Measuring the cosmic shear in Fourier space

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ABSTRACT
We propose to measure the weak cosmic shear using the spatial derivatives of the galaxy surface brightness field. This method is mathematically well defined regardless of the galaxy morphology and the form of the PSF, and involves simple procedures of image processing. Furthermore, with high-resolution galaxy images, this approach allows one to probe the shape distortions of galaxy substructures, which can potentially provide much more independent shear measurements than the ellipticities of the whole galaxy. We demonstrate the efficiency of this method using computer-generated mock galaxy images.

Key words: cosmology: theory.

1 INTRODUCTION
The coherent distortions of background galaxy images by the intervening metric perturbations provide us a direct probe of the large-scale mass distribution (see reviews by Bartelmann & Schneider 2001; Wittman 2002; Refregier 2003). Recently, several groups have claimed positive detections of the weak lensing effect and obtained useful constraints on the cosmological model (Bacon, Refregier & Ellis 2000; Kaiser, Wilson & Luppino 2000; van Waerbeke et al. 2000; Wittman et al. 2000; Maoli et al. 2001; Rhodes, Refregier & Groth 2001; van Waerbeke et al. 2001; Hoekstra, Yee & Gladders 2002; Refregier, Rhodes & Groth 2002; Bacon et al. 2003; Brown et al. 2003; Hamana et al. 2003; Jarvis et al. 2003; Rhodes et al. 2004; Heymans et al. 2005; Massey et al. 2005; van Waerbeke, Mellier & Hoekstra 2005; Dahle 2006; Hoekstra et al. 2006; Jarvis et al. 2006; Semboloni et al. 2006; Hettterscheidt et al. 2007; Schrabback et al. 2007). In future weak-lensing observations (e.g. VST-KIDS, DES,1 VISTA darkCAM, Pan-STARRS,2 LSST,3 DUNE, SNAP,4 JDEM5), if the photometric redshift can be well calibrated, we will be able to study the dark energy properties (its abundance and equation of state) using the redshift dependence of the shear fields (Hu 2002; Abazajian & Dodelson 2003; Jain & Taylor 2003; Bernstein & Jain 2004; Hu & Jain 2004; Song & Knox 2004; Takada & Jain 2004; Takada & White 2004; Ishak 2005; Simpson & Bridle 2005; Zhang, Hui & Stebbins 2005; Hannestad, Tu & Wong 2006; Zhan 2006; Schimd et al. 2007; Taylor et al. 2007). By constraining the growth factor of the mass perturbation and the geometrical distance as functions of redshift separately, weak lensing provides a consistency check of the cosmological model (Kratochvil et al. 2004; Simpson & Bridle 2005; Zhang et al. 2005; Knox, Song & Tyson 2006), and opens a window for testing alternative gravity theories (Acquaviva, Baccigalupi & Perrotta 2004; Song 2005; Ishak, Upadhye & Spergel 2006).

An important and challenging job in weak lensing is to measure the weak cosmic shear (of the order of a few per cent) from the shapes (or ellipticities) of the background galaxy images, which have large intrinsic variations. The existing methods are all based on convoluting the galaxy images with some weighting functions, and are called the INTEGRAL methods hereafter (see Tyson, Wenk & Valdes 1990; Bonnet & Mellier 1995; Kaiser, Squires & Broadhurst 1995; Luppino & Kaiser 1997; Hoekstra et al. 1998; Kaiser 2000; Rhodes, Refregier & Groth 2000; Bernstein & Jarvis 2002; Refregier & Bacon 2003; Massey & Refregier 2005; Kuijken 2006; Nakajima & Bernstein 2007). The INTEGRAL methods typically have disadvantages in three aspects. (1) Since the galaxy images are smeared by the point spread function (PSF) (either instrumental or environmental), the INTEGRAL methods involve at least two folds of convolutions, the math of which is complicated. (2) The details of the methods are often sensitive to the galaxy morphology and the form of the PSF. (3) The shear information from the shape distortions of galaxy substructures is not considered. Strictly speaking, the shapelets method (see e.g. Refregier 2003) may not be called an INTEGRAL method, because the galaxy weighting functions form a complete set of orthonormal shapelets which have very convenient mathematical properties. It also has the potential of measuring the cosmic shears on galaxy substructures. However, since this method requires calibrations of the intrinsic distributions of the shapelet coefficients, it has strong dependence on the galaxy morphology.

