A general scheme for modelling γ-ray burst prompt emission

Pawan Kumar* and Erin McMahon
Astronomy Department, University of Texas, Austin, TX 78712, USA

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ABSTRACT
We describe a general method for modelling γ-ray burst (GRB) prompt emission, and determine the range of magnetic field strength, electron energy, Lorentz factor of the source and the distance of the source from the central explosion that is needed to account for the prompt γ-ray emission of a typical long-duration burst. We find that for the burst to be produced via the synchrotron process unphysical conditions are required – the distance of the source from the centre of the explosion \( R_\gamma \) must be larger than \( \sim 10^{17} \) cm and the source Lorentz factor \( \gtrsim 10^3 \); for such a high Lorentz factor the deceleration radius \( R_d \) is less than \( R_\gamma \) even if the number density of particles in the surrounding medium is as small as \( \sim 0.1 \) cm\(^{-3} \). The result, \( R_\gamma > R_d \), is in contradiction with the early X-ray and optical afterglow data that show that γ-rays precede the afterglow flux that is produced by a decelerating forward shock. This problem for the synchrotron process applies to all long GRBs other than those that have the low-energy spectrum precisely \( \nu^{-1/2} \). In order for the synchrotron process to be a viable mechanism for long bursts, the energy of electrons radiating in the γ-ray band needs to be continuously replenished by some acceleration mechanism during much of the observed spike in GRB light curve – this is not possible if GRB-prompt radiation is produced in shocks (at least the kind that has been usually considered for GRBs) where particles are accelerated at the shock front and not as they travel downstream and emit γ-rays, but might work in some different scenarios such as magnetic outflows.

The synchrotron-self-Compton (SSC) process fares much better. There is a large solution space for a typical GRB-prompt emission to be produced via the SSC process. The prompt optical emission accompanying the burst is found to be very bright \( \lesssim 14 \) mag; for \( z \sim 2 \) in the SSC model, which exceeds the observed flux (or upper limit) for most GRBs. The prompt optical is predicted to be even brighter for the subclass of bursts that have the spectrum \( f_\nu \propto \nu^\alpha \) with \( \alpha \sim 1 \) below the peak of \( f_\nu \). Surprisingly, there are no SSC solutions for bursts that have \( \alpha \sim 1/3 \); these bursts might require continuous or repeated acceleration of electrons or some physics beyond the simplified, although generic, SSC model considered in this work. Continuous acceleration of electrons can also significantly reduce the optical flux that would otherwise accompany γ-rays in the SSC model.

Key words: radiation mechanisms: non-thermal – methods: analytical – cosmology: theory – gamma-rays: bursts.

1 INTRODUCTION

The last ten years have seen a rapid advance in our understanding of γ-ray bursts (GRBs), due mainly to the study of GRB afterglows. We now know that at least some of the long-duration GRBs (that last for more than about 5 s) are produced in the collapse of massive (young) stars (Galama et al. 1998; Della Valle et al. 2003; Hjorth et al. 2003; Kawabata et al. 2003; Stanek et al. 2003; Malesani et al. 2004; Della Valle et al. 2006; Pian et al. 2006) as proposed by Woosley (1993) and Paczynski (1998), and short-duration bursts are associated with old stellar populations and are a likely product of merging neutron star binaries (Paczynski 1991; Narayan, Paczynski & Piran 1992; Berger et al. 2005; Fox et al. 2005; Gehrels et al. 2005; Gorosabel et al. 2006; Nakar 2007); for recent reviews please see Piran (2005), Meszaros (2002), Woosley & Bloom (2006) and Zhang (2007). We also have good estimates of the total energy and beaming for these explosions as well as the

*E-mail: pk@surya.as.utexas.edu

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Figure 1. A schematic representation of our model. Assuming that radiation is synchrotron and IC, the γ-ray source properties can be described by five parameters \((\gamma_i, \Gamma, B, N, \tau)\) that determine the observed flux at one instance in time. We take this time to be the peak of a pulse in a GRB LC. All of the calculations presented in this work apply to one single pulse in a typical GRB-prompt LC, as shown in the top left-hand corner.

The property of the medium within about 1 pc of the explosion (Rhoads 1999; Sari, Piran & Halpern 1999; Frail et al. 2001; Panaitescu & Kumar 2001). However, our understanding of how the prompt γ-ray emission is generated, and the mechanism for energy transport from the central engine (via magnetic field or kinetic energy of proton–neutron and/or electron–positron pairs) remains highly uncertain.

The goal of this paper is to provide a nearly model independent way of modelling the prompt γ-ray emission with synchrotron or synchrotron-self-Compton (SSC) processes. We determine the basic properties of the γ-ray source from the data, and then determine how these can be interpreted in currently popular models such as the internal/external shock model.

In the next section we provide the basic idea and details of the technique we use to model γ-ray emission (the idea in its early form can be found in Kumar et al. 2006), and in Sections 3 and 4 we describe the synchrotron and SSC results, respectively.

2 MODELLING γ-RAY EMISSION: BASIC IDEA AND TECHNICAL FORMALISM

The starting point for our modelling of the prompt γ-ray emission in GRBs is the assumption that the radiation is produced via the synchrotron or SSC processes\(^1\) in a source moving relativistically outward from the inner engine. Fig. 1 provides a cartoon description of our model. For a simple GRB light curve (LC) consisting of a single peak we determine the average source properties corresponding to the time when the observed LC peaks, and for a multipeak GRB LC our calculation applies to individual pulses or spikes in the LC.

The source property can be uniquely described by the following set of five parameters: the magnetic field strength \((B)\) in gauss, the number of radiating particles \((N)\), i.e. electrons and positrons, the optical depth of the source to Thomson scattering \((\tau)\), the Lorentz factor of the source with respect to the rest frame of the GRB host galaxy \((\Gamma)\), and the minimum electron energy \(\gamma_i\) at the location where particles are accelerated (all the variables we use in this paper are defined in Table 1 for easy reference). In addition, the particle distribution above \(\gamma_i\) is taken to be a power-law function: \(dN/d\gamma \propto \gamma^{-p}\). Particles cool as a result of radiative losses and with time, or as they travel away from the acceleration site, and the distribution function becomes steeper than the index \(p\) at some energy where radiative losses become important.

We calculate the modified distribution self-consistently as discussed below. We constrain this 5D parameter space with at least four observed quantities – the \(v_f\), peak frequency \(v_p\), the spectral index below \(v_p\), the flux \(f_\gamma\), at \(v_p\), the decay time of a single pulse in a GRB LC \(t_p\), \(p\), the power-law index, is constrained by the high-energy spectral index, for \(v > v_p\).

\(^1\) Mechanisms such as the inverse-Compton (IC) scattering of ‘photospheric’ emission from a hot fireball (cf. Lazzati et al. 2000; Broderick 2005) are not modelled by the approach we have adopted. And if it were to turn out that the GRB-prompt emission is produced by such a mechanism then the work presented here is of little relevance.

\(^2\) The electron energy is \(\gamma_i m_e c^2\); however, for convenience we suppress the factor \(m_e c^2\).
Modelling γ-ray burst prompt emission

Table 1. Definition of variables.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>γi</td>
<td>Minimum electron LF in source comoving frame</td>
</tr>
<tr>
<td>Γ</td>
<td>bulk LF of the source</td>
</tr>
<tr>
<td>B</td>
<td>Magnetic field strength, in gauss, in source comoving frame</td>
</tr>
<tr>
<td>τ</td>
<td>Optical depth to Thomson scattering</td>
</tr>
<tr>
<td>N</td>
<td>Number of radiating electrons (isotropic equivalent)</td>
</tr>
<tr>
<td>α</td>
<td>The spectral index below the peak of νfc, i.e. fc ∝ ν−α</td>
</tr>
<tr>
<td>νa</td>
<td>LF of electrons emitting synchrotron at νa</td>
</tr>
<tr>
<td>νc</td>
<td>LF of electrons emitting synchrotron at νc</td>
</tr>
<tr>
<td>νsh</td>
<td>LF of the shocked gas with respect to the unshocked gas</td>
</tr>
<tr>
<td>νp</td>
<td>Observed peak frequency of GRB νfc spectrum νp ≡ νfc /105 eV</td>
</tr>
<tr>
<td>νi</td>
<td>Synchrontron injection frequency in observer frame νi ≡ νf /10 eV</td>
</tr>
<tr>
<td>νc</td>
<td>Synchrontron cooling frequency in observer frame νc ≡ νfc /105 eV</td>
</tr>
<tr>
<td>νs</td>
<td>Synchrontron self-absorption frequency in observer frame</td>
</tr>
<tr>
<td>νss</td>
<td>SSC self-absorption frequency, below which the fν spectral index is +1</td>
</tr>
<tr>
<td>Λw</td>
<td>External medium wind parameter, n = (A/m_p) r−2; A w ≡ A /5 × 1011 g cm−1</td>
</tr>
<tr>
<td>dθ sh</td>
<td>Luminosity distance in units of 1028 cm</td>
</tr>
<tr>
<td>E_E</td>
<td>Kinetic energy of electrons and positrons (lab frame; isotropic equivalent)</td>
</tr>
<tr>
<td>E_B</td>
<td>Energy in magnetic field (lab frame; isotropic equivalent)</td>
</tr>
<tr>
<td>E53</td>
<td>Isotropic equivalent of outflow energy in units of 1053 erg</td>
</tr>
<tr>
<td>f_Ec</td>
<td>EνEν – ratio of magnetic to eν energy</td>
</tr>
<tr>
<td>f_E</td>
<td>Synchrontron prompt optical flux (in R band, at 2 eV)</td>
</tr>
<tr>
<td>f_x</td>
<td>Synchrontron prompt X-ray flux, at 1 keV</td>
</tr>
<tr>
<td>f_v</td>
<td>Observed flux (in mJy) at νv</td>
</tr>
<tr>
<td>f_vp</td>
<td>Synchrontron flux at peak – min(νc, νs)</td>
</tr>
<tr>
<td>n_0</td>
<td>Density of circumburst medium</td>
</tr>
<tr>
<td>n_e</td>
<td>Comoving electron density in unshocked shell</td>
</tr>
<tr>
<td>p</td>
<td>Power-law index of electron energy distribution</td>
</tr>
<tr>
<td>R_J</td>
<td>Distance from centre of explosion at which the radiation is produced</td>
</tr>
<tr>
<td>R_d</td>
<td>Deceleration radius</td>
</tr>
<tr>
<td>t_f</td>
<td>Duration of one pulse in GRB LC (observer frame)</td>
</tr>
<tr>
<td>t_a</td>
<td>The time available for electrons to cool before being re-accelerated</td>
</tr>
<tr>
<td>Y</td>
<td>Compton parameter</td>
</tr>
<tr>
<td>z</td>
<td>Redshift</td>
</tr>
<tr>
<td>η_l</td>
<td>γ /γ_c</td>
</tr>
<tr>
<td>η_u</td>
<td>γ /γ_i</td>
</tr>
</tbody>
</table>

A relativistic moving source of finite angular size θj (as seen by an observer at the centre of explosion) can be treated as spherically symmetric as long as Γ−1 < θj. The angular size determined afterglow modelling suggests that θj is larger than about two degrees for all bursts for which we have good data (Frail et al. 2001; Panaitescu & Kumar 2001) and a number of lines of argument and evidence suggests that Γ is greater than about 100 (cf. Piran 1992; Lithwick & Sari 2001). Therefore, we can treat the source for prompt γ-ray emission as spherically symmetric, and the numerical values we quote in this paper are all isotropic equivalent quantities; for instance N is the total number of radiating particles in the source assuming the source to be spherically symmetric.

2.1 Synchrotron and inverse-Compton radiations: basic equations

The synchrotron injection frequency, νi, corresponding to electron minimum energy γi, is

\[ ν_i = \frac{qBγ_i^2Γ}{2πm_ee(1+z)} \]

(e.g. Rybicki & Lightman 1979; Wijers & Galama 1999), where q is electron charge, m_e the electron mass, c the speed of light, and z is the burst redshift. The synchrotron cooling frequency, νc, the characteristic frequency at which electrons cooling on a time-scale t_a (observer frame) radiate, is

\[ ν_c = \frac{18πqγ_i^2c(1+z)}{σ_Γ B^2Γ^2γ_i^2(1+Y)^2}, \]

where σ_Γ is the Thomson scattering cross-section, and Y the Compton parameter.

For most of the calculations in this work we assume that electrons are accelerated only once, and the time-scale for acceleration is taken to be much less than the duration of a pulse in the GRB LC (t_p). One time acceleration is, for instance, believed to apply to shocks where electrons are accelerated at the shock front (by crossing the front back and forth multiple times) and not while they travel downstream; the picture is likely very different in magnetic reconnection/dissipation. To capture some of the effects of multiple-times particle acceleration in time period of a pulse duration in GRB LC we introduce a time-scale, t_a, which is the average time in between two successive episodes of
particle acceleration or the time available for electrons to cool in between acceleration; for one shot acceleration \( t_a = t_r \), and in the opposite limit of continuous acceleration \( t_a = 0 \) when the rate of energy gain is balanced by radiative loss rate. The electron distribution function resulting from the acceleration process is taken to be \( \text{d}n/\text{d}v_e \propto v_e^{-p} \). The distribution function in the source as a whole is different due to the radiative cooling of electrons with time. The electron distribution function averaged over the source is described by two characteristic energies, namely \( \gamma_c \) and \( \gamma_i \), \( \gamma_e m_e c^2 \) is the energy of electrons that cool on time-scale \( t_a \). The electron distribution for \( \gamma_e > \text{max}(\gamma_c, \gamma_i) \) is proportional to \( \gamma_e^{-p-1} \), and the distribution between \( \gamma_c \) and \( \gamma_i \) (for \( \gamma_c < \gamma_i \)) is proportional to \( \gamma_i^{-2} \). Electrons cool via synchrotron and IC losses. The rate of loss of energy is affected by the synchrotron self-absorption frequency \( v_a \) – electrons with characteristic synchrotron frequency below \( v_a \) lose energy only via the IC scattering process. We calculate \( \gamma_c \) and \( \gamma_i \) by solving a coupled set of equations as described in McMahon, Kumar \& Piran (2006).

The synchrotron flux at the peak of the \( f_s \) spectrum, at \( \min(v_i, v_f) \), is given by

\[
f_{s} = \frac{\sqrt{3} q^3 B N \Gamma}{4\pi d_c^2 m_e c^2}.
\]

where \( d_c \) is the luminosity distance to the source. The effect of synchrotron self-absorption is not included in the above expression for \( f_s \), and therefore the observed flux, in general, would be different from \( f_s \). The flux at other frequencies is calculated as described in Sari, Piran \& Narayan (1998).

The IC flux (in observer frame) at frequency \( \nu \), \( f_{ic}(\nu) \), is calculated using the following equation (cf. Rybicki \& Lightman 1979)

\[
f_{\nu}(\nu) = \frac{3}{4} \nu \frac{\sigma_{T} \delta r}{v_i v_f} \int d\nu_i \frac{d\nu}{d\nu_i} f(\nu_i) \int_{\gamma_i}^{\infty} \frac{d\gamma}{\nu_i^2} F\left(\frac{\nu}{4\gamma_i^2 v_i}\right),
\]

where \( f(\nu_i) \) is the synchrotron flux per unit frequency in the observer frame, \( \delta r \) is the radial extent of the source (comoving frame) which is related to the optical depth \( \tau \) (one of the five parameters we use to characterize the source), the function \( F(x) \) is

\[
F(x) = 2 \ln x + x + 1 - 2x^2, \quad \text{for } 0 < x < 1,
\]

and \( \gamma^* \) is the minimum Lorentz factor (LF) for electron distribution. We include the Klein–Nishina correction to the above expression when \( \nu, \gamma_i^2/\Gamma > m_e c^2 \).

The expression for the Compton Y parameter is

\[
Y = \sigma_T \int \text{d}r' \int d\nu_i \gamma_i \frac{d\nu}{d\nu_i} = \frac{4}{3} \pi \left( \frac{p - 1}{p - 2} \right)^{1/2} \left( \frac{v_i}{v_f} \right)^{3-p} \nu_i^{2} \nu_f, \quad (p - 2)<p<3,
\]

where \( p \) is the energy index of the electron energy distribution and \( \nu \) is the characteristic frequency of the electron distribution.

In our 5D parameter space search, we limit the Compton Y parameter for a synchrotron solution to be less than 10 and for an IC solution \( Y \geq 100 \). The rationale for the constraint on the Compton parameter is that we want an efficiency of \( \gtrsim 10 \) per cent in the \( \gamma \)-ray energy band of \( \sim 10-400 \) keV; observations suggest this efficiency for a typical long-duration burst from a comparison of energy in the \( \gamma \)-ray radiation and the kinetic energy of the ejecta determined from the afterglow observations (Panaitescu \& Kumar 2002).

We can calculate the distance of the source from the centre of explosion with two of the five parameters, \( N \) and \( \tau \):

\[
R_y = \left( \frac{N \tau}{4\pi} \right)^{1/2}.
\]

### 2.2 Relation between \( R_y \) and pulse width

We consider a \( \gamma \)-ray source of a finite lifetime, at a distance \( R_y \), from the central explosion, that is responsible for generating one pulse in the observed GRB-prompt LC. Electrons in the source are heated during some time interval, set by the central engine variability/activity time, and subsequently the source undergoes adiabatic expansion. The width of an observed GRB pulse is determined by a number of different factors – the central engine variability time, the adiabatic expansion and cooling time, and the curvature time – which are described below.

