of the inclination is observed.

In Fig. 11 we present two more maps for the cases \( i = 5^\circ \) and \( 30^\circ \), which show the appearance of a broad stability region at large inclinations. For \( i = 30^\circ \) the unstable motion is confined to a narrow region and it is essentially due to close encounters.

**Figure 10.** Upper panel: a map of dynamical stability computed according to the Lagrange–Laplace solution. The initial conditions for the small body are \( \omega = 0^\circ, M = 0^\circ \) and \( i = 0^\circ \). The dotted curve corresponds to the collision line \( q = 1 \). The colour scale is the same as in Fig. 6. Middle panel: graph of the inclination of the small body versus time along three different orbits. ‘Orbit 1’ starts with semimajor axis \( a = 1.3 \) and eccentricity \( e = 0.06 \). These initial conditions belong to the unstable zone appearing in the top panel and they are associated with the \( v_{18} \) secular resonance. ‘Orbits 2’ and ‘3’ start outside the unstable zone and have initial conditions \( a = 1.2, e = 0.05 \) and \( a = 1.4, e = 0.05 \), respectively. Bottom panel: graph of the eccentricity versus time for the three orbits mentioned above.

**Figure 11.** Map of dynamical stability using the Lagrange–Laplace solution for the motion of Neptune. The initial conditions are (a): \( i = 5^\circ, \dot{\omega} = 0^\circ, M = 0^\circ \); (b): \( i = 30^\circ, \dot{\omega} = 0^\circ, M = 0^\circ \). The colour scale is the same as in Fig. 6.
6.2 The 1:2 Uranus–Neptune resonance and an application to the Kuiper Belt population

We now evaluate \( L_{\text{max}} \) in the framework of the Uranus–Neptune resonant model described in Section 2.3. A comparison with the circular case (see Fig. 5) shows that the effect of the Uranus–Neptune resonance contributes to regularizing the motion for any value of the eccentricity (see Fig. 12, top panel). Such results do not change in a meaningful way when considering the elliptic problem instead of the circular one.

Let us now briefly discuss an application to the Kuiper Belt. As we mentioned in Section 1 two distinct populations seem to crowd the Kuiper Belt. They are called the cold population, including all objects whose inclination is less than 4\(^\circ\), and the hot population for inclinations greater than 4\(^\circ\); the current data available on the Kuiper objects show that the inclinations of the hot population do not exceed 30\(^\circ\).

We present in Fig. 13(a) the value of the maximum LCN versus the semimajor axis for the true (cold and hot) Kuiper Belt population. The results show that the plutinos (\( a \approx 1.3 \)) are characterized by a relatively high value of the LCN, thus indicating their possible escape or weak diffusion from the present dynamical state, unless some resonant mechanism acts to stabilize their motion. The behaviour reported in Fig. 13(b) confirms that for small eccentricities the maximum LCN keeps a small value.

In order to evaluate the effect of the 1:2 resonance between Uranus and Neptune, we assume the model introduced in Section 2.3 and we compute the stability region by varying the semimajor axis and the eccentricity. The results are presented through the histogram of Fig. 14, which is obtained as follows. In the interval [0, 0.5] we consider the eccentricities \( e_k = 10^{-2}k \) for \( k = 1, \ldots, 50 \); for each eccentricity value \( e_k \) we compute the maximum LCN and we mark the value of the semimajor axis (in the range [1.1, 2.1]) at which the stability region starts (i.e. at which the LCN is definitely small – see e.g. the behaviour of the LCN presented in Fig. 5). Therefore the lower part of the histogram provides the stable region, while the upper part corresponds to chaotic dynamics. We superimpose on the histogram the cold and hot Kuiper populations, which are almost entirely contained in the stable region. Indeed, with the exception of some plutinos, only four hot objects fall outside the stable region.