In this paper, we propose to use the spatial derivatives of the galaxy surface brightness field to measure the cosmic shear. This method...
was first used by Seljak & Zaldarriaga (1999) on cosmic microwave background lensing. We generalize their analysis by including the PSF and carrying out the measurement in Fourier space. This approach is well defined regardless of the galaxy morphology and the form of the PSF, and involves simple image-processing procedures. Given a high image resolution, the method can potentially probe the cosmic shear from galaxy substructures, greatly suppressing the shape noise.

We begin by introducing the method in Section 2. In Section 3, this approach is shown to work well on different types of computer-generated mock galaxy images with general forms of PSF. A brief summary is given in Section 4.

2 THE METHOD

We derive the relation between the cosmic shear and the spatial derivatives of the galaxy surface brightness field without a PSF in Section 2.1. In the presence of an isotropic Gaussian PSF, the relation is modified and shown in Section 2.2. In Section 2.3, Fourier transformation is introduced not only to simplify the measurement of the spatial derivatives, but also to deal with general forms of PSF.

2.1 Without the PSF

The surface brightness on the image plane \(f_i(\theta^\prime)\) and on the source plane \(f_s(\theta^S)\) are related through a simple relation

\[
f_i(\theta^\prime) = f_s(\theta^S),
\]

where \(A_i = \delta_{ij} + \Phi_{ij}\), and \(\Phi_{ij} = \partial \delta^\prime_i / \partial \theta^S_j\) are the spatial derivatives of the lensing deflection angle, which can be expressed in terms of the convergence \(\kappa = \left(\Phi_{xx} + \Phi_{yy}\right)/2\) and the shear components \(\gamma_1 = \left(\Phi_{xx} - \Phi_{yy}\right)/2\) and \(\gamma_2 = \Phi_{xy}\). Using equation (1), we get

\[
\frac{\partial f_i}{\partial \theta^\prime} = \frac{\partial f_s}{\partial \theta^S} = (\delta_{ij} - \Phi_{ij}) \frac{\partial f_s}{\partial \theta^S},
\]

where we have implicitly assumed that \(\Phi_{ij}\) is small, which is true for weak lensing. Assuming that the original surface brightness field \(f_s\) is isotropic on the source plane, the quadratic combinations of the derivatives of the lensed image provide a direct measure of the shear components (Seljak & Zaldarriaga 1999):

\[
1 \left(\langle \partial_{ij} f_s \rangle^2 - \langle \partial_i f_s \partial_j f_s \rangle \right) = -\gamma_1,
\]

\[
2 \left(\langle \partial_i f_s \partial_j f_s \rangle^2 + \langle \partial_i f_s \partial_j f_s \rangle \right) = -\gamma_2,
\]

where the averages are taken over the whole galaxy.

2.2 With an isotropic Gaussian PSF

The presence of PSF brings both advantages and disadvantages. On the positive side, the PSF smoothes out the galaxy surface brightness field, which is originally not differentiable due to structures on arbitrarily small scales. On the other hand, the convolution of the galaxy image with the PSF leads to a nontrivial modification to equation (3), the form of which is calculated in this section. For simplicity, we assume that the PSF is isotropic and Gaussian. General forms of PSF will be discussed in Section 2.3.

The observed galaxy surface brightness distribution \(f_o\) is related to \(f_i\) via

\[
f_o(\theta) = \int d^2 \theta^\prime W_\beta (\theta - \theta^\prime) f_i(\theta^\prime),
\]

where \(W_\beta\) is the Gaussian PSF with scalelength \(\beta\):

\[
W_\beta(\theta) = \frac{1}{2\pi\beta^2} \exp \left( -\frac{\theta^2}{2\beta^2} \right).
\]