1. Central engine variability time: It sets the observed GRB pulse width if the variability time is larger than \( \sim R_y/2c \Gamma^2 \) and if the distance where \( \gamma \)-ray photons are generated does not increase with time; in this case GRB pulse duration is independent of \( R_y \). However, when the source is turned off the \( \gamma \)-ray flux would decline on a time-scale of \( R_y/2c \Gamma^2 \), the adiabatic expansion time-scale; for all the calculations in this work we use the LC decline time although for simplicity we continue to refer to it as pulse width.
(2) Curvature time-scale: It is the time interval between arrival of photons with angular separation of $\Gamma^{-1}$ as seen by an observer at the centre of explosion. It is the minimum time-scale for a $\gamma$-ray pulse width, as long as the outflow from the GRB has an angular size larger than $\Gamma^1$, and is equal to $R_s/c(\Gamma^2)$.

(3) Adiabatic expansion time-scale: This is the time-scale for electrons/protons to cool because of expansion of the source. As the distance of the source from the centre doubles its volume increases by a factor of $\gtrsim 4$, and electron/proton energy drops by a factor of 2. This time-scale in the observer frame is $\sim R_s/c(\Gamma^2)$.

We note that if electrons are heated by coupling with protons, then the time-scale for electrons to cool down can be larger than $R_s/c(\Gamma^2)$; electromagnetic energy/electron energy) provided that protons transfer energy to electrons for this time duration, and the energy transfer rate balances the loss of energy for electrons to adiabatic and radiative coolings. However, the coupling between $p^+$ and $e^+$ is unlikely to increase the pulse width by a large factor ($\gtrsim 5$) unless the energy in electrons is much smaller than in $p^+$, but in that case the efficiency of $\gamma$-ray generation would be small which is not supported by observations.

(4) Radiative cooling time-scale: For $v_e > v_i$, the radiative cooling time-scale is larger than the adiabatic time-scale and in that case GRB pulse width is equal to the adiabatic or curvature time. For $v_e < v_i$, electrons cool on a smaller time-scale, and once electron heating stops, the LC falls off on the curvature time-scale. We note that the observed pulse duration cannot be larger than $R_s/(2c\Gamma^2)$ which corresponds to an elapsed time of $R_s/c$ in the centre of explosion frame, and during this time the source has moved to $R_s$ from the centre.

Thus we see that the observed decay time for a pulse in GRB LC, produced by a relativistic source, is

$$t_r \approx \frac{R_s}{2c\Gamma^2}(1 + z),$$

when the GRB redshift is $z$.

### 2.3 Energy etc.

The total energy in the source consists of the kinetic energy of electrons and positrons ($E_e$) and the magnetic field ($E_B$):

$$E_e = N(p-1)\gamma_e m_e c^2 \Gamma/(p-2), \quad E_B = R_s^4 B^2 / 6.$$  

Note that $(p-1)\gamma_e m_e c^2 / (p-2)$ is the average energy per electron/positron in the source comoving frame at the acceleration site, and in the calculation of $E_B$ we took the comoving radial thickness of the source to be $R_s / \Gamma$ which is roughly what one expects for a causally connected source where the signal speed is close to the speed of light.

We do not make any assumptions regarding the energy in protons since protons do not contribute to the observed $\gamma$-ray radiation. This has the effect that the parameter space we determine is larger than it would be if protons carried a substantial amount of energy since the energy available to $e^+$ would be smaller than the upper limit of $10^{55}$ erg (isotropic equivalent) we impose in our search for solutions in the 5D parameter space.

For a $\gamma$-ray source that arises from shock-heated gas, the minimum electron energy behind the shock front, $\gamma_{i,\text{sh}}$ (one of the five parameters we use), can be related to the LF of the shocked gas with respect to the unshocked gas, $\Gamma_{i,\text{sh}}$. The minimum $\Gamma_{i,\text{sh}}$ needed to produce $\gamma_i$ is

$$\Gamma_{i,\text{sh}} = \left[ \frac{m_e (p-1)}{m_i (p-2)} \right]^{2/3} \gamma_i,$$

where $m_e$ ($m_i$) is proton (electron) mass. The factor of 2 in the above expression is for the case where there is an energy equipartition between electrons and protons and there are no $e^+e^-$ pairs in the plasma; $\Gamma_{i,\text{sh}}$ will be larger if there are pairs or if electrons have less energy than protons.3

### 2.4 The basic technique for finding source properties

We determine the properties of the $\gamma$-ray source for a GRB by finding the region in the 5D parameter space ($\gamma_i$, $\Gamma$, $B$, $N$, $\tau$) that satisfies the following set of observational constraints: (1) the frequency at the peak of the $v_f$ spectrum ($v_i$); (2) the peak flux at $v_i$; (3) the spectral index above $v_f$, $\Gamma_i - \gamma$; (4) the spectral index below $v_i$; (5) the burst duration – for a GRB with a single pulse in the LC – or the duration of an individual pulse ($t_p$) for GRBs with complicated LC; (5) optical and X-ray prompt flux or limit if available. The flux at a given observer time reflects the property of the source averaged over equal-arrival-time volume, therefore, the observed peak flux depends on the evolution of the source and this introduces uncertainty in the flux calculation by a factor of about 2. For this reason we only require the theoretical flux to match the observed value to within a factor of $\sim 2$.

We now use this technique to find the 5D solution space and source property for GRBs produced via synchrotron (Section 3) and SSC (Section 4).

3 If there are ‘cold’ protons and electrons in the shocked gas, i.e. only a fraction of particles in the shocked gas are accelerated, and electrons have more than $m_e c^2 \Gamma_{i,\text{sh}}$ energy, we would in that case overestimate $\Gamma_{i,\text{sh}}$. However, in this work we do not consider that there is a cold component to the $\gamma$-ray source since such a component would not radiate and affect observations and the solutions in the 5D parameter space. Therefore, such a cold component, if present, would have little effect on all of the major results in this work; the only quantity affected by the cold component is the value for $\Gamma_{i,\text{sh}}$ which is a peripheral quantity and not part of the central flow of the logic in this paper.
3 SYNNCHROTRON SOLUTIONS

We consider in this section the parameter space of solutions when the observed γ-rays are produced via the synchrotron process. First, we determine approximate solutions by analytically solving a system of equations for our 5 parameters ($\gamma_s$, $\Gamma$, $B$, $N$, $\tau$) for the generic synchrotron case. The solutions for each parameter are expressed in terms of the Compton $Y$ parameter and three observed quantities: the frequency $v_p$ where $v_f$ peaks, the γ-ray flux at this frequency ($f_f$ in mJy), and the duration of a pulse in GRB LC ($\tau_f$); $Y$ is a convenient and useful parameter because its value is expected to lie in a limited range, e.g. $Y \lesssim 1$ for the synchrotron solutions and $1 \lesssim Y \lesssim 10$ for the SSC process. Having the general synchrotron solutions in hand, we then apply the analytical results to the low-energy spectral index cases of $\alpha = 1/3$, $\alpha = -1/2$ and $\alpha = -(p - 1)/2$, compare the analytical and numerical results, and draw conclusions as to the process by which γ-rays are generated in GRBs; the spectral index $\alpha$ is defined by $f_f \propto v^\alpha$ for $v < v_p$.

The five equations that we solve are those for the observer frame synchrotron injection frequency $v_i$ (1), the cooling frequency $v_c$ (2), the pulse duration ($\tau_f$) (8 and 9), synchrotron flux $f_{gi}$ in mJy at $v_p = \min(v_f, v_c)$ (3) and the Compton $Y$ parameter (6):

$$v_i = 1.1 \times 10^{-13} B Y^2 \Gamma (1 + z)^{-1}.$$  \hspace{1cm} (12)

$$v_{cs} = 6.6 \times 10^4 (1 + z) B^{-1} \Gamma^{-1} t_c^{-2} (1 + Y)^{-2}.$$  \hspace{1cm} (13)

$$\tau \approx 1.5 \times 10^8 N_{ss} (1 + z)^2 \Gamma^{-1} d_c^{-2}.$$  \hspace{1cm} (14)

$$f_y = 110 B \Gamma N_{ss} d_c^{-2} (1 + z) \text{ mJy},$$  \hspace{1cm} (15)

where $v_i \equiv v_i/10^6 \text{ eV}$, $v_{cs} \equiv v_{cs}/10^8 \text{ eV}$, and $N_{ss} \equiv N/10^{55}$.

To solve equations (12)–(15) and (6), we first eliminate $N_{ss}$ from equation (15) using equation (14), then eliminate $\tau$ using equation (6) to find

$$B Y^2 \Gamma^{-1} \approx 1.9 \times 10^6 f_y d_{18}^{-2} (1 + z) \xi^{-2} Y^{-1} \xi^{-4}.$$  \hspace{1cm} (16)

Next, combining (16) and (12) we get

$$\gamma_i Y^{-1} \Gamma^4 \approx 2.1 \times 10^{-7} v_i^{-1} f_y d_{18}^{-2} Y^{-1} \Gamma^{-1}. Y^{-1}.$$  \hspace{1cm} (17)

Multiplying the square root of equations (12) and (13) together, we have

$$B Y^2 \Gamma^{-1} \approx 8.5 \times 10^{-5} t \xi^{-2} v_i^{-1/2} v_{cs}^{-1/2} (1 + Y)^{-1}.$$  \hspace{1cm} (18)

We can eliminate $\gamma_i$ from equations (17) and (18) by dividing equation (17) by (18) to the fourth power:

$$\Gamma^4 B^{-4} \approx 4.0 \times 10^5 v_i^4 v_{cs}^4 v_f d_{18}^{-4} Y^{-4} (1 + Y) \xi^{-4} \xi^{-4} = t^{-4}.$$  \hspace{1cm} (19)

And finally, if we multiply equation (19) by the fourth power of (16) and divide by the square of (17) we find the solution for $\Gamma$ to be

$$\Gamma \approx 10^{3 \xi^4} v_i^{3/16} v_{cs}^{1/8} f_y^{1/8} t_c^{1/4} Y^{-1/16} (1 + Y)^{1/4} \xi^{1/8} d_{18}^{1/4} (1 + z)^{1/4}.$$  \hspace{1cm} (20)

Using $\Gamma$, we can solve for $\gamma_i$ in terms of $\Gamma$, $B$ and $t$:

$$\gamma_i \approx 4.7 \times 10^5 v_i^{7/16} v_{cs}^{1/8} f_y^{-1/8} t_c^{1/8} Y^{1/16} (1 + Y)^{1/4} \xi^{-1/8} d_{18}^{-1/8} (1 + z)^{1/4},$$  \hspace{1cm} (21)

$$B \approx 4.0 v_i^{3/16} v_{cs}^{3/8} f_y^{-1/8} t_c^{1/4} Y^{1/16} (1 + Y)^{-1/4} \xi^{-1/8} d_{18}^{-1/8} (1 + z)^{1/4}.$$  \hspace{1cm} (22)

$$\tau \approx 3.3 \times 10^{-40} v_i^{-7/16} v_{cs}^{1/4} f_y^{-1/4} t_c^{-1/4} Y^{1/8} (1 + Y)^{-1/8} \xi^{-7/8} d_{18}^{-1/4} (1 + z)^{-1/2}.$$  \hspace{1cm} (23)

Equations (20)–(23) provide approximate solutions for ($\gamma_s$, $\Gamma$, $B$, $N$, $\tau$) when the synchrotron process produces the observed γ-ray radiation; more accurate solutions for these parameters are obtained by numerical calculations and the results are shown in Figs 2–8. These general solutions can be used to investigate different cases of low-energy spectral indices ($\alpha$) by adopting appropriate values for $v_i$ and $v_{cs}$. The full dependences on these two frequencies are not completely shown here – each case of $\alpha$ has a different functional dependence on $\xi$, and $\xi$ is a function of $v_i$ and $v_c$.

Note that $f_y$ is not the observed flux at $v_y$, the peak of γ-ray spectrum, but is the flux at $min(v_f, v_c) = v_p$, and the effect of synchrotron-self-absorption, if any, at $v_p$ has been ignored. Since the dependence of the parameters $\Gamma, \gamma_i, etc$ on $f_y$ is very weak (equations 20–23), we do not worry about the difference between $f_y$ and $f_f$ at this point, even though $f_y$ can be much greater than $f_f$ (the flux at $v_f$); $v_f$ is the peak of $v_f$ for $p < 3$, $v_f = \max[v_f, v_i]$ and for $p > 3 v_f = \min[v_f, v_i]$.

Using the parameter solutions, we can derive the distance of the γ-ray source from the centre of explosion ($R_f$), and the energy in the magnetic field and electrons. The radius $R_f = 2 c \Gamma^2 t_f (1 + z)^{-1}$ is found to be

$$R_f \approx 6.0 \times 10^{16} v_i^{1/8} v_{cs}^{1/4} f_y^{3/8} t_c^{-1/2} Y^{-3/8} (1 + Y)^{1/2} \xi^{3/8} d_{18}^{3/4} (1 + z)^{-1/2} \text{ cm}. $$  \hspace{1cm} (24)

4 It has been suggested that another radiation process, called jitter, might be responsible for γ-ray generation for those bursts that have low-energy spectrum $f_c \propto v$ (Medvedev 2000). We show in Appendix B that whenever jitter radiation dominates the observed flux to produce a $f_c \propto v$ spectrum the Compton $Y$ parameter is extremely large – $Y \gtrsim 10^6$ – and most of the energy of the explosion comes out in $\sim 100$ GeV SSC photons.
and should be compared to the deceleration radius \( R_d \) of the GRB outflow in both a homogeneous external medium with particle number density \( n_0 \), and a wind external medium where the particle number density is given by \( (A/m_p)r^{-2} \) (these are two special cases of a power-law density stratification – the density varying as \( r^{-e} \) – corresponding to \( s = 0 \) and \( 2 \))

\[
R_d = \begin{cases} 
1.2 \times 10^{17} E_{33}^{1/3} n_0^{-1/3} \Gamma_2^{-2/3} & \text{cm} \quad s = 0 \\
1.8 \times 10^{15} E_{33} A_s \Gamma_2^{-2} & \text{cm} \quad s = 2 
\end{cases}
\]

(25)

where \( E_{33} \) is the isotropic equivalent energy in GRB ejecta in units of \( 10^{53} \) erg, \( \Gamma_2 = \Gamma/100 \) and \( A_s = A/(5 \times 10^{11} \text{ g cm}^{-1}) \). Substituting in the solution for \( \Gamma \), we find \( R_d \) to be

\[
R_d \approx \begin{cases} 
2.6 \times 10^{16} E_{33}^{1/3} n_0^{-1/3} v_i^{-1/8} f_{\nu_0}/f_{\nu_{\gamma}}^{1/4} t_{s3}/y^{1/4} (1 + Y)^{-1/8} \tilde{Y}^{-1/8} d_{28}^{-1/4} (1 + z)^{-1/6} & \text{cm} \quad s = 0 \\
1.8 \times 10^{15} E_{33} A_s^{-1/4} v_i^{-3/8} f_{\nu_0}/f_{\nu_{\gamma}}^{3/4} A_s^{-1/2} \tilde{Y}^{3/8} (1 + Y)^{-1/2} \tilde{Y}^{-3/8} d_{28}^{-3/4} (1 + z)^{-1/2} & \text{cm} \quad s = 2 
\end{cases}
\]

(26)

The magnetic and \( e^\pm \) energies, given by equation (10), are found to be

\[
E_B \approx 5.8 \times 10^{50} v_i f_{\nu_0} \tilde{Y}^{-1/4} d_{28}^2 (1 + z)^{-1} \text{ erg} \quad \text{and} \quad E_{e^\pm} \approx 8.5 \times 10^{50} \left( \frac{p - 1}{p - 2} \right) v_i^{1/2} f_{\nu_0} t_{s} (1 + Y) d_{28}^2 (1 + z)^{-1} \text{ erg}.
\]

(27)

(28)

Since the dependence of the above two quantities on \( f_{\nu_0} \) is linear, we should replace \( f_{\nu_0} \) with \( f_{\nu_{\gamma}} \), the flux observed at \( \nu_{\gamma} \). This will be done in the following sections, since the expression for \( f_{\nu_{\gamma}} \) depends on \( \alpha \).

We can relate the solution subspace we find to parameters for the shock model for GRBs; if electrons are accelerated in a relativistic shock then the LF of shock front (\( \Gamma_{\text{sh}} \)) with respect to the unshocked material is related to \( \gamma_{\gamma} \) (one of the five parameters) and is given by equation (11)

\[
\Gamma_{\text{sh}} \approx 50 e_{\gamma}^{-1} \left( \frac{p - 1}{p - 2} \right) v_{\gamma}^{7/16} v_i^{1/8} f_{\nu_0} f_{\nu_{\gamma}}^{1/4} \gamma_{\gamma}^{1/4} (1 + Y)^{-1/4} \tilde{Y}^{-1/4} d_{28}^{-3/4} (1 + z)^{-1/4}.
\]

(29)

where \( e_{\gamma} \) is the ratio of energy in electrons and the total thermal energy in the \( \gamma \)-ray source.