7 EXOPLANETARY SIMULATION: A PRELIMINARY ANALYSIS

We now proceed to perform a preliminary analysis concerning the role of the perturbing body with possible applications to exoplanetary systems. To this end we consider fictitious values for the mass and eccentricity of the perturber in the framework of the planar, elliptic RTBP and we compute the maps of dynamical stability.
Let $r_H$ be Hill’s radius defined as (see Murray & Dermott 1999)

$$r_H = a' \left( \frac{\epsilon}{3} \right)^{1/3},$$

where $a'$ is the semimajor axis of the planet and $\epsilon$ denotes the planet-to-star mass ratio. According to Menou & Tabachnik (2003) we define the influence region of a planet as the region included between the following inner and outer radii:

$$R_{in} = (1 - \epsilon')a' - 3r_H, \quad R_{out} = (1 + \epsilon')a' + 3r_H,$$

where $\epsilon'$ is the eccentricity of the planet. Since we are interested in the region outside the planet, we concentrate on the value of $R_{out}$. Moreover, we select the following values of the mass of the primaries: the value corresponding to Neptune ($\mu = \mu_N$), the value of Jupiter ($\mu = \mu_J$) and a mass ratio which is greater than that associated to Jupiter by the amount $\mu_J/\mu_N$, that is, $\mu_N = \frac{\mu_J}{\mu_J + \mu_N}$. For these mass ratios we draw in Fig. 15 the value $R_{out}$ as a function of the semimajor axis. The crosses refer to the known exoplanets; the lines denote the values associated, respectively, from bottom to top, to ($\mu$, $e$) = ($\mu_N$, 0), ($\mu_J$, 0), ($\mu_N$, 0), ($\mu_J$, 0.6). The figure shows that most of the exoplanets are actually confined between the lines associated to ($\mu_N$, 0) and to ($\mu_J$, 0.6), thus validating our choice of the above samples.

The maps of dynamical stability are presented in Fig. 16, where the sample planets are considered on a circular orbit or on an elliptic orbit with eccentricity equal to Jupiter’s value (i.e. $\epsilon' = 0.048$); in the last case we consider also $\epsilon = 0.6$. The vertical line corresponds to $R_{out}$ as in (2). The quantity $\mu$ increases from top to bottom; for the circular case corresponding to $\mu_N$ we refer to Fig. 6. When the eccentricity of the primary increases there is a loss of regularity as witnessed by Figs 16(b) and (c), as well as by Figs 16(d) and (e); in particular Fig. 16(c) shows that a large gap appears around the 1:2 resonance. The last panel refers to the case $\mu_J$ with $\epsilon' = 0.6$, showing that unstable trajectories dominate the dynamics.

It is worth noticing that for any fixed value of the eccentricity the size of the stability region shrinks as the mass ratio gets larger (compare Figs 16a, c and e). Furthermore, the thin zones of stability above the collision line disappear. This result suggests that exoplanets with mass ratio bigger than Jupiter–Sun value have a little chance to be surrounded by a Kuiper Belt-like system at least at distances comparable to the actual location of the Kuiper Belt. Moreover, the increase of the eccentricity contributes to strengthen the degree of chaoticity in the outer neighbourhood of the perturber.

**Figure 15.** The external radius $R_{out}$ (in au) of the influence region versus the semimajor axis (in au). The crosses mark the values of the known exoplanets; the straight lines refer to the pairs (from bottom to top) ($\mu$, $e$) = ($\mu_N$, 0), ($\mu_J$, 0), ($\mu_N$, 0), ($\mu_J$, 0.6).

**Figure 16.** Maps of dynamical stability for the planar elliptic RTBP. From top to bottom: (a) $\mu_N$, $\epsilon' = 0.048$. (b) $\mu_J$, $\epsilon' = 0$. (c) $\mu_J$, $\epsilon' = 0.048$. (d) $\mu_N$, $\epsilon' = 0$. (e) $\mu_N$, $\epsilon' = 0.048$. (f) $\mu_N$, $\epsilon' = 0.6$. The initial phases are: $\omega = 0^\circ$, $\lambda = 0^\circ$. The dotted curve corresponds to the collision line $q = 1$ and the vertical lines refer to $R_{out}$ as in (2). The colour scale is the same as in Fig. 6.
8 CONCLUSIONS

The dynamics of small bodies, moving in the Kuiper Belt region between 33 and 63 au has been studied using simple models; in particular, we considered the RTBP and later we included the effect of Uranus using the Lagrange–Laplace solution or the resonant normal form. These models need the knowledge of the data concerning only one or two planets. The main question is whether our models are sufficiently accurate to provide a good description of the stability of a trans-Neptunian belt, with respect to a full exploiting of the influence due to all giant planets. To this end we performed a wide exploration of the long-term stability of a large set of initial conditions by computing the maximum LCN.