Using equation (1) to replace \(f_i\) with \(f_s\) and \(\theta^\prime\) with \(\theta^S\) in equation (4), we get

\[
f_o(\theta) = |\text{det}(A)| \int d^2 \theta^S W_\beta(\theta - A\theta^S) f_s(\theta^S)
\]

or, equivalently,

\[
f_o(A\theta) = |\text{det}(A)| \int d^2 \theta^S W_\beta[A(\theta - \theta^S)] f_s(\theta^S)
\]

\[
= |\text{det}(A)| \int d^2 \theta^S f_s(\theta^S) W_\beta(\theta - \theta^S)
\]

\[
\times [1 - (\theta - \theta^S) \cdot (A - I) \cdot (\theta - \theta^S)/\beta^2],
\]

where \(I\) is the \(2 \times 2\) unitary matrix. Note that the second part of equation (7) is a result of Taylor expansion of the term \(W_\beta[A(\theta - \theta^S)]\) due to the small amplitudes of the lensing components \(\Phi_{ij}\). For convenience, let us define

\[
F_s(\theta) = \int d^2 \theta^S f_s(\theta^S) W_\beta(\theta - \theta^S)
\]

which is the surface brightness field we would observe in absence of lensing. Equation (7) can then be re-written as

\[
|\text{det}(A)| \int d^2 \theta^S f_s(\theta^S) W_\beta(\theta - \theta^S)
\]

\[
\left(1 - (\Phi_{xx} - \Phi_{yy})\right) F_s(\theta)
\]

\[
- \beta^2 \left( \Phi_{xx} \frac{\partial^2 F_s}{\partial \theta^2_{xx}} + 2\Phi_{xy} \frac{\partial^2 F_s}{\partial \theta^2_{xy}} + \Phi_{yy} \frac{\partial^2 F_s}{\partial \theta^2_{yy}} \right).
\]

(9)

Let \(\theta^S = A\theta\), then

\[
|\text{det}(A)| \frac{\partial f_o(A\theta^S)}{\partial \theta^S} = (A^{-1})_{ij} \frac{\partial f_o(A\theta)}{\partial \theta^S},
\]

(10)

Using equations (9) and (10), it is not hard to express the derivatives of \(f_o\) in terms of the derivatives of \(F_s\):

\[
\frac{\partial f_o}{|\text{det}(A)|} = (1 - 2\Phi_{xx} - \Phi_{yy}) F_s - \Phi_{xy} F_y
\]

\[
- \beta^2 (\Phi_{xx} F_{xxx} + 2\Phi_{xy} F_{xxy} + \Phi_{yy} F_{yyy}),
\]

\[
\frac{\partial f_o}{|\text{det}(A)|} = (1 - 2\Phi_{yy} - \Phi_{xx}) F_y - \Phi_{xy} F_x
\]

\[
- \beta^2 (\Phi_{yy} F_{yyy} + 2\Phi_{xy} F_{yxy} + \Phi_{xx} F_{xxy}),
\]

(11)

where

\[
F_{i+} = \frac{\partial^2 F_s}{\partial \theta^2_{i+}}.
\]

Note that we have implicitly assumed that the spatial fluctuation of the cosmic shear is negligible on galactic scales. Assuming that the distribution of \(F_s\) is isotropic, we obtain the following relation between the shear components and the spatial derivatives of the surface brightness field:

\[
1 \left(\langle \partial_{ij} F_s \rangle^2 - \langle \partial_i F_s \partial_j F_s \rangle \right) = -\gamma_1,
\]

\[
2 \left(\langle \partial_i F_s \partial_j F_s \rangle^2 + \langle \partial_i F_s \partial_j F_s \rangle \right) = -\gamma_2,
\]

(13)
where
\[ \Delta = \frac{\beta^2}{2} \langle \nabla f_0 \cdot \nabla (\nabla^2 f_0) \rangle. \] (14)

The derivation of equations (13) and (14) is shown in Appendix A. Note that in the limit when the galaxy image is very smooth over the scalelength, the correction \( \Delta \) approaches zero, equation (13) then reduces to equation (3).

### 2.3 Fourier transform and general PSF

For the method to become useful, there are at least two remaining issues to be addressed. (1) How to measure the spatial derivatives of the surface brightness field? (2) How to deal with other forms of the PSF? It turns out that Fourier transformation provides a solution to both problems.

Since convolutions in real space correspond to multiplications in Fourier space, one can easily transform the PSF to a desired form (an isotropic Gaussian form in our case) by multiplying the Fourier modes of the observed image with the ratios between the Fourier modes of the desired PSF and those of the original PSF (known from calibrations with stars). This operation is usually well defined if the scalelength of the desired PSF is larger than that of the original PSF. Moreover, it turns out that for the purpose of measuring the cosmic shear, one does not need to transform the new image back to real space, because the derivatives of the surface brightness field can be more easily measured in Fourier space.