We now apply the results obtained in this section to each possible synchrotron low-energy spectral index \( \alpha \).

### 3.1 Synchrotron solutions when the low-energy spectrum is \( \nu^{-(p-1/2)} \)

We use equation (7) to eliminate \( \xi \) from the analytical solutions given by equations (20)–(23) for the \( \gamma_{\gamma} \ll \gamma_{\epsilon} \) and \( 2 < p < 3 \) case, and substitute \( v_{\gamma} = v_i \) and \( f_{\nu_{\gamma}} = f_{\nu_{\gamma}}(v_{\gamma}/v_i)^{(p-1)/2} \), to find that synchrotron solutions for \( \alpha = -(p-1)/2 \) are:

\[
\Gamma \approx 10^{13} v_{\gamma}^{13/2} f_{\nu_{\gamma}}^{3/8} \beta_{\gamma}^{1/4} v_i^{1/4} Y^{-3/16} (1 + Y)^{1/4} (1 + z)^{1/4} d_{28}^{-3/8} A_{1p}^{-1/4},
\]

(30)

\[
\gamma_{\gamma} \approx 4.7 \times 10^{13} v_{\gamma}^{13/2} f_{\nu_{\gamma}}^{1/2} \gamma_{\gamma}^{1/4} \beta_{\gamma}^{1/4} v_i^{1/4} Y^{-3/16} (1 + Y)^{1/4} (1 + z)^{1/4} d_{28}^{-3/8} A_{1p}^{-1/4},
\]

(31)

\[
B \approx 4.0 v_{\gamma}^{17} f_{\nu_{\gamma}}^{1/8} \beta_{\gamma}^{1/4} v_i^{1/4} Y^{3/16} (1 + Y)^{-3/4} (1 + z)^{1/4} d_{28}^{-1/8} A_{1p}^{-1/4} G,
\]

(32)

\[
\tau \approx 3.3 \times 10^{-10} v_{\gamma}^{13/2} \beta_{\gamma}^{1/4} f_{\nu_{\gamma}}^{1/4} \gamma_{\gamma}^{1/4} \beta_{\gamma}^{1/4} v_i^{1/4} Y^{-7/8} (1 + Y)^{-1/2} (1 + z)^{-1/2} d_{28}^{-1/4} A_{1p}^{-1/8},
\]

(33)

where

\[
A_{1p} \equiv \left( \frac{(p - 1)}{(p - 2)(3 - p)} \right)
\]

(34)

and we should emphasize that \( t_{\gamma} \) is not the burst duration – it is the width of a single spike in the GRB-prompt LC.

For a typical long-duration GRB with \( f_{\nu_{\gamma}} = 1 \text{ mJy} \), \( v_{\gamma} = 100 \text{ keV} \), \( t_{\gamma} = 0.1 \text{ s} \), \( z = 1 \), \( d_{28} = 2 \) and \( t_{s} \sim t_{\gamma} \) – henceforth we will refer to a GRB with these observed parameters as \( GRB_{-2} \) – the five parameters of the \( \gamma \)-ray source (\( \gamma_{\gamma}, \Gamma, B, N, \tau \)) are obtained from equations (30)–(33) and are given by

\[
\Gamma \gtrsim 3.2 \times 10^{13} Y^{-3/16} (1 + Y)^{1/4},
\]

(35)

\[
\gamma_{\gamma} \lesssim 6.0 \times 10^{13} Y^{1/16} (1 + Y)^{1/4},
\]

(36)

\[
B \lesssim 1.6 Y^{1/16} (1 + Y)^{-3/4} G,
\]

(37)

\[
\tau \gtrsim 4.7 \times 10^{-5} Y^{7/8} (1 + Y)^{-1/2}.
\]

(38)

In deriving these inequalities we took \( p = 2.5, v_{\gamma} = v_{\gamma} = 100 \text{ keV} \), and \( v_{\gamma} \ll 0.1 \).
The dependence of $\Gamma$, $\gamma$, $B$ and $\tau$ on $Y$ is weak, so the coefficients in above expressions are reasonable estimates for the $\gamma$-ray source basic physical parameters for GRB-z. We see that the $\gamma$-ray source LF, $\Gamma$, is required to be rather large $-\Gamma \gtrsim 3 \times 10^3$ – if the radiation is to be produced via the synchrotron process. This large $\Gamma$ is not consistent with afterglow modelling, which gives a value of a few hundred or less (Panaitescu & Kumar 2002). Furthermore, as shown below, the distance of $\gamma$-ray source from the centre of explosion turns out to be larger than the deceleration radius for this large $\Gamma$, unless $n_0$ is very small. This suggests that the synchrotron solution is internally inconsistent; after the deceleration radius $\Gamma$ is a function of $N$, $\gamma_i$ and $n_0$ and is no longer an independent parameter as considered in these derivations. The possibility that $R_\gamma > R_d$ is also ruled out by early optical afterglow data – e.g. GRBs 050801, 050820A, 060124, 060418, 060607A, 060614, 060714 – that show that $\gamma$-rays precede a rising afterglow flux that is produced by a decelerating forward shock. Moreover, if $R_\gamma \gtrsim R_d$, then in this case of a decelerating source we should see an increasing GRB pulse duration with time, which is not observed.

The distance of the $\gamma$-ray source from the centre of explosion, $R_\gamma \approx 2ct_\gamma \Gamma^2/1(1+z)$, is calculated using equation (30), and is given by

$$R_\gamma \approx 6 \times 10^{16} \frac{19-3p}{\nu_{39}^{3p-9} f_\gamma^{1/3} t_\gamma^{1/2} \nu_{39}^{3p-9} t_\gamma^{1/2} Y^{-3/8} (1 + Y)^{1/2} (1 + z)^{-1/2} d_{L,28}^{1/4}}{a_{19}^{7/8}} \text{ cm}$$

(39)

or $R_\gamma \sim 3 \times 10^{16} Y^{-3/8} (1 + Y)^{1/2} \text{ cm}$ for GRB-z.

We now compare these analytical estimates to the numerically computed solution space for synchrotron radiation. A numerical search of the allowed region of the 5D parameter space that satisfies the observational constraints ($\nu_\gamma$, $f_\gamma$ and $t_\gamma$; the same constraints that we used in the derivation of analytical expressions), confirms that for synchrotron solutions $R_\gamma > 10^{16}$ cm, $\Gamma > 10^3$ and $10^3 < \gamma_i < 10^2$ (see Fig. 2). We have considered a wide range of values of peak frequency ($\nu_\gamma$), $\gamma$-ray flux at the peak, and pulse duration, to see if we can find some viable synchrotron solutions for any GRBs with $\alpha = -(p - 1)/2$. These solutions are shown in Fig. 2. We find that by decreasing any of the observable parameters $R_\gamma$ decreases, but the dependence is weak in agreement with the scaling given in equation (39). Furthermore, a decrease in $t_\gamma$ reduces $\Gamma$ as expected from equation (30), but even for $t_\gamma = 10$ ms, $\Gamma$ is still $\gtrsim 10^3$.

![Figure 2](https://example.com/figure2.png)  
*Figure 2. Results of numerical calculation for the allowed synchrotron solution space when the spectrum below the peak of $\nu f_\gamma$, at $v_\gamma$, is: $f_{\gamma} \propto \nu^{-0.5-1/2}$ for $\nu < v_{\gamma}$. A point in the 5D parameter space ($\gamma_i$, $\Gamma$, $B$, $N$, $\tau$) is considered an allowed solution for the observed GRB parameters ($\nu_{\gamma}$, $f_\gamma$, $t_\gamma$, $\alpha$) provided that $v_{\gamma}$ is within a factor of 2 of the observed value, the pulse duration ($t_\gamma$) and flux at $v_\gamma$($f_\gamma$) are within a factor of 1.5 and 3 of the observed value, respectively; the larger tolerance on flux is due to larger error in flux calculation. The $x$-axis shows the distance of the $\gamma$-ray source from the centre of the explosion. The top left-hand panel is $\gamma_i$ – the minimum LF of electrons in source comoving frame at the site where they are accelerated (electron distribution function for $\gamma > \gamma_i$ is: $d\nu/d\nu_{\gamma} \propto \nu_{\gamma}^{-2}$, i.e. $p = 2.5$). The top right-hand panel shows the bulk LF of the source, the bottom left-hand panel shows the comoving magnetic field in gauss, and the bottom right-hand panel shows the ratio of energy in the magnetic field and electrons. For all of the numerical calculations we took the burst redshift $z = 1$. Legend shows several different cases of GRBs corresponding to different observed values for $v_{\gamma}$, $f_\gamma$ and $t_\gamma$. Only one observational parameter – that noted in the legend – is changed at a time, all the remaining parameters are left unchanged; the base value for the parameters is the same as we took for GRB-z, i.e. $\nu_{\gamma} = 100$ keV, $f_\gamma = 1$ mJy, $t_\gamma = 0.1$ s and $t_\gamma = t_{\gamma}$. For instance, for the 20 keV case, denoted by the solid black line, $\nu_{\gamma} = 20$ keV, and $f_\gamma$ and $t_\gamma$ are same as for GRB-z, i.e. 1 mJy and 0.1 s, respectively.*

for internal and external shocks, then density for the circumstellar medium compared with \( n_R \) and the ratio of heating of particles we expect to be modified, when \( t_a \approx t_f/100 \). Note that by decreasing the amount of time electrons have to radiate away their energy before being re-accelerated \( (t_a) \) increases the max \( (n_0) \) roughly as \( t_a^{-1} \). The \( n_0 \) upper limit decreases when any of the GRB parameters \( (v_f, t_f, t_a) \) is increased; \( n_0 \ll f^{-1} \).

Figure 3. Left-hand panel: the upper limit to the ISM density \( (n_0) \) for synchrotron solutions with \( \alpha = -(p - 1)/2 \) for a burst with \( v_f = 100 \) keV, \( f_{B} = 1 \) mJy, \( t_f = 1 \) s and \( t_a = t_f/100 \). Right-hand panel: same as the left-hand panel except that \( t_a = t_f/100 \). Note that by decreasing the amount of time electrons have to radiate away their energy before being re-accelerated \( (t_a) \) increases the max \( (n_0) \) roughly as \( t_a^{-1} \). The \( n_0 \) upper limit decreases when any of the GRB parameters \( (v_f, t_f, t_a) \) is increased; \( n_0 \ll f^{-1} \).

We next calculate the deceleration radius and compare it with \( R_d \) to ensure \( R_f < R_d \) for self-consistent solutions. The deceleration radius for GRB ejecta is calculated using equation (26) and is given by

\[
R_d \approx \begin{cases} 
2.6 \times 10^6 E_{55}^{-1/3} n_0^{-1/3} \frac{10^{-12}}{\nu_{38}} f_{-1}^{-1/8} v_{15}^{-1/8} t_f^{-1/6} Y^{1/6}(1 + Y)^{-1/6}(1 + z)^{-1/6} d_{L_{28}}^{-1/4} A_1^{-1/8} \text{ cm} & s = 0 \\
1.8 \times 10^7 E_{55} A_1^{-1/3} n_0^{-1/3} \frac{10^{-12}}{\nu_{38}} f_{-1}^{3/8} Y^{3/8} v_{15}^{-3/8} t_f^{-3/8} Y^{1/8}(1 + Y)^{-1/8}(1 + z)^{-1/8} d_{L_{28}}^{-3/4} A_1^{-1/8} \text{ cm} & s = 2
\end{cases}
\]

and the ratio of \( R_f \) and \( R_d \) is

\[
\frac{R_f}{R_d} \approx \begin{cases} 
2.3 E_{55}^{-1/3} n_0^{-1/3} \frac{10^{-12}}{\nu_{38}} f_{-1}^{1/2}(1 + Y)^{-1/2}(1 + z)^{-1/2} d_{L_{28}} A_1^{-1/2} \text{ s} = 0 \\
3.3 \times 10^3 E_{55}^{-1/3} n_0^{-1/3} \frac{10^{-12}}{\nu_{38}} f_{-1}^{3/4} Y^{1/2} v_{15}^{3/4} t_f^{-3/4} Y^{3/4}(1 + Y)^{-1/2}(1 + z)^{-1/2} d_{L_{28}} A_1^{3/4} \text{ s} = 2
\end{cases}
\]

Substituting in the observable parameters for GRB-51 into the above equation and solving for \( n_0 \) and \( A_1 \) such that \( R_f/R_d < 1 \), we find

\[
n_0 < 0.057 E_{55}^{-1/2} Y^{1/2}(1 + Y)^{1/2} \text{ cm}^{-3} \quad s = 0
\]

\[
A_1 < 5.5 \times 10^{-5} E_{55}^{-1/2} Y^{1/4}(1 + Y)^{1/4} \quad s = 2
\]

Note that \( n_0 \) and \( A_1 \) must be very small to ensure that \( R_f < R_d \), especially for \( Y < 1 \) expected of synchrotron solutions. Fig. 3 shows the results of numerical calculations which confirms these analytical estimates. Moreover, if we want \( R_f/R_d \ll 0.5 \), in order to have a clear separation between internal and external shocks, then \( n_0 \ll 10^{-2} \text{ cm}^{-3} \). Therefore, self-consistent synchrotron solutions with \( R_f \ll R_d \) require very low density for the circumstellar medium compared with \( n_0 \sim 1 \text{ cm}^{-3} \) obtained from afterglow modelling (Panaitescu & Kumar 2001). The limit on \( n_0 \) can be increased by decreasing \( t_a \) (see equation 42). Numerical result for the upper limit on \( n_0 \) when \( t_a = t_f/100 \) is shown in Fig. 3. It confirms the analytical result that \( n_0 \sim 1 \text{ cm}^{-3} \) can give \( R_f < R_d \) provided that \( t_a \ll t_f \). It should be noted that for systems involving shock heating of particles we expect \( t_a \sim t_f \) because electrons are accelerated at the shock front and there is no subsequent acceleration as particles travel downstream; in magnetic reconnections or dissipation it is natural to expect \( t_a \ll t_f \).

We now estimate the ratio of energy in \( e^\pm \) and magnetic field to find out if it is much less than unity or not when \( t_a \ll t_f \) (a small value for \( E_{\pm}/E_B \) results in low efficiency for \( \gamma \)-ray generation). The ratio \( E_{\pm}/E_B \) can be calculated using equations (27) and (28) and is given by

\[
\frac{E_{\pm}}{E_B} \approx 1.5 (3 - p) \left( \frac{v_{15}}{\nu_{38}} \right)^{-p/2} \left( \frac{t_f}{t_a} \right) Y(1 + Y),
\]

for \( 2 < p < 3 \) (numerical calculations take \( p = 2.5 \)). For the solution space corresponding to \( \alpha = -(p - 1)/2, 0.1 \ll v_f/\nu_{38} \ll 10^5 \) and so \( E_{\pm} > E_B \) even when \( t_f/t_a \sim 10^{-2} \). Therefore, small \( t_f/t_a \) solutions are fine from the point of radiative efficiency; the above equation needs to be modified, when \( t_a \ll t_f \), to include the total energy input in electrons during a GRB pulse width of \( t_f \), which will further improve the radiative efficiency when \( t_f/t_a \) is very small.

The reason that these synchrotron solutions have large \( R_f \) is not hard to understand. It requires a certain minimum number of electrons to produce the observed flux of \( f_{B} \sim 1 \) mJy at \( v_f \sim 100 \) keV: \( N \sim 10^{33}/(B_\gamma) \) – see equation (15). And in order to keep the Compton \( Y \) parameter,
3.2 Synchrotron solutions when the low-energy spectrum is $\nu^{1/3}$

This is a special case of $\alpha = -(p - 1)/2$ analysed in the previous subsection (Section 3.1) when $\gamma_{\gamma} \sim \gamma_t$; the solutions are a subset of those found in Section 3.1. The analytical solutions for this case, obtained by substituting $\xi = p/(p - 2)$ (see equation 7), $v_{cs} \sim v_{\gamma\nu}$, and $f_{\gamma} \sim f_{\nu\gamma}$, into equations (20)–(23) are

$$\Gamma \approx 10^3 v_{\gamma\nu}^{1/16} f_{\gamma\nu}^{3/16} t_{\gamma\nu}^{-3/8} t_d^{1/4} Y^{-3/16}(1 + Y)^{1/4} (1 + z)^{1/4} d_L^{3/16} \left( \frac{p}{p - 2} \right)^{3/16},$$

$$\gamma_t \approx 4.7 \times 10^3 v_{\gamma\nu}^{1/16} f_{\gamma\nu}^{1/16} t_{\gamma\nu}^{1/8} Y^{1/16}(1 + Y)^{1/4} (1 + z)^{1/4} d_L^{1/8} \left( \frac{p}{p - 2} \right)^{-1/16},$$

$$B \approx 4.0 v_{\gamma\nu}^{1/16} f_{\gamma\nu}^{1/16} t_{\gamma\nu}^{1/8} Y^{1/16}(1 + Y)^{-3/4} (1 + z)^{1/4} d_L^{1/8} \left( \frac{p}{p - 2} \right)^{-1/16},$$

$$\tau \approx 3.3 \times 10^{-10} v_{\gamma\nu}^{1/16} f_{\gamma\nu}^{1/16} t_{\gamma\nu}^{1/4} Y^{1/16}(1 + Y)^{-1/2} (1 + z)^{-1/2} d_L^{1/4} \left( \frac{p}{p - 2} \right)^{-7/8}.$$

Substituting $f_{\gamma\nu} = 1$, $v_{\gamma\nu} = 1$, $t_{\gamma\nu} = 0.1$ s (the observed parameters for GRB-32), and $t_\nu \sim t_\gamma$, in these equations, we find

$$\Gamma \sim 2.5 \times 10^3 Y^{-3/16}(1 + Y)^{1/4},$$

$$\gamma_t \sim 2 \times 10^3 Y^{1/16}(1 + Y)^{1/4},$$

$$B \sim 18 Y^{1/16}(1 + Y)^{-3/4} G,$$

$$\tau \sim 6.3 \times 10^{-10} Y^{7/8}(1 + Y)^{-1/2},$$

and indeed, the solutions are a subset of the $\alpha = -(p - 1)/2$ solution space – these have smaller $\tau$ and larger $\Gamma$ and $\gamma_t$. The distance of the source from the centre of the explosion is

$$R_\gamma \approx 6.0 \times 10^6 v_{\gamma\nu}^{3/8} f_{\gamma\nu}^{3/8} t_d^{1/4} Y^{1/4} (1 + Y)^{1/2} (1 + z)^{-1/2} d_L^{3/8} \left( \frac{p}{p - 2} \right)^{3/8} \text{ cm}$$

or $R_\gamma \sim 2 \times 10^6 Y^{-3/8}(1 + Y)^{1/2} \text{ cm}$ for GRB-32. This case has the same problems as $\alpha = -(p - 1)/2$ case discussed in Section 3.1, i.e. large $R_\gamma$ and $\Gamma$, and requiring extremely small external density in order that $R_\gamma \lesssim R_d$. Also, the conclusions drawn in Section 3.1 regarding $t_\nu/t_\gamma \ll 1$ offering a way out of this problem apply here as well.