In the framework of the planar circular RTBP we obtained a wide stability region below the collision line. In this model the topology of the phase space and the global dynamics are strongly related to periodic orbits. Although there exist unstable periodic orbits in the stability region, the Poincaré maps show that they occupy tiny portions of the phase space. However, for eccentricities above the collision line, the stability occurs only in the resonant zones, which extend around the families of stable periodic orbits. Indeed, it is clearly shown that along the family II of periodic orbits the collisions with Neptune are avoided and the zones of stability are extended along the whole range of eccentricities. It is worth noticing that the stability zone associated to the planar circular RTBP (compare with Fig. 6) includes many real Kuiper objects, thus showing that the simplest three-body model already provides a strong indication of the possible location of a Kuiper Belt-like region.

Next step is to evaluate the role of the eccentricity and of the inclination. As far as the elliptic planar RTBP is considered, we found that the eccentricity of Neptune does not affect significantly the shape of the stability region displayed in the circular case. Nevertheless one has to take into account that Neptune’s eccentricity is rather small. If the circular case is compared to an elliptic model with larger eccentricity, then the stability region considerably changes and it eventually develops instability zones around some resonances (compare, e.g. panels b and c of Fig. 16).

The same behaviour is found when the inclination is considered: small values do not affect the shape of the stability region, while a large inclination contributes to the formation of a remarkable stability zone above the collision line (see Figs 8 and 9).

We underline that the results based on the elliptic inclined RTBP are in good agreement with those found in the literature (Duncan et al. 1995; Kuchner et al. 2002; Gomez 2003; Jones et al. 2005). Nevertheless the RTBP is not able to detect some gaps which are present at low eccentricity and inclination, when the four giant planets are included in the model (in particular between 40 and 43 au). The introduction of the perturbation of Uranus on Neptune, as well as a model based on the 1:2 resonant normal form, do not account for these gaps (see Figs 11, 12 and 14). This is certainly a limitation of our models, showing that the effect of all giant planets is definitely responsible for a finer description of the stability behaviour within the Kuiper Belt. Let us remark that the introduction of Uranus’ perturbation causes a mixing of the phases: initial conditions of the small body at $M = 0$ or 180° lead to qualitatively similar results; therefore, for small eccentricity and inclination the resonant stability zones above the collision line are destroyed, while they reappear as the inclination increases (compare Figs 11a and b). For $i > 30°$ chaos is restricted to a small region of the phase space where the orbits show short-term close encounters with Neptune.

Concerning exoplanetary systems we know that they show a wide range of masses, semimajor axes and orbital eccentricities. Therefore we decided to analyse some samples with different parameter values. As far as the masses are concerned, if $\mu_S$ and $\mu_J$ denote the Neptune–Sun and Jupiter–Sun mass ratios, we varied the mass to include also $\mu_J$ and a planet with mass $(\mu_J/\mu_S)$ $\mu_J$. In terms of the influence region introduced in Menou & Tabachnik (2003) these samples represent good starting points for a preliminary analysis of the exoplanets known to date (see Fig. 15). We emphasize that the stability zones above the collision line are gradually destroyed, while the main stability region below the collision line considerably shrinks (compare with Fig. 16). This is due to the increase of the mass as well as to the higher eccentricity, which contribute to the increase of the outer radius of the region of influence. Such radius moves outward, thus confining the stability region to higher values of the semimajor axis (compare panels a–f of Fig. 16). This result suggests that a Kuiper Belt-like region can survive up to a planetary mass comparable with Jupiter’s value. For bigger masses the stability zone is destroyed and eventually displaced at larger distances from the primary. A more detailed description of exoplanetary Kuiper Belts, using some of the methods presented in the present work, is currently under investigation.

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