As an example, we show how to measure quantities such as \( \langle |\nabla f|^2 \rangle \), where \( f \) is the surface brightness field of interest. First of all, the distribution \( f \) in real space should be sampled with an interval \( \Delta \) which is a few times less than the size of the PSF to avoid translating high-frequency power into the frequency range determined by the sampling resolution through discrete Fourier transform (Press et al. 1992). In other words, the galaxy image should be ‘oversampled’ to avoid aliasing power from small scales in the discrete Fourier transform. For undersampled images, one can smooth the images with an additional large enough PSF, which can be treated as a part of the PSF from the instrumentation, and therefore does not affect our discussion below. Similarly, to avoid such aliasing power at low frequency, the box size for the Fourier transform should be a few times larger than the image size. Given this setup, the Fourier transform of the image is defined as
\[ \tilde{f}(l_i, l_j) = 2\pi N \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(\theta_m, \theta_n) \exp[i(\theta_m l_i + \theta_n l_j)]. \] (15)

where
\[ \theta_{m(n)} = m(n) \times \Delta \theta, \quad m(n) = 0, 1, \ldots, N - 1, \]
\[ l_i(j) = i(j) \times \Delta l, \quad i(j) = -N/2, \ldots, N/2, \]
\[ \Delta l = 2\pi/(N \Delta \theta), \] (16)

where \( N \) is the box size, chosen to be a power of 2 for the fast Fourier transform. It is now straightforward to show that \( \langle |\nabla f|^2 \rangle \) can be expressed as the sum over the Fourier modes weighted by the wave numbers:
\[ \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} |\nabla f(\theta_m, \theta_n)|^2 \]
\[ = \frac{1}{N^2 \Delta \theta^2} \sum_{i=-N/2}^{N/2} \sum_{j=-N/2}^{N/2} |\tilde{f}(l_i, l_j)|^2 (l_i^2 + l_j^2). \] (17)

Equation (17) gives exactly the quantity \( \langle |\nabla f|^2 \rangle \) multiplied by the number of bright pixels covered by the galaxy, because the dark pixels have no contributions. Similarly, we can calculate the other terms in equation (13) in Fourier space. Note that for the purpose of obtaining \( \gamma_1 \) and \( \gamma_2 \), it is not necessary to calculate the number of bright pixels because it appears in both the nominator and the denominator in equation (13).

### 3 THE TEST

This section is organized as follows. In Section 3.1, we test the method using mock regular galaxies smeared by different forms of PSF; in Section 3.2, using mock irregular galaxies generated by 2D random walks, we further demonstrate the usefulness of this approach on galaxies with a different morphology, and explore the possibility of suppressing the shape noise in the shear measurements by including the information from galaxy substructures.

#### 3.1 With mock regular galaxies

Each regular galaxy in our simulation contains a thin circular disc with an exponential profile and a coaxial de Vaucouleurs-type spheroidal component (de Vaucouleurs et al. 1991). When viewed face-on, the surface brightness distribution (before lensing and smearing by the PSF) of the galaxy can be parametrized as
\[ f(r) = \exp \left( -\frac{r}{r_e} \right) + f_{i/d} \exp \left[ -\left( \frac{r}{r_{i/d}} \right)^{1/4} \right], \] (18)

where \( r \) is the distance to the galaxy centre, \( r_e (r_{i/d}) \) is the scalelength of the disc (spheroid), and \( f_{i/d} \) determines the relative importance of the spheroid. The overall luminosity of the galaxy is only important in the presence of noise, which will be discussed in a future paper.

Our simulation box is 128 × 128. We choose \( r_e \) to be 1/32 of the box size of the simulation, \( r_e = r_{i/d}/2 \), and \( f_{i/d} = 1 \). Note that changing these particular numbers does not affect our main conclusions. Once the galaxy’s face-on image is generated, it is projected on to the source plane with a random inclination angle along a random direction perpendicular to the line of sight.\(^7\)

The projected galaxy image is subsequently distorted by a constant cosmic shear and smeared by the PSF in real space. We consider two PSF models given by the following forms rotated by certain angles (shown in Fig. 1):
\[ W^{(1)}(x, y) \propto \exp \left[ -\left( \frac{x - y}{8r^2} \right)^2 \right] \]
\[ W^{(2)}(x, y) \propto \exp \left[ -\left( \frac{x^2 + 0.8y^2}{2r^2} \right) \right], \] (19)

where \( r \) is the scalelength, which is equal to six times the grid size, comparable to the galaxy size. The shear components \( (\gamma_1, \gamma_2) \) are chosen to be \((-0.012, 0.035), (-0.032, -0.005)\) and \((0.01, 0.02)\) for \( W^{(1)} \), and \((0.015, -0.024), (0.05, 0.01)\) and \((-0.04, -0.04)\) for \( W^{(2)} \).