The numerical calculation of the hypersurface in 5D parameter space allowed by GRB observations – $v_{\gamma\nu}$, $f_{\gamma\nu}$, and $t_\gamma$ – for GRB-32 finds $\gamma_t \gtrsim 10^3$, $\Gamma \gtrsim 10^5$, $200 \lesssim \gamma_t t_\gamma \lesssim 700$, source radius ($R_\gamma$) $10^{18}$–$10^{19} \text{ cm}$, and $B$ between 1 and $10^2$ G for the entire solution space (see Fig. 4) – which is in very good agreement with analytical estimates. For a wide range of values for the three observable parameters we find the GRB source to be located between $\sim 10^{32}$ and $10^{34} \text{ cm}$, $\gamma_t \gtrsim 3 \times 10^3$, and $\Gamma \gtrsim 10^5$ (Fig. 4). In order that $R_\gamma/R_d < 1$, the density of the surrounding medium ($n_0$) is required to be less than $\sim 0.1 \text{ cm}^{-3}$ which is much smaller than the value inferred from late-time afterglow modelling for long-duration GRBs. The density requirement is relaxed if $t_\nu \ll t_\gamma$ (see Fig. 5).

In conclusion, the synchrotron process, in a shock-heated medium, has serious problems accounting for prompt $\gamma$-ray emission for those bursts that have spectrum below the peak frequency ($v_{\gamma\nu}$) scaling as $f_{\gamma\nu} \propto \nu^{-1/3}$ or $\nu^{-(p-1)/2}$. A possible resolution is provided if electrons are more or less continuously accelerated while they are radiating $\gamma$-ray photons during the entire time period of a spike in the observed GRB LC; in other words $t_\nu \ll t_\gamma$ It should be pointed that $t_\nu \sim t_\gamma$ in shocks whereas continuous acceleration might be possible in regions of magnetic reconnection/dissipation.

3.3 Synchrotron solutions when the low-energy spectrum is $\nu^{-1/2}$

Substituting $f_{\gamma\nu} = f_\nu(v_{\gamma\nu}/v_{\nu})^{1/2}$, $v_{\gamma\nu} = v_{\nu}$ and $\xi$ from equation (7) for the case where $v_{\gamma\nu} < v_{\nu}$, into equations (20)–(23) we find the allowed part of the 5D parameter space when the spectrum below $v_{\nu}$ is $f_\nu \propto \nu^{-1/2}$;

$$\Gamma \approx 10^3 v_{\gamma\nu}^{1/8} f_{\gamma\nu}^{3/8} t_{\gamma\nu}^{3/16} Y^{3/8}(1 + Y)^{1/4} (1 + z)^{1/4} d_L^{3/8} \left( \frac{p}{p - 2} \right)^{3/16},$$

Modelling γ-ray burst prompt emission

Figure 4. Synchrotron solution space when the spectrum below $\nu_\gamma$, the peak of $\nu f_\nu$, is $f_\nu \propto \nu^{1/3}$, i.e. $\alpha = 1/3$. See Fig. 2 caption for details.

Figure 5. Left-hand panel: the maximum density of the circumburst medium so that $R_\gamma < R_d$ for synchrotron solutions with $\alpha = 1/3$ and for a burst with $\nu_\gamma = 100$ keV, $f_\nu = 1$ mJy, $t_\gamma = 1$ s and $t_a = t_\gamma$. Right-hand panel: same as the left-hand panel except that $t_a = t_\gamma / 100$; note that by decreasing the amount of time electrons have to radiate away their energy before being re-accelerated ($t_a$) increases the upper limit for $n_0$ roughly as $t_a^{-1}$. The $n_0$ upper limit is weakly dependent on $\nu_\gamma$, and decreases with increasing $t_\gamma$ and $f_\nu$ being most sensitive to $f_\nu - \max(n_0) \propto f_\nu$.

\begin{align}
\nu_\gamma &\approx 4.7 \times 10^{3} \nu_{7/8}^{7/16} f_{\gamma}^{-1/16} t_{\gamma}^{1/8} v_{c5}^{1/8} Y_{1/16}^{1/4}(1 + Y)^{1/4}(1 + z)^{1/4} d_{28}^{-1/8} A_{2p}^{-1/16}, \\
B &\approx 4.0 \nu_{7/8}^{1-1/16} f_{\gamma}^{-1/16} t_{\gamma}^{1/8} v_{c5}^{1/8} Y_{1/16}^{3/4}(1 + Y)^{-3/4}(1 + z)^{1/4} d_{28}^{-1/8} A_{2p}^{-1/16} G, \\
\tau &\approx 3.3 \times 10^{-10} \nu_{7/8}^{3/8} f_{\gamma}^{1/8} t_{\gamma}^{-1/4} v_{c5}^{3/4} Y_{7/8}^{1/2}(1 + Y)^{-1/2}(1 + z)^{-1/2} d_{28}^{1/4} A_{2p}^{-7/8},
\end{align}

where

\begin{equation}
A_{2p} \equiv \frac{p - 1}{p - 2}.
\end{equation}
For \textit{GRB}-2 - $t_p = 0.1$ s, $v_y \approx 100$ keV, $f_y = 1$ mJy and $t_o = t_p$ - and taking $v_{cs} < 10^{-4}$ (in agreement with the numerical calculation) we find $\Gamma \lesssim 720 \gamma / Y_{16}$, $\gamma / Y_{16} \lesssim 6.7 \times 10^3$ and $B \lesssim 590 (1 + Y)^{-3/4}$. In contrast to the previous two cases considered in Sections 3.1 and 3.2, $\Gamma < 1000$ in this case. We find the $\gamma$-ray source distance, $R_{\gamma}$, to be

$$R_{\gamma} \approx 6.0 \times 10^{16} v^{-3/4}_y f_{\gamma}^{-1/2} \Gamma^{-1/2} Y_{16}^{-1/2} v_{cs}^{-1/2} t_o^{-1/2} (1 + Y)^{-1/2} d_{20}^{-3/4} A^{-5/8}_{2p} \text{ cm}$$

(59)

which is $R_{\gamma} \approx 2 \times 10^{-3} Y^{-3/4} (1 + Y)^{1/2}$ cm for \textit{GRB}-2. We compare this radius to the deceleration radius, $R_d$, which is obtained from equation (26) and is given by

$$R_d \approx \begin{cases} 2.6 \times 10^{16} \frac{E_{53} R_0}{v^2_y} v_{cs}^{-1/3} f_{\gamma}^{-1/4} t_o^{-1/2} Y_{16}^{-1/2} (1 + Y)^{-1/6} d_{20}^{-1/2} A^{-1/4}_{2p} \text{ cm} & \text{for } \gamma = 0 \\ 1.8 \times 10^{13} E_{53} A_1^{-1/3} v_{cs}^{-1/3} f_{\gamma}^{-1/4} t_o^{-1/2} Y_{16}^{-1/2} d_{20}^{-1/2} A^{-1/4}_{2p} \text{ cm} & \text{for } \gamma = 2 \end{cases}$$

(60)

and for the ratio $R_o/R_d < 1$, we find that $n_0 \lesssim S_{53} Y_{16}^{3/2} (1 + Y)^{-2} \text{ cm}^{-3}$ and $A_1 < 0.014 E_{53} Y_{16}^{3/4} (1 + Y)^{-1}$ if $v_{cs} \sim 0.1$; the limits on $n_0$ and $A_1$ are much higher for $v_{cs} \ll 0.1$ and poses no problem for synchrotron solutions in a shock-heated source.

If the synchrotron solutions were to arise in a shock-heated medium, we can calculate the LF of the shock front with respect to the unshocked fluid, $\Gamma_{sh}$, using equation (11):

$$\Gamma_{sh} \approx 26 \epsilon_5^{-1} v^{-1}_y f_{\gamma}^{-1/4} t_o^{-1/2} Y_{16}^{-1/2} (1 + Y)^{1/2} v_{cs}^{1/4} d_{20}^{-1/2} A^{-15/16}_{2p}$$

(61)

or $\Gamma_{sh} \lesssim 32 Y_{16}^{1/4} (1 + Y)^{1/4}$ for \textit{GRB}-2, assuming that electrons receive half of the shock energy and that there are no $e^\pm$ pairs. Note that as long as $v_{cs} \sim 1$, $\Gamma_{sh}$ is pretty high (~20), and it is insensitive to $v_{cs}$ (and the other quantities). In order to produce $\Gamma_{sh} \sim 20$ in internal shocks, we need the relative LF of the two colliding shells $\Gamma_{rel} \approx 2 \Gamma_8^2 (n_1/n_2)^{1/2}$ (Sari & Piran 1995; Panaitescu & Kumar 2004), where $n_1$ and $n_2$ are the comoving densities of the two colliding shells (see Appendix A for a discussion of how we calculate $\Gamma_{rel}$). For $\Gamma_{rel} < 5$, so that the ratio of the LFs of the colliding shells is not larger than 10, $n_1/n_2 \lesssim \sim 10^{-5}$ is required (see Appendix A); $\Gamma_{rel} > 5$ is an unlikely situation to be realized in nature.

We calculate the total energy in electrons ($E_e$) and magnetic field ($E_B$) to determine the efficiency for synchrotron radiation – if there is a lot more energy in magnetic field than that in the electrons, the efficiency for $\gamma$-ray radiation would be small. The magnetic and electron energies for the case of $v^{-1/2}$ spectra are obtained from equation (10) and the solutions for $B$, $\Gamma$, $\gamma$, and $R_{\gamma}$ derived above, and are given by

$$E_B = B^2 R^2_y / 2 \approx 6.0 \times 10^{10} v_{cs} f_{\gamma} t_o Y^{-1} (1 + z) d_{20}^{-1/4} A^{-1/2}_{2p} \text{ erg}$$

(62)

$$E_e = N(p - 1) m_e c^2 \gamma / (p - 1) \approx 8.7 \times 10^{10} v_{cs} f_{\gamma} t_o (1 + Y) (1 + z) d_{20}^{-1/4} A^{-1/2}_{2p} \text{ erg}.$$}

(63)

The ratio $E_B/E_e$ is

$$f_{B/e} \approx \frac{f_{B/e}}{f_{B/e}} \approx 0.68 t_o^{-1} Y^{-1} (1 + Y)^{-1}.$$}

(64)

For $t_o \ll t_p$, the above expression for the ratio $f_{B/e}$ would need to be modified to include the total energy deposited in $e^\pm$ as a result of multiple acceleration episodes during a GRB pulse time period of $t_p$; for $t_o/t_p \sim 1$ the expression for $f_{B/e}$ reduces to the familiar form that depends only on the Compton $Y$. For $Y \ll 1$ most of the energy is in the magnetic field and for these solutions the radiative efficiency to produce a GRB is very small.

We numerically search the 5D parameter space subject to the three observational constraints ($v_y, f_{\gamma}, t_o$) and find solutions with $10^{13} \text{ cm} < R_p < 10^{17} \text{ cm}$, $10 < B < 10^5 \text{ G}$, $100 < \gamma / Y_{16} < 3 \times 10^{4}$, $80 < \Gamma < 3000$ and $10^{-3} < Y < 7$ (see Fig. 6) – all in good agreement with analytical estimates presented above. We find $2 < \Gamma_{sh} < 100$ and $10^{-2} < f_{\gamma} \lesssim 10^4$. So it would seem that we have solutions with $f_{\gamma} \lesssim 1$ and $\Gamma_{sh}$ of the order of a few – however, it turns out that for $f_{\gamma} \lesssim 10$, $\Gamma_{sh} \lesssim 5$. For $\Gamma_{sh} \lesssim 5$, $10^{-3} < n_1/n_2 < 0.1$, and the ratio of the LFs of two colliding shells, $\Gamma_{rel}$, to produce this $\Gamma_{sh}$ is greater than 20 (see Fig. 7) – fluctuations in the LF of the outflow with $\Gamma_{rel}$ of the order of a few are typically expected in internal shocks.

Numerical solutions for the allowed part of the 5D space for a range of observable parameters are shown in Fig. 6. An increase in $v_y$ leads to a slight increase of $\gamma / Y_{16}$ and $R_p$ whereas $\Gamma$ is quite insensitive to it. These behaviours are consistent with our analytical calculations (equations 54–57). The decrease of $f_{\gamma}$ with $v_y$ (Fig. 6) is due to an increase of $Y$. An increase of $f_{\gamma}$ has little effect on $\gamma / Y_{16}$ (for the allowed solution space), $\Gamma$ and $Y$ increase a little, and $f_{\gamma}$ decreases; these parameters have a very weak dependence on $f_{\gamma}$ (see equations 54–57). And finally, when $t_o$ is increased, $\Gamma$ and $Y$ increase, and $\Gamma$ and $f_{\gamma}$ decrease. This is again in agreement with the analytical estimates – $\gamma / Y_{16} \propto t_o^{-1}$. The large decrease in $\Gamma$ with $t_o$ is due to an increase of $Y$ with $t_o \propto t_o^{-1} Y^{-3/4}$.

We have looked at the variation of $\Gamma_{sh}$, $\Gamma_{rel}$ and $f_{\gamma}$ with $v_y$, $f_{\gamma}$ and $t_o$. The results are shown in Fig. 7. The minimum value of $\Gamma_{rel}$ has a weak dependence on $v_y$, $f_{\gamma}$ and $t_o$. $\Gamma_{rel}$ is between 5 and 20 for $f_{\gamma} \lesssim 10$; $\Gamma_{rel} \sim 5$ solutions are only present when $t_o \lesssim 0.01 \text{ s}$, $f_{\gamma} \lesssim 0.1 \text{ mJy}$ or $v_y < 20 \text{ keV}$ (note that we only alter one of the three at a time, i.e. for $t_o \sim 0.01 \text{ s}$, $v_y \sim 100 \text{ keV}$ and $f_{\gamma} \sim 1 \text{ mJy}$). $\Gamma_{rel}$ may be small enough, then, that the synchrotron mechanism can produce GRBs with $v^{-1/2}$ spectra if it has very short pulse duration, small peak frequency or small flux.
Figure 6. Synchrotron solution space when the spectrum below the peak of \( \nu f_\nu \) is \( f_\nu \propto \nu^{-1/2} \) for \( \nu < \nu_\gamma \). \( \gamma_i \), \( \Gamma_1 \), \( Y \) and \( f_B/ke \) are plotted against the distance of the source from the centre of explosion \( (R_\gamma) \). The solution spaces for several different values of \( \nu_\gamma \), \( f_\gamma \) and GRB pulse duration \( (t_\gamma) \) are shown in the four panels. Please see the Fig. 2 caption for details.

Figure 7. This figure, unlike all the previous ones, shows model dependent results. For each point in the allowed region of the 5D parameter space, corresponding to synchrotron solutions for \( \alpha = -1/2 \) and for given \( \nu_\gamma, f_\gamma \) and \( t_\gamma \), we calculate parameters for the popular internal shock model for GRBs (see Appendix A for details). The left-hand panel shows the LFs \( \Gamma_1 \) and \( \Gamma_2 \) of the two colliding shells in the internal shock model as a function of \( R_\gamma \), the distance from the centre where the shells collide. The right-hand panel shows \( \Gamma_{rel} = \Gamma_1 \Gamma_2 (1 - v_1 v_2) \) versus \( f_B/ke \) (the ratio of energy in magnetic field and electrons). Different \( \nu_\gamma, f_\gamma \) and \( t_\gamma \) cases are displayed as described in Fig. 2. Note that \( \Gamma_{rel} \) is correlated with \( f_B/ke \); for high \( \Gamma_{rel} \), \( f_B/ke \) is small (<1), and for low \( \Gamma_{rel} \), \( f_B/ke \gg 1 \). Low \( \Gamma_{rel} \) and \( f_B/ke \gg 1 \) solutions might correspond to a highly magnetized outflow.