To measure the cosmic shear, we follow the procedures described in Section 2.3. The desired PSF has an isotropic Gaussian form with a scalelength of about 4/3 times that of the original PSF. The results are plotted in Fig. 2. The results are consistent within 1\( \sigma \) error regardless of the form of the PSF.

\(^6\)In the presence of noise, extra procedures may be required to clean the galaxy map before the Fourier transform. We will discuss this in a future paper.

\(^7\)The intrinsic flattening parameter \( q \) of the spheroid part is set to 1 for simplicity.
3.2 With mock irregular galaxies

Our irregular galaxies are generated using 2D random walks. The random walk starts from the centre of the simulation box for 20 000 steps, each of which is equal to the grid size of the simulation box (which is now 1024 × 1024). Once the distance from the centre is more than 1/6 of the box size, the walk starts from the centre again to finish the rest of the steps. The surface brightness of the galaxy is equal to the density of the trajectories. Note that these galaxies naturally have abundant substructures, which are useful not only for further testing the method, but also for illustrating how much lensing information may be contained in the substructures. We caution that our random-walk-type galaxies are not based on any physical models, therefore they do not necessarily mimic observed irregular galaxies. In a future paper, more realistic galaxy models will be adopted to study this topic.

For the purpose of this section, we smooth the galaxies directly with the isotropic Gaussian PSF of different scalelengths, which correspond to different angular resolutions. The scalelength $\beta$ (defined in equation 5) is chosen to be $1/256$, $1/128$, $1/64$ and $1/32$ of the box size (roughly corresponding to $1/85$, $1/43$, $1/21$ and $1/10$ of the galaxy size). Fig. 3 shows typical images of our irregular galaxy under these four different angular resolutions. For convenience, we plot the minimum $\beta$ as unity in the figures of this section.

The shear component $\gamma_2$ is set to zero, and $\gamma_1$ is fixed at 0.03. After averaging over 10 000 irregular galaxies, we find that the measured $\gamma_1$ is $0.0324 \pm 0.0029$ for $\beta = 8$, $0.0301 \pm 0.0022$ for $\beta = 4$, $0.0291 \pm 0.0015$ for $\beta = 2$, and $0.0293 \pm 0.0010$ for $\beta = 1$. More interestingly, as shown in Fig. 4, the statistical error bar is found to decrease significantly when the angular resolution is increased. This is further illustrated in Fig. 5, which shows an approximate power-law relation between the measured variance of $\gamma_1$ and $\beta$, the exponent of which is close to one. Note that as the angular resolution increases, one gets additional information on the cosmic shears from the galaxy substructures. If we naively assume that each bright pixel on the galaxy map provides an independent measurement of the cosmic shear, we expect the variance of the measured cosmic shear to scale as the inverse of the number of the bright pixels, or $\beta^d$, where

\[ P(\gamma_1) \]

\[ \beta = 1 \quad \beta = 2 \]

\[ \beta = 4 \quad \beta = 8 \]

\[ \gamma_1 \]

\[ \gamma_1 \]

\[ P(\gamma_1) \]

\[ \beta = 1 \quad \beta = 2 \]

\[ \beta = 4 \quad \beta = 8 \]

\[ \gamma_1 \]

\[ \gamma_1 \]

\[ P(\gamma_1) \]
$d$ is the Hausdorff dimension of the galaxy image (Hausdorff 1919). Since the Hausdorff dimension of our random-walk-generated irregular galaxies is 2 (Falconer 1986), the variance of $\gamma_1$ should scale as $\beta^2$, which is not too far from what we have observed in our numerical experiments. In reality, substructures generated by the 2D random walks are correlated at some unknown level; therefore, the observed exponent indicated in Fig. 5 is less than the Hausdorff dimension.