3.3.1 X-ray flux during the GRB when \( \alpha = -1/2 \)

So far, we have only considered prompt \( \gamma \)-ray emission due to the synchrotron process. We now calculate emission in other wavelengths, particularly the X-ray and optical, that should accompany \( \gamma \)-ray photons. In this subsection, and in Section 3.3.2, we relate the solutions we
found in Section 3.3 to the internal shock model for GRBs (Rees & Meszaros 1994; see Piran 1999, for complete references) according to which shells of material ejected in the explosion undergo collisions and the resulting shocks convert part of the kinetic energy of the outflow to radiation. Throughout this section we assume that the γ-ray emission is produced in shell ‘1’ which is taken to be the faster of the two shells. The results are essentially identical if we assume that the GRB is produced in the outer, slower, shell, which we shall refer to as shell ‘2’. The X-ray flux from shell ‘1’; however, is independent of the internal shock model, and is expected to accompany the prompt synchrotron ν−1/2 γ-ray emission.

The X-ray and optical flux from shell ‘1’ lie on the ν−1/2 extrapolation of the γ-ray flux and therefore it is straightforward to calculate these using \( f_{\nu} \) and the information that \( v_{\gamma} \lesssim 2 \text{ eV} \) and \( v_{\gamma} \lesssim 5 \text{ eV} \) for the entire solution subspace of the 5D parameter space. The calculation of emission from shell ‘2’ is more involved and also a bit uncertain. We provide here (and in Section 3.3.2) a lower limit to the X-ray and optical flux from shell ‘2’ for each point in the 5D space that satisfies the three observational constraints \( (\nu_\gamma, f_\nu, t_c) \). The calculation of flux from shell ‘2’ requires the knowledge of the LF of the shock front moving into this shell as well as the ratio of densities \( (n_1/n_2) \). The calculation for these quantities is described in Appendix A. The synchrotron injection frequency in shell ‘2’, \( \nu_{\text{synch}} \), is smaller than that in shell 1 by a factor of \( (\Gamma_2 - 1)/(\Gamma_1 - 1)^2 \); \( \Gamma_1 \) and \( \Gamma_2 \) are shock front LFs into shell ‘1’ and ‘2’ with respect to unshocked gas in shells ‘1’ and ‘2’, respectively. This factor is approximately \( n_1/n_2 \) for \( \Gamma_1, \Gamma_2 \gg 1 \), but the approximation breaks down for \( n_1/n_2 \lesssim 10^{-2} \) (see Fig. A2) since the shock in shell ‘2’ becomes mildly relativistic (we note that this approximation is not used in our numerical calculations). The peak flux of the synchrotron spectrum at \( v = \min(v_\gamma, v_{\text{synch}}) \) in shell ‘2’, \( f_{\nu_{\text{synch}}} \), is larger than \( f_{\nu_{\gamma}} \) by a factor of \( (n_1/n_2)^{1/2} \). The magnetic field is assumed to be the same in the two shells, and therefore the difference between the cooling frequencies in the shells is due to different \( Y \) parameters; since synchrotron dominates over SSC here by design, the difference between \( v_{\gamma} \) and \( v_{\text{synch}} \) ends up being very small. The shell ‘2’ synchrotron self-absorption frequency, \( v_{\text{abs}} \), is larger than that in shell 1 by a factor of \( (n_1/n_2)^{3/2} \), or a factor of a few.

The 1-keV synchrotron and SSC flux from shells ‘1’ and ‘2’ for GRB-2 is shown in the top left-hand panel of Fig. 8. The shell ‘2’ synchrotron flux contributes the most to the prompt X-ray flux, and the shell 1 synchrotron flux contributes a slightly smaller amount. SSC flux from either shell is negligible. There is a weak dependence of X-ray flux on \( \Gamma_{\text{iso}} \). The shell ‘1’ synchrotron flux at 1 keV is about 10 mJy. This is simply the extrapolation of the 100-keV flux back to 1 keV, with a spectral index of \(-1/2\) or \((1/100)^{-1/2}\) 1 mJy ~ 10 mJy for GRB-2.

Shell ‘2’ synchrotron X-ray flux ranges from 1 to 100 mJy. We expect the flux from shell ‘2’ at 1 keV to be

\[
f_{2\nu} \sim f_{\nu} \left( f_{\nu_{\text{synch}}} \right)^{-1/2} \left( \frac{v_{\gamma}}{100 \text{ keV}} \right)^{-1/2} \sim f_{\nu} \left( \frac{n_1}{n_2} \right)^{1/4} \left( \frac{1 \text{ keV}}{v_{\gamma}} \right)^{-5/4} \text{ mJy}. \tag{65}
\]

Figure 8. Prompt X-ray (1 keV) and optical (2 eV) flux associated with points in the solution subspace of 5D parameter space when cast in terms of the internal shock model or two colliding shells; the solutions are for synchrotron radiation with low-energy spectrum \( f_{\nu} \propto \nu^{-1/2} \). Top left-hand panel: contributions of SSC and synchrotron to X-ray flux at 1 keV from shells ‘1’ and ‘2’ (see legend) for GRB-2. Top right-hand panel: sum of all contributions is shown for the cases that have been described in Fig. 2. Bottom left-hand panel: contributions of synchrotron emission from shells ‘1’ and ‘2’ to the optical flux; IC contributions are negligible from either shell. Bottom right-hand panel: sum of all contributions in the optical, for the same cases as the top right-hand panel. The results shown here assume that the γ-ray flux is produced in shell ‘1’; production of the GRB in shell ‘2’ does not change the results significantly.

Using the above arguments and assuming that $v_\gamma < 1$ keV (valid when $n_1/n_2 \leq 0.01$ – satisfied by roughly half of the solution points in the 5D space), $v_\gamma \sim v_{\gamma 2}$, and $p = 2.5$. From this equation, we estimate that the flux at 1 keV should be between $\sim 20$ and 300 mJy (for $f_\nu = 1$ mJy and $v_\gamma \sim 100$ keV) – in agreement with the numerical results in Fig. 8.

The shell ‘2’ SSC flux at 1 keV ranges from $10^{-5}$ to almost 10 mJy. The SSC peak frequency ($\sim v_\gamma y_2^2$) ranges from about 100 eV to very high values, so over a large part of the solution space, the expected SSC flux at 1 keV in comparison to the synchrotron 100-keV flux from shell ‘1’ is (assuming that $v_\gamma > v_{\gamma 2}$)

$$\frac{f_\nu^{\text{SSC}}}{f_\nu^{\text{syn}}} \approx \frac{f_{\nu 2} Y_2}{f_{\nu 1} Y_1} \left( \frac{1 \text{ keV}}{100 \text{ keV}} \right)^{-1/2} \left( \frac{v_{\gamma 1}}{10^2 \text{mJy}} \right)^{-1/2} \sim 10 \left( \frac{n_2}{n_1} \right) \frac{y_1}{y_2}$$

which, after substituting in the solutions for $\rho$ and $y_\gamma$, is

$$f_\nu^{\text{SSC}} \sim 1.5 \times 10^{-4} \left( \frac{n_2}{n_1} \right)^{1/2} v_\gamma^{-21/8} f_{\nu 1}^{15/8} t_{\nu 1}^{1/8} Y_3^{1/4} \nu_3^{1/4} (1 + \gamma_3)^{-1/4} d_{28}^{1/8} A_2^{15/16} \text{mJy};$$

with $Y \sim 1$ and $1 < n_2/n_1 < 10^5$, the range for the X-ray flux obtained from the above equation is in agreement with the numerical solutions.

The sum of synchrotron and SSC contributions to flux at 1 keV from both shells for various values of $v_\gamma f_{\nu 1}$ and $t_{\nu 1}$ are shown in the upper right-hand panel of Fig. 8. The 1-keV flux ranges from 0.1 to a few thousand mJy, and is most sensitive to $f_{\nu 1}$ and $v_\gamma$ – in agreement with equation (65), $f_{\nu 1} \propto f_{\nu 1}^{15/4}$, since synchrotron emission dominates.

The early 0.2–10 keV X-ray flux as observed by the Swift X-ray telescope ranges from $10^{-12}$ to $10^{-8}$ erg cm$^{-2}$ s$^{-1}$, which corresponds to 1-keV flux of about $10^{-4}$ to a few mJy (assuming $v^{-1/2}$ in the X-ray band). These observations are made at roughly 100 s after the GRB trigger. The X-ray LC from the γ-ray source should peak at about the same time as the GRB LC. After the peak, assuming that the outflow opening angle is greater than $1/\Gamma$, the emission should be dominated by off-axis emission and the LC should fall off as $t_r^{-2.5}$ (Kumar & Paniautescu 2000), which in this case is $t_r^{-2.5}$ since $f_\nu \propto v^{-1/2}$. Extrapolating the observed 1-keV flux of $\sim 10^{-4}$ to 1 mJy backwards in time from 100 to 10 s, we find the X-ray flux during the GRB to be consistent with values shown in the upper right-hand panel of Fig. 8.

### 3.3.2 Prompt optical emission when the GRB index $\alpha = -1/2$

In the bottom left-hand panel of Fig. 8, the R band (2 eV) flux from shell ‘1’ and ‘2’ are plotted against $\Gamma_{\text{rel}}$. Optical flux from the γ-ray source can be pretty bright during the burst for these solutions ranging from $10^{-3}$ and 100 mJy (24th to 11th magnitude in the R band). The synchrotron flux is smaller for smaller $\Gamma_{\text{rel}}$ solutions. The SSC makes negligible contribution to the optical flux compared to the synchrotron process, because $v_\gamma y_2^2$ is well above the optical.

If we extrapolate the shell ‘1’ 100-keV flux back to 2 eV using the spectral index $\alpha = -1/2$, we expect $f_{\nu R} \sim 225 v_f^{1/2} f_{\nu 1} \sim 225$ mJy for GRB-2 whereas for most of the solution space the optical flux for shell ‘1’ falls below 10 mJy (Fig. 8, bottom left-hand panel) – this is because the synchrotron self-absorption frequency is larger than the R-band frequency by a factor of $\sim 10$ or more. The range of shell ‘2’ R-band flux is higher than shell ‘1’ by a factor of $f_{\nu 2}^{1/2} f_{\nu 1} \sim \sqrt{n_2/n_1} \sim 30$.

In the bottom right-hand panel of Fig. 8, we show the affect of varying $v_\gamma f_{\nu 1}$ and $t_{\nu 1}$ on the prompt optical flux; the total flux – obtained by adding the contributions for the two colliding shells – ranges from $10^{-4}$ to 10 mJy, or $R$ magnitude of 26–9. The optical flux increases with $v_\gamma f_{\nu 1}$ and $t_{\nu 1}$ – longer GRB pulses with higher peak frequency and/or flux tend to be brighter in the optical band. Synchrotron self-absorption is larger at smaller $\Gamma_{\text{rel}}$, and that makes the optical flux smallest at the minimum of $\Gamma_{\text{rel}}$. There are $\Gamma_{\text{rel}}$ ~ 5 solutions with $f_{\nu 1}^{15/4} \sim 1$, that have small enough optical flux to be in accord with the observed upper limits, especially for smaller $v_\gamma f_{\nu 1}$ and $t_{\nu 1}$. Although $v_\gamma > 2$ eV for much of the solution space, the optical LC should peak at the same time as the GRB LC, since $v_\gamma < 2$ eV. After the peak, the LC should fall off – if dominated by off-axis emission – as $t_r^{-2.5}$.

### 3.4 Synchrotron solutions for $f_{\nu} \propto v^{-p/2}$

If the observed spectral index is steep and consistent with $v^{-p/2}$ and no break is detected in the observed energy band of 15–150 keV, for instance, then this is a special case of either $v^{-1/2}$ or $v^{-p-1/2}$ low-energy spectrum discussed in Sections 3.1 and 3.3 – with $v_\gamma = \max(v_\gamma, v_c) \lesssim 0.15$. The allowed solution space for this situation should be close to the $v_\gamma = 20$ keV case in Figs 2 and 6; specifically, $v_c \lesssim 10^3$, $\Gamma > 100$ and $2 \lesssim \Gamma_{\text{rel}} \lesssim 200$.

### 4 SYNCHROTRON-SELF-COMPTON – SSC – SOLUTIONS

In this section, we present solutions for the prompt γ-ray emission to be produced via the synchrotron-self-IC radiation or the SSC process. The basic approach is same as in section 3. We determine the hypersurface in the 5D parameter space $(v_\gamma, \Gamma, R, N, \tau)$ that has SSC emission consistent with the three observational constraints $v_\gamma f_{\nu 1}$ and $t_{\nu 1}$. Since different cases of low-energy spectral index have different ordering for the characteristic synchrotron frequencies ($v_\gamma, v_c, v_\nu$), we do not consider a general SSC solution, but describe analytical and numerical
solutions for the positive low-energy spectral index case, i.e., \( f_c \propto v^\alpha \) with \( \alpha > 0 \) for \( v < v_c \), and the negative index case, i.e., \( \alpha < 0 \), separately in several subsections below.

### 4.1 SSC solutions: positive low-energy spectral index

This section is broken up into two subsections. One dealing with the special case of \( f_c \propto v^{1/3} \) is discussed below. All the other cases of \( \alpha > 1/3 \) are discussed in Section 4.1.2.

#### 4.1.1 SSC solutions: spectral index \( \alpha \approx +1/3 \)

The SSC spectrum \((v, f_c)\) peaks at \( v_p \sim 4 \max(v_c, v_i) \max(\gamma_c, \gamma_i)^2 \); where \( v_i \) and \( v_c \) are the injection and cooling frequencies of the underlying synchrotron radiation, \( \gamma_i \) is the minimum LF of electrons in the source comoving frame and \( \gamma_c \) is the LF of electrons that cool on time-scale \( t_i \) available since last accelerated. For the spectrum below \( v_p \) to be \( \sim v^{1/3} \), we must have \( \gamma_i \sim \gamma_c \). In that case, \( v_p \sim 4v_i^2 \gamma_i^2 \). The IC flux at \( v_p \) is \( f_p \sim f_p(\gamma_{ic} \text{ is the synchrotron flux at } v_i) \). The equations for pulse duration \( t_p \) and Compton \( Y \) are same as in Section 3. The equations for \( v_i \sim v_c \), the peak IC frequency \( v_p \), the IC flux \( f_p \) at \( v_p \), and the Compton \( Y \) parameter are given below:

\[
B^2 \Gamma \gamma_i \approx 7.7 \times 10^9 (1 + z) \Gamma_1 (1 + Y)^{-1}, \tag{68}
\]

\[
B \gamma_i \gamma \approx 2.3 \times 10^5 v_i (1 + z), \tag{69}
\]

\[
B \Gamma^3 \tau_i \approx 1.6 \times 10^6 \left( f_i t_i \gamma_i^2 (1 + z) d_{LS}^2 \right), \tag{70}
\]

\[
\tau \gamma_i^2 \approx \frac{3}{4} \left( \frac{p - 1}{p - 2} \right) Y. \tag{71}
\]

We first eliminate \( \tau \) from equation (70) using (71) to get

\[
B \Gamma^3 \gamma_i^{-4} \approx 2.8 \times 10^5 f_i t_i Y^{-2} (1 + z) d_{LS}^2 Y^{-2} \left( \frac{p - 1}{p - 2} \right) \frac{Y^2}{Y}. \tag{72}
\]

Next, we divide equations (68) and (69) to eliminate \( \Gamma \):

\[
B \gamma_i \gamma \approx 3.3 \times 10^{-4} \Gamma_1 (1 + Y)^{-1} v_i^{-1}, \tag{73}
\]

divide equation (69) by (72) to eliminate \( B \):

\[
\gamma_i \Gamma^{-4} \approx 8.2 \times 10^5 v_i f_i^{-1} t_i^{-2} d_{LS}^{-2} Y^{-2} \left( \frac{p - 1}{p - 2} \right) \frac{Y^2}{Y}. \tag{74}
\]

and combine equations (68) and (72) to obtain

\[
B \Gamma \approx 4.6 \times 10^4 t_i^{-4/9} (1 + Y)^{-4/9} f_i^{1/9} Y^{-2/9} (1 + z)^{5/9} d_{LS}^{2/9} Y^{-2/9} \left( \frac{p - 1}{p - 2} \right)^{2/9}. \tag{75}
\]

Equations (73) and (75) give

\[
\Gamma \gamma_i \approx 1.4 \times 10^4 v_i f_i t_i (1 + Y)^{5/9} f_i^{1/9} Y^{-2/9} (1 + z)^{5/9} d_{LS}^{2/9} Y^{-2/9} \left( \frac{p - 1}{p - 2} \right)^{2/9}. \tag{76}
\]

and substituting this into equation (74), we find the solution for \( \gamma_i \) to be

\[
\gamma_i \approx 84 v_i^{3/4} f_i^{1/3} t_i^{1/6} \gamma_i^{1/3} Y^{1/6} (1 + Y)^{1/6} d_{LS}^{1/6} (1 + z)^{1/6} A_p^{-1/18}. \tag{77}
\]

Note that the electron LF \( \gamma_i \) has a very weak dependence on the observed quantities as well as the Compton \( Y \) parameter, and therefore \( \gamma_i \sim 80 \) for the entire SSC solution space. By plugging equation (77) back into equations (73), (74) and (71), we find the remaining parameters:

\[
B \approx 200 v_i^{14/9} f_i^{1/9} Y^{7/18} (1 + Y)^{-2/9} d_{LS}^{7/18} (1 + z)^{-2/9} A_p^{7/18}, \tag{78}
\]

\[
\Gamma \approx 240 v_i^{14/9} f_i^{1/9} Y^{7/18} (1 + Y)^{2/9} d_{LS}^{7/18} (1 + z)^{2/9} A_p^{-7/18}, \tag{79}
\]

\[
\tau \approx 1.1 \times 10^{-4} v_i^{1/2} f_i^{1/18} Y^{1/9} (1 + Y)^{2/9} d_{LS}^{1/9} (1 + z)^{2/9} A_p^{-8/9}. \tag{80}
\]

All of these parameters are weakly dependent on the three observable quantities, namely \( v_i, f_i \) and \( t_i \). Substituting the observed values for GRB-2, i.e., \( v_i = 1, f_i = 1 \text{ mJy and } t_i \sim t_p = 0.1 \text{ s, } z = 1 \), and taking \( p = 3.2 \), we find that \( \gamma_i \sim 58 Y^{1/6} (1 + Y)^{-1/6}, B \sim 1.1 \times 10^3 Y^{1/6} (1 + Y)^{2/3} \text{ G, } \Gamma \sim 110 Y^{7/18} (1 + Y)^{2/9} \text{ and } \tau \sim 1.3 \times 10^{-4} Y^{8/9} (1 + Y)^{-2/9}. \)

The distance of the \( \gamma \)-ray source from the centre of the explosion, \( R_p \), is given by

\[
R_p \approx 3.3 \times 10^5 v_i^{1/2} f_i^{1/18} Y^{7/18} (1 + Y)^{2/9} (1 + z)^{-5/9} d_{LS}^{1/9} A_p^{-2/9} \text{ cm}, \tag{81}
\]

or \( R_p \sim 1.3 \times 10^{25} Y^{-7/9} (1 + Y)^{4/9} \text{ cm for GRB-2}. \) This distance is smaller than the deceleration radius for a homogeneous or a wind external medium, unlike the situation when \( \gamma \)-rays are produced via the synchrotron process (see Section 3).