4 SUMMARY

We have presented a simple approach of measuring the weak cosmic shear using the spatial derivatives of the galaxy surface brightness field. The measurement should be carried out in Fourier space, in which it is easy to evaluate the spatial derivatives and to transform the PSF to a desired form. The accuracy of the method is demonstrated using computer-generated mock regular and irregular galaxies. We find no systematic errors on the measured shear components in the numerical experiments.

Given high image resolutions, this new method may reduce the shape noise in the shear measurement significantly, because it takes into account the shape information on the galaxy substructures. Using the mock irregular galaxies generated by 2D random walks, we have shown that the variance of the measured shear is indeed suppressed by a large factor when the image resolution is increased. This example encourages us to test this method on real galaxies of a wide range of morphology classes in a future paper by joining the Shear TEsting programme (Heymans et al. 2006; Massey et al. 2007), the results of which may be useful for optimizing the signal-to-noise ratio in shear measurements and planning future weak-lensing surveys.

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APPENDIX A: RELATING THE COSMIC SHEARS WITH THE SPATIAL DERIVATIVES OF THE SURFACE BRIGHTNESS FIELD

From equation (11), we have

\[
\frac{1}{|\text{det}(A)|^2} \left[ (\partial_x f_0)^2 - (\partial_y f_0)^2 \right]
= (1 - 6\kappa)(F_x^2 - F_y^2) - 2\gamma_1(F_x^2 + F_y^2) - \beta^2[(2\kappa \Pi_1 + \gamma_1(\Lambda + \Upsilon_1) + \gamma_2(\Upsilon_2 - \Lambda)],
\tag{A1}
\]

and

\[
\frac{2}{|\text{det}(A)|^2} \partial_x f_0 \partial_y f_0
= 2(1 - 6\kappa) F_x F_y - 2\gamma_2(F_x^2 + F_y^2) - \beta^2[2\kappa \Pi_2 + \gamma_1(\Lambda + \Upsilon_2) + \gamma_2(\Lambda - \Upsilon_1)],
\tag{A2}
\]

where

\[
\Lambda = F_x F_{xxx} + F_y F_{xyy} + F_x F_{xyy} + F_y F_{xxx},
\]
\[
\tilde{\Lambda} = F_y F_{xx} + F_x F_{yy} - F_x F_{yy} - F_y F_{xx},
\]
\[
\Pi_1 = F_x F_{xxx} + F_y F_{xyy} - F_x F_{xyy} - F_y F_{xxx},
\]
\[
\Pi_2 = F_x F_{xyy} + F_y F_{xxx} + F_x F_{xyy} + F_y F_{xxx},
\]
\[
\Upsilon_1 = F_x F_{xxx} - 3F_y F_{xyy} - 3F_x F_{yy} + F_y F_{xxx},
\]
\[
\Upsilon_2 = F_y F_{xxx} - 3F_x F_{xyy} + 3F_x F_{yy} - F_y F_{xxx}.
\tag{A3}
\]

Note that according to the definitions in equation (22), \(\Lambda\) is a scalar, \(\tilde{\Lambda}\) is a pseudo-scalar, \(\Pi_1 + i\Pi_2\) is a spin-2 field, and \(\Upsilon_1 + i\Upsilon_2\) is a spin-4 field. If the intrinsic surface brightness distribution is isotropic, the spatial averages of \(\tilde{\Lambda}\), \(\Pi_1\), \(\Pi_2\), \(\Upsilon_1\), and \(\Upsilon_2\) must vanish. As a result of this, we have

\[
\frac{1}{2} \left( (\partial_x f_0)^2 - (\partial_y f_0)^2 \right)
= -\gamma_1 \left( F_x^2 + F_y^2 \right) + \frac{\beta^2}{2} (\Lambda),
\tag{A4}
\]

\[
(\partial_x f_0 \partial_y f_0)
= -\gamma_2 \left( F_x^2 + F_y^2 \right) + \frac{\beta^2}{2} (\Lambda).
\]

We have neglected the factor \(|\text{det}(A)|\) which is equal to unity to the zeroth order. Using the fact that \(\Lambda = \nabla F_S \cdot (\nabla^2 F_S)\), and \(F_S = f_0\) to the zeroth order, it is now straightforward to prove equation (13).

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