One constraint that we have not yet considered is that the SSC self-absorption frequency, \(v_a \sim 4v_c\gamma_c^2\), must be smaller than \(\sim 20\) keV otherwise the low-energy spectral index, obtained by Band function fit to the BATSE or Swift/BAT data, would be steeper than \(\alpha = 1/3\) we are considering in this subsection. The expression for \(v_a^{\text{ssc}}\) valid for \(v_a < v_c, v_c\) is

\[
v_a^{\text{ssc}} \sim \frac{1.7 \times 10^{-14} \gamma_c^2 T_{10}^{6/5}}{(1+z)} \left( \frac{f_p v_i^{1/3}}{2\gamma_m c} \right)^{3/5},
\]

where \(f_p = \sqrt{3} q B \tau_4 \gamma_m c^2\) is the comoving synchrotron flux. Substituting for \(\gamma_c, \Gamma, B, \text{and } \tau\) using equations (77)–(80) we find

\[
v_a^{\text{ssc}} \sim 2.2 \times 10^5 \nu_i^{1/10} f_p^{1/15} t_4^{1/15} \gamma_m^{4/15} (1 + Y)^{-1/15} (1 + z)^{2/15} d_{18}^{1/3} A_{2p}^{-4/15} \text{eV}
\]

which is very insensitive to \(\alpha\) of the observed quantities and for a wide range of observables \(v_a^{\text{ssc}} \sim 100\) keV which is too large to produce an SSC spectrum with \(f_a^{\text{ssc}} \propto v^{1/3}\) below the peak. We now try relaxing one of the constraints we had imposed to simplify the analytical calculation, i.e. \(\gamma_c \sim \gamma\). This approximation was guided by the observational result that the observed spectrum for \(v > v_c\) is almost always \(\sim v^{-1.5}\) for GRBs with \(\alpha = 1/3\). This result suggests \(\gamma_c \sim \gamma\), provided that \(p \approx 3\). However, if the electron distribution is steeper, \(p \approx 4\), then \(v_c\) can be much greater than \(v_c\), and a high-energy spectrum of \(v^{-p-1/2} \sim v^{-1.5}\) would be consistent with observations. We now investigate this possibility and determine if letting \(v_c > v_c\) would allow for a smaller \(v_a^{\text{ssc}}\) and hence \(\alpha = 1/3\) solutions. Note that the opposite arrangement of frequencies (\(v_c < v_c\)) is uninteresting, since the low-energy index is \(-1/2\) in this case.

For \(v_c > v_c\), equation (68) is modified to read

\[
\gamma_c B^2 \Gamma = \frac{7.7 \times 10^7 (1+z)}{\eta_{ic}(1+Y)},
\]

where \(\eta_{ic} = \gamma_c/\gamma\). We also need to use the appropriate expression for \(Y\) when \(\gamma_c \gg \gamma\), and \(p > 3\) (see equation 6)

\[
Y = \frac{4}{3} \frac{(p-1)}{(p-2)(p-3)} \frac{\gamma_c^2}{\eta_{ic}^2},
\]

which apart from a factor of \((p-3)\) is same as equation (71). We solve the above two equations together with equations (70) and (71) to find that \(v_c^{\text{ssc}} \propto \gamma_a^{-1/5};\) so \(v_c^{\text{ssc}}\) does not decrease by much even if we take \(\gamma_c\) to be larger than \(\gamma\) by many orders of magnitude, and therefore there are no SSC solutions with low-energy spectrum of \(v^{1/3}\) between \(\sim 15\) and 200 keV.

The above analytical calculation is based on a number of approximations for the SSC spectrum and flux. We check the validity of analytical results using numerical calculations and by searching the 5D parameter space for SSC solutions with low-energy index \(\alpha \approx 1/3\). It turns out that numerically also we find no solutions – \(v_c^{\text{ssc}}\) is indeed too high to produce a GRB with a low-energy spectrum of \(v^{1/3}\) in the \(\sim 15\)–200 keV band.

The only other possibility is that the \(\alpha \approx 1/3\) index is transitory, i.e. the spectrum is changing continuously from \(\alpha \approx 1\) at \(v \sim 20\) keV to \(\alpha \approx -1\) at \(v = v_c\) and that \(\alpha \approx 1/3\) is realized at some intermediate frequency. This might, however, pose a problem for those GRBs with \(v_c \gg 100\) keV, since the spectrum would be steeper than \(v^{1/3}\) near 15 keV and therefore a Band function fit to the spectrum will yield \(\alpha > 1/3\).

### 4.1.2 SSC solutions: spectral index \(1/3 < \alpha \leq 1\)

The analytical solution for this case is similar to the SSC \(\alpha = 1/3\) case analysed in Section 4.1.1. We take \(\gamma_c \sim \gamma\), in order that the high-energy spectrum is \(\propto v^{-4/3} \sim v^{-1.3}\). The equations we solve are for \(v_c, \gamma_c\), Compton Y and \(\gamma_c \sim \gamma\):

\[
B^2 \Gamma \gamma_c \approx 7.7 \times 10^7 (1+z) \gamma_a^{-1}(1+Y)^{-1},
\]

\[
B \gamma_c^4 \Gamma \approx 2.3 \times 10^5 \nu_i(1+z) \eta_{ic}^{-2},
\]

\[
B \gamma_c^4 \gamma_c^2 \approx 1.6 \times 10^6 \nu_i \tau_c^{-2}(1+z) \nu_i A_{2p}^{-1/2} \eta_{ic}^{-2},
\]

\[
\tau \gamma_c^2 \approx \frac{3}{4} \frac{Y}{\frac{p-1}{p-2}} \frac{1}{(1+z) A_{2p}^{-1/2} \eta_{ic}^{-1}}
\]

where \(\eta_{ic} = \gamma_c/\gamma_i\). Since \(v_c \gg v_c, v_c\). The above equations are solved in the exact same way that we solved them in Section 4.1.1, and we find that the solutions are:

\[
\gamma_c \approx 84 v^{1/10} f_p^{1/15} t_4^{1/15} \nu_i^{1/15} Y^{1/15} (1 + Y)^{1/15} d_{18}^{-1/18} (1 + z)^{1/18} A_{2p}^{-1/18} \eta_{ic}^{-2/18},
\]

\[
B \approx 200 v^{1/10} f_p^{1/15} t_4^{1/15} \nu_i^{1/15} Y^{1/15} (1 + Y)^{1/15} d_{18}^{-1/18} (1 + z)^{1/18} A_{2p}^{-1/18} \eta_{ic}^{-2/18},
\]

\[
\Gamma \approx 240 v^{1/10} f_p^{1/15} t_4^{1/15} \nu_i^{1/15} Y^{1/15} (1 + Y)^{1/15} d_{18}^{-1/18} (1 + z)^{1/18} A_{2p}^{-1/18} \eta_{ic}^{-2/18},
\]

\[
\tau \approx 1.1 \times 10^{-4} v^{1/10} f_p^{1/15} t_4^{1/15} \nu_i^{1/15} Y^{1/15} (1 + Y)^{1/15} d_{18}^{-1/18} (1 + z)^{1/18} A_{2p}^{-1/18} \eta_{ic}^{-2/18}.
\]
dependence of the X-ray flux on these quantities agree with equation (94), which gives $f \sim f_\gamma$ or $f / \Gamma_1 \eta$ in parameter space (see Fig. 9); we also find $\alpha \leq 4.1$.

Optical emission for $1/3$ SSC model of GRBs, is consistent with the observed data. The LFs of shells before collision (assuming that $\gamma_{s}$ solutions. The LFs of shells before collision (assuming that $\gamma$-rays are produced in the inner, faster, shell) is 300 $\leq \Gamma_1 \leq 5000$ and 100 $\leq \Gamma_2 \leq 1000$; the ratio $\Gamma_1 / \Gamma_2 > 2$ and the efficiency for producing $\gamma$-rays is $\geq 10$ per cent for the allowed 5D parameter space for SSC. The bulk LF of post-shock gas 100 $< \Gamma < 1000$ is compatible with late-time afterglow modelling.

We now calculate the X-ray and optical emissions accompanying the $\gamma$-ray pulse.

### 4.1.2a. X-ray emission for $1/3 < \alpha \leq 1$ SSC solutions

The 1 keV prompt emission from SSC and synchrotron processes is shown in the top two panels of Fig. 10. The contributions of SSC and synchrotron to 1-keV flux is shown separately in the top left-hand panel for SSC- and the sum of the two for a variety of GRB parameters can be found in the top right-hand panel.

The X-ray flux can be estimated analytically using the expression for synchrotron flux $f_x = f_{\gamma}(1 \text{ keV}/\nu_t)^{-p/2}$, since $\nu_t \sim \nu_c \sim \nu_a$; $f_x$ can also be expressed as $f_x \sim f_\gamma \tau^{-1} Y_1^{7/18} (25)^{p/2} v_{\gamma}^{p/2}$, or in terms of observable parameters

$$f_x \sim \frac{9 \times 10^3}{\Gamma_p^p} Y_\eta^2 f_\gamma^{\frac{10 + p}{2}} \frac{\nu_{\gamma}^{1 + p}}{v_{\gamma}^{1 - p}} \frac{v_{\gamma}^{1 + p}}{(1 + Y)^{1 + p}} d_5^{1/2} A_{\nu_0}^{1/2} A_{\eta_0}^{1/2} \eta_\eta^{1/2} \text{ mJy,} \quad \text{(94)}$$

or $f_x \sim 9 Y^{-16/35} (1 + Y)^{-2/5}$ mJy for GRB- with $\eta_\eta > 1, p = 3.2$ and $t_\eta = t_{\gamma}$; this is roughly consistent with the numerically calculated flux shown in Fig. 10.

In the top right-hand panel of Fig. 10, the total 1 keV prompt flux is shown for a number of different values of $v_{\gamma}, f_\gamma$, and $t_{\gamma}$. The dependence of the X-ray flux on these quantities agree with equation (94), which gives $f_x \propto v_{\gamma}^{13/10} f_\gamma^{3/10} t_{\gamma}^{-1/15}$ for $p = 3.2$: an increase in $v_{\gamma}$ or $f_\gamma$ leads to an increase of $f_x$, and an increase in $t_{\gamma}$ has little effect on the X-ray flux (Fig. 10). The 1-keV flux for all of these cases ranges from 0.01 to 10 mJy during the burst; the flux at 100 s, the time when the X-ray telescope aboard the Swift satellite starts looking at the burst, would be smaller by a factor of $\sim 10^{-10}$ depending on GRB pulse duration. So, the X-ray flux accompanying the $\gamma$-ray radiation, for the SSC model of GRBs, is consistent with the observed data.

### 4.1.2b. Optical emission for $1/3 < \alpha \leq 1$ SSC solutions

The prompt optical flux accompanying $\gamma$-rays, in the SSC model, is shown in the bottom left-hand panel of Fig. 10. Analytically we find the prompt $R$-band flux due to the synchrotron component underlying the SSC model to be $f_R \sim f_\gamma \sim f_\gamma \tau^{-1}$, or

$$f_R \sim 9 \times 10^{13/2} f_\gamma^{17/18} v_{\gamma}^{19/2} Y^{-8/9} (1 + Y)^{2/9} d_5^{1/2} (1 + z)^{2/9} A_{\nu_0}^{1/2} A_{\eta_0}^{1/2} \eta_\eta^{1/2} \text{ mJy} \quad \text{(95)}$$

For $p = 3.2, t_\eta \sim t_{\gamma}, \eta_\eta \geq 1$, and parameters corresponding to GRB- we find from the above equations that $\Gamma \gtrsim 110 Y^{-7/18}, \nu_t \lesssim 100$ and $\tau \gtrsim 1.3 \times 10^{-4} Y^{3/4} (1 + Y)^{-5/9}$. These analytical results are roughly consistent with numerical determination of the allowed region in 5D parameter space (see Fig. 9); we also find $\eta_\eta \approx 1$ numerically.

The distance of the $\gamma$-ray source from the centre of the explosion is shown in Fig. 9 for various GRB parameters and is greater than $\sim 10^{14}$ cm for $Y \lesssim 10$. The ratio of magnetic to electron energy is small – $f_B / f_{\gamma} \leq 0.1$ for the entire solution space (Fig. 9) since $Y \gtrsim 1$.

The SSC solutions we have found can be related to the internal shock model. The relative LF of collision between shells – obtained from $\gamma_{s}$ (see equation 11 and Appendix A) – is found to be between 2 and 10, which is significantly less than what we were finding for synchrotron solutions. The results are found in the top right-hand panel.

The X-ray flux can be estimated analytically using the expression for synchrotron flux $f_x = f_{\gamma}(1 \text{ keV}/\nu_t)^{-p/2}$, since $\nu_t \sim \nu_c \sim \nu_a$; $f_x$ can also be expressed as $f_x \sim f_\gamma \tau^{-1} Y_1^{7/18} (25)^{p/2} v_{\gamma}^{p/2}$, or in terms of observable parameters

$$f_x \sim \frac{9 \times 10^3}{\Gamma_p^p} Y_\eta^2 f_\gamma^{\frac{10 + p}{2}} \frac{\nu_{\gamma}^{1 + p}}{v_{\gamma}^{1 - p}} \frac{v_{\gamma}^{1 + p}}{(1 + Y)^{1 + p}} d_5^{1/2} A_{\nu_0}^{1/2} A_{\eta_0}^{1/2} \eta_\eta^{1/2} \text{ mJy,} \quad \text{(94)}$$

or $f_x \sim 9 Y^{-16/35} (1 + Y)^{-2/5}$ mJy for GRB- with $\eta_\eta > 1, p = 3.2$ and $t_{\gamma} \sim t_\eta$; this is roughly consistent with the numerically calculated flux shown in Fig. 10.

In the top right-hand panel of Fig. 10, the total 1 keV prompt flux is shown for a number of different values of $v_{\gamma}, f_\gamma$, and $t_{\gamma}$. The dependence of the X-ray flux on these quantities agree with equation (94), which gives $f_x \propto v_{\gamma}^{13/10} f_\gamma^{3/10} t_{\gamma}^{-1/15}$ for $p = 3.2$: an increase in $v_{\gamma}$ or $f_\gamma$ leads to an increase of $f_x$, and an increase in $t_{\gamma}$ has little effect on the X-ray flux (Fig. 10). The 1-keV flux for all of these cases ranges from 0.01 to 10 mJy during the burst; the flux at 100 s, the time when the X-ray telescope aboard the Swift satellite starts looking at the burst, would be smaller by a factor of $\sim 10^{-10}$ depending on GRB pulse duration. So, the X-ray flux accompanying the $\gamma$-ray radiation, for the SSC model of GRBs, is consistent with the observed data.
which for GRB-\(\gamma\) reduces to \(f_R \approx (8 \times 10^7 \, \text{mJy}) \, Y^{-5/9} (1 + Y)^{2/9}\). There are, however, many numerical solutions corresponding to \(Y \gtrsim 10\) for which \(f_R \approx 70 \, \text{mJy}\). It turns out that \(v_a > 2 \, \text{eV}\) for the low radius solutions, up to a factor of 3. If \(v_L < 2 \, \text{eV} < v_a\), then the expression for optical flux is

\[ f_R \approx 50 v_{75}^{-1/4} f_Y^{20/36} t_4^{7/18} t_9^{7/9} (1 + Y)^{7/9} d_{23}^{-7/18} (1 + z)^{7/9} A_{36}^{11/18} \eta_a^{-(7/9+12/9)} \, \text{mJy} \]

(96)
giving \(f_R \lesssim 7 \, Y^{-11/18} (1 + Y)^{7/9} \, \text{mJy}\) for GRB-\(\gamma\).

The optical flux, obtained by numerical calculations, is shown in the bottom right-hand panel of Fig. 10, for several sets of \((v_L, f_Y, t_f)\).

The results are consistent with the dependences found in equations (95) or (96) when \(v_a > 2 \, \text{eV}\). The reason that the self-absorbed \(f_R\) increases with \(v_L\) numerically while equation (96) shows a decrease is that \(\eta_a \approx v_L^{-1/2}\) (confirmed numerically), and the huge dependence of \(f_R \propto \eta_a^{4/9}\) gives positive dependence of \(f_R \propto v_L^{4/9}\). The dependence of optical flux on the duration of the GRB pulse is due to the fact that longer pulses have larger \(R_F\) and \(v_L < 2 \, \text{eV}\). The range of optical flux for SSC solutions is between 0.01 mJy to a few times \(10^5\) mJy (96)

\(f_R \approx 3 - 6 \, \text{mag}\). There is an approximately linear relationship between \(f_R\) and \(R_F\). Solutions with \(R\) magnitude of above \(9\) mag (1 Jy) are most likely ruled out. In particular, this rules out the 1 s pulse duration solutions that have \(R_F \gtrsim 2 \times 10^{15} \, \text{cm}\). If the pulse width were \(10\) s, the SSC solutions would have prompt optical of between 1 and 4 Jy, or \(R \approx 7\) mag, which is too bright to have been missed in optical follow-up observations. We note that if GRB dissipation radius is \(\sim 10^{16} \, \text{cm}\) as found in Kumar et al. (2007), then bright optical flux of \(R > 9\) mag is expected in every GRB produced via SSC. Since this bright optical emission is not seen, this may pose major problems for the SSC process to produce GRBs with positive \(\alpha\).

If, however, electrons are accelerated multiple times during the course of a \(\gamma\)-ray pulse in the GRB LC, i.e. \(t_a \ll t_f\), the optical flux can be reduced significantly. The dependence \(f_R \propto \tau_a^{1/9}\) (equation 95) itself does not reduce \(f_R\) by much, but since \(v_a \propto \tau_a^{2/3} \approx t_a^{-0.3}\) as well, a smaller \(t_a\) gives a larger \(v_a\) and that reduces the optical flux by an additional factor of \(\sim t_a^{-0.6}\) and results in \(f_R \propto \tau_a^{1/9}\); for \(t_a \approx t_f/100\), the optical flux is reduced by a factor of \(\sim 10^2\) compared with the case where \(t_a \sim t_f\), in agreement with the numerical results found in the lower left-hand panel of Fig. 10. We have numerically searched the whole range of observable parameters and find that the scaling \(f_R \propto \tau_a^{1/9}\) is valid through the entire solution space, and even the highest optical flux levels of 10 Jy (for the \(t_f \sim 1\) s case) is reduced below 0.1 Jy, or \(R > 11\) mag, if \(t_a \lesssim t_f/10^2\). Multiple acceleration episodes for electrons is therefore, a possible way of reducing the excessive optical flux that otherwise necessarily accompanies SSC solutions for \(\gamma\)-ray emission.

The optical flux should peak at the same time the GRB LC peaks, since \(v_a\) is of the order of 2 eV. The temporal decay of optical flux in this case is dictated by the curvature or the off-axis emission (Kumar & Panaitescu 2000) – the optical LC should fall off as \(t^{-2a-\nu /2} \sim t^{-3.6}\) as long as \(v_a\) is below the \(R\) band. At first glance, it might seem that if \(v_a > 2\) eV, and the synchrotron spectrum \(\propto \nu^{-3/2}\) or \(v^2\) in the optical band, the optical LC would be flat \(\sim \nu^0\) or even rise as \(t^{1/2}\). This behaviour, however, lasts for a very short time since \(v_a/2\) eV \(\lesssim \eta\) a few, and \(v_a \propto t^{-1}\)
for off-axis emission; once $v_a$ drops below 2 eV the optical flux would start falling off as $\sim t^{-3.6}$. The upper limit of $V \sim 18.5$ mag (0.2 mJy) at 100 s for many Swift detected bursts (e.g. Roming et al. 2006) is a lot smaller than the flux expected during the burst for the SSC model. If the pulse occurs at $\sim 1$ s post-trigger then the optical flux at 100 s would be smaller than the prompt optical flux by a factor of $\sim 10^7$ and that is quite consistent with observational upper limits for almost the entire solution space for the SSC model.

We emphasize that a bright optical flash ($R \lesssim 14$ mag) concurrent with $\gamma$-ray emission is a generic prediction of the SSC model for GRBs with positive low-energy index. Bright, prompt, optical radiation has been reported for a few bursts with positive $\alpha$ – e.g. 990123 (Akerlof et al. 1999; Briggs et al. 1999), 061007 (Golenetskii et al. 2006; Yamaoka et al. 2006), and the second emission episodes of 050401 (Golenetskii et al. 2005) and 050820A (Cummings et al. 2005) – however, if future observations fail to detect prompt optical with $R \lesssim 14$ mag then that will suggest that one of the assumptions of the model developed in this work has to be abandoned – the most likely possibility, in our view, is to discard the assumption that $t_a \sim t_\gamma$ and that suggests that $\gamma$-rays are not generated in a shock-heated medium.

4.2 SSC solutions: negative low-energy spectral index

In this section we consider SSC solutions when the spectrum below $v_\gamma$, the peak of $v \gamma_i^\nu f_\nu$, is $f_\nu \propto \nu^\alpha$ with $\alpha < 0$. There are two different class of solutions in this case – those with the underlying seed synchrotron spectrum having $v_c < v_\gamma$ and vice versa. We treat the two cases separately analytically, but plot the numerical solutions for both cases together in Fig. 11. We use one vital piece of information gained from the numerical solutions to simplify our analytical calculations: the synchrotron self-absorption frequency, $v_a$, is larger than $\max(v_\gamma, v_c)$, and therefore $v_\gamma \sim 4v_a \max(v_\gamma, v_c)^2$; note that even though $v_a > \max(v_\gamma, v_c)$, the spectral index below $v_\gamma$ is negative down to the frequency $\sim v_a \min(v_\gamma, v_c)^2 < 10$ keV.

4.2.1 $v_c < v_\gamma$ case

The equations that are solved for this case can be cast in a form very similar to the SSC $\alpha \approx 1/3$ case (considered in Section 4.1.1) by introducing two variables: $\eta_i \equiv \gamma_i / \gamma_a$, and $\eta_a \equiv \gamma_a / \gamma_i$. The equations for $v_\gamma, v_\gamma, f_\nu$ and Compton $Y$ expressed in terms of $\eta_i$ and $\eta_a$ are

$$B^2 \Gamma \gamma_i \sim 7.7 \times 10^8 \eta_i (1+z) t_a^{-1} (1+Y)^{-1},$$

$$B^2 \gamma_i^2 \Gamma \sim 2.3 \times 10^{12} v_a (1+z) \eta_a^{-2},$$

$$B^{5/3} \tau_i^2 \sim 1.6 \times 10^6 f_\nu t_\gamma^{-2} (1+z) d_{28}^2, \eta_a^2 \eta_i^2,$$

$$\gamma_i \sim 10 \eta_a^{-1} \sim 10 \eta_i^{-1}$$

Figure 11. SSC solutions when the spectral index, $\alpha$, below $v_\gamma$ (the peak of $v \gamma_i^\nu f_\nu$) is less than 0. There are two branches of the solutions both of which are shown in the figure. For one of these branches $v_c < v_\gamma$ and for the other $v_c > v_\gamma$; $\gamma$-ray sources corresponding to the $v_c < v_\gamma$ branch lie at a smaller distance from the centre of explosion ($R_\gamma$) than the other branch. Allowed regions for the 5D parameter space is shown for a number of different sets of GRB observable parameters (see Fig. 2 caption for details). The top right-hand panel shows the LF of the $\gamma$-ray source (one of the five basic parameters we use to describe the source) as well as the $\Gamma_{rel}$ – the relative LF of collision of two shells obtained by mapping the 5D parameter solution to internal shocks (see Appendix A).
\[ \tau \gamma^2 \sim 0.75Y \left( \frac{p-1}{p-2} \right)^{-1} \eta_i. \]  

These equations are solved the same way as outlined in the previous section, and we find the solutions to be

\[ \gamma_i \sim 84 \nu_i^{1/4} P_i f_i^{-1/36} Y^{1/18} (1 + Y)^{1/9} \ell_{19}^{-1/18} Y_{18}^{1/9} \ell_{28}^{1/18} Y_{28}^{1/9} \eta_i^{-1/4} \eta_a^{7/18} \frac{\nu_i}{\nu_p} \gamma_a^{1/9} \gamma_p^{1/9} \eta_i^{7/18} \gamma_a^{1/9} \gamma_p^{1/9} \eta_i^{7/18}, \]

\[ B \sim 200 \nu_i^{1/4} P_i f_i^{-1/12} Y_{18}^{1/6} \ell_{19}^{-1/6} (1 + Y)^{-2/3} \gamma_a^{1/6} Y_{28}^{1/6} \eta_i^{-1/3} \eta_a^{1/3} \gamma_a^{1/3} \gamma_p^{1/3} \eta_i^{1/3} \gamma_a^{1/3} \gamma_p^{1/3} \eta_i^{1/3}, \]

\[ \Gamma \sim 240 \nu_i^{1/4} P_i f_i^{1/36} Y_{18}^{1/6} Y_{28}^{1/6} (1 + Y)^{2/9} \ell_{28}^{1/18} Y_{28}^{1/18} \eta_i^{-1/6} \eta_a^{1/18} \gamma_a^{1/18} \gamma_p^{1/18} \eta_i^{1/18} \gamma_a^{1/18} \gamma_p^{1/18} \eta_i^{1/18}, \]

\[ \tau \sim 1.1 \times 10^{-9} \nu_i^{1/4} P_i f_i Y_{18}^{1/6} Y_{28}^{1/6} Y_{28}^{1/6} (1 + Y)^{-2/9} \ell_{28}^{1/18} (1 + z)^{-2/9} \gamma_a^{1/18} \gamma_p^{1/18} \eta_i^{1/18} \gamma_a^{1/18} \gamma_p^{1/18} \eta_i^{1/18}. \]

The dependence of \( B, \Gamma, \gamma, \) and \( \tau \) on the observables is same as in equations (77–80) – the difference is in the dependence on \( \eta_i \) and \( \eta_a \).

The distance of the \( \gamma \)-ray source from the centre of the explosion is \( R_v \propto t_v \Gamma^2 \propto \eta_i^{-3/4} \eta_a^{1/4} \gamma_a^{1/4} \gamma_p^{1/4} \eta_i^{1/4} \gamma_a^{1/4} \gamma_p^{1/4} \eta_i^{1/4}. \) We use the constraint that \( v_i^{\text{keV}} \sim 4 \nu_i \gamma_i^2 \sim 10^3 \nu_i \gamma_i^{-2} < 10 \text{ keV} \) to infer that \( \gamma_i \gtrsim 3.2 \) for these solutions; we numerically find that \( \gamma_i \gtrsim 10 \).

Numerical calculations of the allowed region in the 5D parameter space for \( \text{GRB-z} \) give \( \gamma_i \gtrsim 50, B \gtrsim 200 \text{ G}, \Gamma \lesssim 300 \) and \( \tau \gtrsim 3 \times 10^{-3}. \)

There is good agreement between the numerical and analytical \( \gamma_i \) and \( \Gamma \) solutions (see Fig. 11), although analytical and numerical solutions for \( B, \tau \) and \( R_v \) can differ by a factor of a few due to the difference of a factor of a few in the analytical and numerical value of \( \eta_i \).

In Fig. 11, we show the numerical solutions for the allowed region in the 5D space for the SSC model for a wider range of observables. Note that results for both \( v_i > v_p \) and \( v_i < v_c \) (to be considered analytically in the next subsection) are plotted together in Fig. 11. The allowed range for electron LF \( \gamma_i \) is 3–200 (Fig. 11) which is characteristic of mildly relativistic shocks. If we cast the 5D parameter solutions in terms of colliding shells as described in Appendix A, we find the relative LF of collision between shells to be less than a few. For \( \Gamma \gtrsim 100 \) and \( \Gamma_{\text{rel}} \sim 1 \), there is little chance of an external forward-shock origin for these SSC photons. However, the 5D solutions we find appear to be consistent with an internal shock; \( R_v \) is smaller than the deceleration radius and \( f_{\text{keV}} \lesssim 1 \) for the entire SSC solution space.

The dependence of the solution space on \( v_i, f_p \) and \( t_p \) is in agreement with analytical estimates; for example, we have verified the analytical dependence, \( \gamma_i \propto v_i^{1/4} P_i^{1/36} Y_{18}^{1/6} \) (for \( t_i = t_p \)), given in equation (101), and \( \Gamma \propto v_i^{1/4} P_i^{1/36} Y_{18}^{1/6} \) (equation 103). We find that SSC can produce a wide variety of GRBs with the low-energy spectral index \( \alpha \sim -1/2 \) without requiring extreme parameter values.

In order for these solutions to remain viable, the prompt optical and X-ray fluxes need to be in accord with observations and/or upper limits. We next look at the prompt X-ray and optical emission from these solutions.

### 4.2.1a. Prompt X-ray emission for SSC solutions with \( \alpha \lesssim -1/2 \)

In the top two panels of Fig. 12, we have plotted the synchrotron and SSC contributions to the X-ray (1-keV) flux accompanying the \( \gamma \)-ray emission during the burst. In the top-left hand panel, we see that the 1-keV flux ranges from 0.1 to over 100 mJy for \( \text{GRB-z} \) and much of it is due to the underlying synchrotron emission.

There are at least a few solutions for each value of \( v_i, f_p \) and \( t_p \) we have considered with X-ray flux less than 10 mJy (Fig. 12) with the exception of the \( f_p \sim 10 \text{ mJy} \) case. The high end of this range of X-ray flux is above the value typically observed by \( \text{Swift} \) at 100 s following the burst (10-6 to a few mJy); but we know from early X-ray observations that this flux is initially falling off very steeply, \( \sim t^{-3} \), and therefore X-ray flux of \( \sim 10^5 \) mJy during the burst, \( t_p \lesssim 10 \text{ s} \), would be less than 1 mJy at 100 s, or within the observed flux range of the X-ray telescope aboard \( \text{Swift} \).

We expect the X-ray flux to be

\[ f_x = f_{\nu_x} \left( \frac{1 \text{ keV}}{\nu_x} \right)^{-p/2} \left( \frac{v_i}{v_p} \right)^{-1/2} \]

(since synchrotron dominates). In terms of the observable quantities the flux is

\[ f_x \approx 9 \times 10^3 \nu_x^{p/2} f_x^{3/2} Y_{18}^{2/3} \ell_{28}^{1/3} Y_{28}^{1/3} (1 + Y)^{2/3} \ell_{28}^{1/3} Y_{28}^{1/3} \eta_i^{-2/3} \gamma_a^{-2/3} \gamma_p^{-2/3} \eta_i^{2/3} \gamma_a^{2/3} \gamma_p^{2/3} \text{ mJy,} \]

so for \( p = 2.5 \) and \( t_i \approx t_p f_i \approx \nu_i^{9/8} f_i^{3/72} t_i^{-1/12} \). This analytical formula is in good agreement with numerical solutions shown in Fig. 12.

The upper limit to the synchrotron flux can be understood from the limit we place on the synchrotron 10-keV flux. We restrict synchrotron flux at 10 keV to be less than the 10-keV SSC flux, in order that the GRB spectrum is SSC dominated. The 1-keV synchrotron flux is then \( \lesssim f_{\nu_x} (10 \text{ keV}/\nu_x)^{p-1/2} \sim 55 \text{ mJy for } \text{GRB-z}, \) in agreement with Fig. 12.

### 4.2.1b. Prompt optical emission for SSC solutions with \( \alpha \lesssim -1/2 \)

In the bottom two panels of Fig. 12, we plot the optical emission accompanying the \( \gamma \)-ray radiation for the SSC solutions. In the optical (2 eV), we look only at the synchrotron emission, since the SSC flux is highly suppressed as \( v_x \gg 2 \text{ eV} \). The optical flux is between 0.2 and 100 mJy, or between \( R \sim 18 \) and 11 mag for \( \text{GRB-z} \) (Fig. 12). For the range of \( v_i, f_p \) and \( t_p \) we have considered, we find the optical flux to be between 0.02 mJy and 50 Jy, or \( R \sim 20–40 \text{ mag} \) (bottom right-hand panel of Fig. 12). Observational limits on \( R \) band flux, 100 s after the burst, are \( \gtrsim 18.5 \text{ mag} \), or \( \lesssim 0.2 \text{ mJy} \), for approximately half of \( \text{Swift} \) bursts (Roming et al. 2006). For small values of \( v_i, f_p \) and \( t_p \) there is no problem satisfying this upper limit, but large \( v_i, f_p \) or \( t_p \) might exceed the observed optical flux limit. Since the prompt optical flux falls off...
Since the ratio $f_{R}$ numerically is 1.08, and therefore the analytical expression for

If we assume that $\eta$ not accelerated just once, but multiple times, during the course of a pulse duration of $t_{\gamma}$, we have a population of electrons that do not radiate synchrotron emission and we need to reduce the number of radiating particles. This is done by using $f_{\gamma}$ and $t_{\gamma}$, but not $v_{r}$. Numerically, $f_{\gamma}$ increases with $v_{r}$, while this expression shows a decrease. This difference is caused by the sensitivity of $f_{\gamma} \sim \eta_{a}^{-4}$ – numerically we find that $\eta_{a} \propto v_{r}^{-1/2}$, changing the above dependence on $v_{r}$ to be $f_{\gamma} \propto v_{r}^{5/2}$, in accord with results shown in Fig. 12.

For GRB-5672, $f_{\gamma} \lesssim 120 \gamma^{-11/18} (1 + Y)^{1/9}$. This estimation for $f_{\gamma}$ is larger by a factor of $\sim 10$ than what we find numerically (see Fig. 12). This factor of 10 difference is due to the fact that $y_{c} \sim 1$ for many of these solutions. When $y_{c} < 2$, we have a population of electrons that do not radiate synchrotron emission and we need to reduce the number of radiating particles. This is done by using $v_{r} \propto (y_{c} - 1)^{2}$. Since $f_{\gamma} \propto v_{r}^{5/2}$, the correct value of $f_{\gamma}$ is a factor of $y_{c}/(y_{c} - 1)$ smaller than the crude analytical estimate above. The smallest value of $y_{c}$ that we find numerically is 1.08, and therefore the analytical expression for $f_{\gamma}$ overestimates the true flux, calculated numerically, by a factor of about $\sim 14$. This indeed reconciles the analytical and numerical results.

The $R$-band flux increases with increasing $t_{\gamma}$. This is because increasing the pulse duration increases the radius at which the GRB emission is produced, and the synchrotron self-absorption frequency is smaller at larger radii, and therefore brighter optical flux is observed. Thus, a prediction of the SSC model is that brighter optical flux accompanies wider GRB pulses – very spiky LCs (with short variability time-scale) will have small optical flux that can escape detection, but those with wide pulses should have bright early optical afterglows. If a pulse duration were to be 10 s, we should expect prompt optical flux of 100 mJy or larger ($R$-band magnitude smaller than 11th). If this optical emission is not detected, it will point to one of the assumptions in our model for the SSC emission being violated – most likely $t_{\gamma} \neq t_{\gamma}$, i.e. electrons are not accelerated just once, but multiple times, during the course of a pulse duration of $t_{\gamma}$.

### 4.2.2 $v_{r} < v_{c}$ case

Analytically this case is very similar to the case of $v_{r} < v_{c}$ discussed in Section 4.2.1. We require this time that $y_{c} / y_{r} = \eta_{r} < 1$ and $\eta_{a}$ is still $> 1$. Since the ratio $y_{a} / y_{c} = \eta_{a} \eta_{r}$, we will see that some of the solutions can be expressed in terms of $\eta_{a} \eta_{r}$ instead of using $\eta_{a}$ and $\eta_{r}$ separately.
The equations that we are solving for, in this case are

\[ B^2 \Gamma Y \sim 7.7 \times 10^8 \eta_i (1 + z) \eta_i^{-1} (1 + Y)^{-1}, \]

\[ B \gamma_i^4 \Gamma \sim 2.3 \times 10^{12} v_{\gamma i} (1 + z) \eta_i^{-2} \eta_i^2, \]

\[ B \gamma_i^4 \tau \sim 1.6 \times 10^6 f_\gamma^{-2} (1 + z) d_{L_{28}}^{-1} \eta_i^{-2}; \]

\[ \tau_{\gamma i}^2 \sim 0.75Y \left( \frac{p - 1}{p - 2} (3 - p) \right)^{-1} \eta_i^{-3} \gamma_i. \]

These equations are almost identical to the SSC \( v^4/\gamma \) case of Section 4.1.1, with exception of the dependence on the variables \( \eta_i \) and \( \eta_\gamma \) and a slight change in the \( Y \) expression with an additional factor of \( (3 - p) \). These equations are solved same way as outlined in Section 4.1.1. We find that the solutions have the same dependence on \( v_i, f_\gamma \) and \( t_\gamma \) as in equations (101)–(104). And the dependence on \( \eta_i \) and \( \eta_\gamma \) are: \( \gamma_i \propto \eta_i^{-6(p + 18)/36}, \eta_\gamma \propto \eta_\gamma^{-6(p - 1)/36}, \Gamma \propto \eta_i^{-7p/18} \) and \( \tau \propto \eta_i^{-7p/18} \). The distance of \( \gamma \)-ray source \( R_\gamma \propto \Gamma^2 \propto \eta_i^{(7p - 18)/18} \).

We constrain \( \eta_i \) by requiring \( v_i^2 \sim 4v_i \gamma_i^2 \) or \( \eta_i \lesssim 10 \) keV, which suggests that \( \eta_i \lesssim 0.3 \), in accord with the numerical calculation.

Numerically, we find \( \gamma_i \lesssim 20, B \lesssim 2 \times 10^3 \) G, \( \Gamma \lesssim 10^3 \) and \( \tau \gtrsim 10^{-3} \) for \( \text{GRB}\text{--}2 \) (Fig. 11). The \( \gamma \)-ray source radius \( R_\gamma \lesssim 2 \times 10^{15} \gamma^{-7/9} \left( 1 + Y \right)^{2/9} \) cm is in good agreement with \( R_\gamma \lesssim 5 \times 10^{15} \) cm obtained by numerical calculations. Note that \( R_\gamma \) for this case is a factor of a few higher than the \( v_i < v_c \) case discussed in Section 4.2.1.

The numerical results for the allowed region of 5D space, for \( v_i > v_c \), are also shown in Fig. 11. The solutions corresponding to \( v_i > v_c \) are those at larger radius – the right-hand side of each solution contour, or bubble, shown in the figure. These solutions have smaller \( \gamma_i \), larger \( \Gamma \), smaller \( B \) and \( \tau \), and a little bit smaller \( Y \). Since these optical flux too seem viable for shock models, we explore below the X-ray and optical flux accompanying \( \gamma \)-ray emission.

4.2.2a. Prompt X-ray and optical flux for SCC solutions with \( \alpha \lesssim \eta_\gamma \)

The analytical expression for the X-ray flux is almost identical to that found for the \( v_i < v_c \) case (equation 106) – the only difference is in the dependence on \( \eta_\alpha \) and \( \eta_i \). We find that

\[ f_\alpha \propto (\eta_i/\eta_\gamma)^{(p - 2)(p + 18)/36}, \]

and the lower limit on the X-ray flux is \( \sim 50Y^{-37/36} \left( 1 + Y \right)^{-1/18} \) mJy; this is in agreement with numerical results shown in Fig. (12).

The \( R \)-band flux, dominated by synchrotron emission, is

\[ f_R \sim \left( \frac{2eV}{v_i} \right)^2 \left( \frac{v_\gamma}{v_i} \right) \eta_i^{-p/2} \left( \frac{v_c}{v_i} \right)^{-(p - 1)/2} f_\gamma, \]

which is, in terms of the observed quantities, given by

\[ f_R \sim 3 \times 10^3 v_{\gamma i}^{-1/2} f_\gamma^{5/2} v_i^{1/2} \left( 1 + Y \right)^{-2/3} d_{L_{28}}^{-1/2} (1 + z)^{2/3} A_{Lp}^{-1/3} (\eta_i/\eta_\gamma)^{-p/2} \]

with the value of \( f_R \lesssim 1700 \) mJy for \( \text{GRB}\text{--}2 \). This is higher than the numerical results that give \( f_R \lesssim 200 \) due to the sensitivity of \( f_R \) on \( \eta_i, \eta_\gamma \gtrsim 2 \). If we assume that \( t_\gamma \sim t_\gamma \), as has been done numerically, we find \( f_R \propto v_i^{-1/2} f_\gamma^{5/2} \). Numerically, we find that \( f_R \) increases with \( v_i, f_\gamma \) and \( t_\gamma \), and is most sensitive to \( t_\gamma \). The increase of \( f_R \) with \( v_i \) is again due to the high sensitivity on \( \eta_i, \eta_\gamma \) which numerically we find is \( \alpha \propto v_i^{-1/2} \).

We note that the optical flux for this case \( \gamma = \eta_\gamma \) is larger than when \( v_i < v_c \) \(-10^3 \) mJy – since \( R_\gamma \) is larger for these solutions. For \( t_\gamma /t_\gamma = 10^{-2} f_R \) is reduced by a factor of \( \sim 50 \). This brings the highest flux for the \( t_\gamma \sim 1 \) s solutions down to about 140 mJy – still a bit bright, but there are many other solutions for this case that have flux smaller than about 10 mJy, and probably in accord with observations and upper limits. Thus, bright optical flux is a generic prediction of the SSC model for \( \gamma \)-ray generation whether the low-energy spectral index is positive or negative; the optical is particularly bright for \( \alpha > 0 \). This is a problem for the SSC model if \( R_\gamma \sim 10^{16} \) cm, as found in Kumar et al. (2007), as the brightest optical flux is produced at the larger \( R_\gamma \). One of the, possibly only, ways to avoid bright, prompt, optical emission is if \( t_\gamma \ll t_\gamma \).

5 GEV PHOTON SIGNAL FOR SYNCHROTRON AND SSC SOLUTIONS

Detection of GeV photons by the Gamma-ray Large Area Space Telescope (GLAST) (McEnery & GLAST Mission Team 2006) will be a useful piece of evidence to use to determine if GRBs are produced by synchrotron or SSC emission mechanisms. The IC scattering of \( \gamma \)-ray photons produced by synchrotron will peak above the GLAST band, \( \gg 100 \) GeV, while the second IC scattered SSC photons will peak at \( \sim 1 \) GeV.

For the synchrotron cases \( \alpha = 1/3 \) and \(-p - 1/2, \) max(\( \gamma_i, \gamma_\gamma \)) \( \sim 10^4 \) and \( \tau \lesssim 10^{-6} \), giving the peak of IC scattered flux \( v f_\gamma \) at \( v_\gamma \sim \min(v_\gamma, \gamma_\gamma), \) max(\( v_\gamma, \gamma_\gamma \)) \( \sim 10^4 \) GeV and flux \( v f_\gamma \sim \min(v_\gamma, \gamma_\gamma) \gamma_\gamma \sim 10^{-7} \) erg s\(^{-1}\) cm\(^{-2}\). We need to ensure that such high-energy photons can escape the source region and are not converted to electron–positron pairs. This effect is incorporated in our numerical calculations and is discussed below. Moreover, photons of energy larger than \( \sim 1 \) TeV are converted to \( e^\pm \) by collision with infrared photons and therefore we would not see \( \gtrsim 1 \) TeV photons from GRBs at cosmological distances.

The SSC $\alpha > 0$ solutions have $\gamma_1 \approx \gamma_c \lesssim 10^2$ and $\tau \gtrsim 10^{-4}$ giving the second scattering peak of $v_G \sim 4 v_\gamma \gamma_c^2 \sim 1$ GeV and the flux $v_f \sim 10^{-7}$ erg s$^{-1}$ cm$^{-2}$. The SSC $\alpha < 0$ solutions have very similar $v_G$ and $v_f$. The SSC signals are well above the GLAST threshold, so we expect to see a GeV signal from GRBs produced by the SSC process. For synchrotron solutions, however, the IC flux might be below the GLAST threshold. The spectral shape will also be different -- $v_G$ is well above the GLAST band for the synchrotron case, while SSC should peak at the low end of the GLAST band.

Using analytical results for the synchrotron $\alpha = 1/3$ case (equations 45–48), we find that the IC scattered signal peaks at

$$v_G^c \sim \gamma_1 \Gamma m_e c^2/(1 + z) \sim 2 \times 10^{43} \frac{v_f}{v_\gamma} \frac{t_\gamma}{t_a} \frac{1}{1 + Y} Y^{3/4} (1 + Y) Y_t d_{28} (1 + z) \frac{p}{p - 2} \frac{1}{(1 + \frac{p}{p - 2})^{3/4}} \text{ GeV.}$$

This is due to the Klein–Nishina reduction to cross-section above electron rest-frame photon energy of $m_e c^2$.

The frequency above which the high-energy spectrum is attenuated due to $\gamma \gamma \rightarrow e^- e^+$ within the GRB source is $v_\pm \sim (\Gamma m_e c^2)^2 v_{\nu^{-1}}$, where $v_{\nu^{-1}}$ is the frequency of the synchrotron photon at which the optical depth to pair production with $v_\pm$ photons is 1. The optical depth to pair production is $\tau_{\gamma \gamma} \sim \sigma_T n_e R_c / \Gamma$, where $n_e \sim L_{\text{iso}}(v)/4\pi c R_c^2$ is the comoving number density of photons between $v'$ and $2v'$, $R_c/\Gamma$ is the comoving shell width, and $L_{\text{iso}}(v)$ is an important isotropic luminosity per unit frequency. To find $v_{\nu^{-1}}$, we set $\tau_{\gamma \gamma} \sim 1$ and solve the equation using the observed $\gamma$-ray spectrum; $v_\pm$ is calculated from $\Gamma$ and $v_{\nu^{-1}}$. In terms of the observable parameters for the synchrotron $\alpha = 1/3$ case, we find

$$v_\pm \sim 4 \times 10^{43} \sqrt{v_f} \frac{1}{v_\gamma} \frac{t_\gamma}{t_a} Y Y_t d_{28} (1 + z) \frac{p}{p - 2} \frac{1}{(1 + \frac{p}{p - 2})^{3/4}} \text{ GeV.}$$

where we have assumed that the synchrotron GRB spectrum is $L_{\text{iso}}(v) \propto v^{-2}$ for $v > v_\gamma$. Since $v_\pm < v_G^c$ (calculated above), and the spectrum falls off very steeply above $v_\pm$, the IC spectrum will peak at $v_\pm$ for many of the synchrotron solutions.

The flux at $v_G^c$, with appropriate Klein–Nishina cross-section, is

$$[v_f f'_\gamma] ^\gamma \sim 8 \times 10^{-10} \sqrt{v_f} \frac{1}{v_\gamma} \frac{t_\gamma}{t_a} Y Y_t d_{28} (1 + z) \frac{p}{p - 2} \frac{1}{(1 + \frac{p}{p - 2})^{1/2}} \text{ erg s}^{-1} \text{ cm}^{-2}.$$

This flux is probably just below the GLAST threshold for detection. If $v_\pm < v_G^c$, the $v_f f'_\gamma$ spectrum peaks at $v_\pm$, and the flux at this frequency will be smaller than that in equation (118); the attenuation of $\gtrsim 1$ GeV photons as they propagate through the intergalactic medium would further reduce the observed flux. The results are very similar for the $\alpha = -(p - 1)/2$ case, since the $\alpha = 1/3$ case is a subset of the $\alpha = -(p - 1)/2$ solutions with $\gamma_1 \sim \gamma_c$.

For the SSC case, using the $\alpha > 0$ analytical results in equations (90)–(93), the second IC peak is

$$v_G^{\text{IC}} \sim 3 \frac{1}{v_\gamma} \frac{t_\gamma}{t_a} Y Y_t d_{28} (1 + z) \frac{p}{p - 2} \frac{1}{(1 + \frac{p}{p - 2})^{3/4}} \text{ GeV,}$$

and the flux at this peak is

$$[v_f f'_\gamma] ^\gamma \sim 7 \times 10^{-7} \sqrt{v_f} \sqrt{Y} A_{28} ^{-1} \text{ erg s}^{-1} \text{ cm}^{-2}.$$

$V_G$ is smaller than $v_G^c$ for the majority of the SSC solution space, so the second IC scattering spectrum will indeed peak at $v_G$. For the SSC $\alpha < 0$ case, the expressions are very similar -- the only difference is the dependence on $\eta_1$, $\eta_2$, and $p$:

$$v_G^{\text{IC}} \sim 3 \frac{1}{v_\gamma} \frac{t_\gamma}{t_a} Y Y_t d_{28} (1 + z) \frac{p}{p - 2} \frac{1}{(1 + \frac{p}{p - 2})^{3/4}} \text{ GeV,}$$

and the flux at this peak is

$$[v_f f'_\gamma] ^\gamma \sim 7 \times 10^{-7} \sqrt{v_f} \sqrt{Y} \eta_1 ^{\text{IC}} A_{28} ^{-1} \text{ erg s}^{-1} \text{ cm}^{-2}.$$

We show numerical results for $v_G$ and $v_f$ for four cases in Fig. 13 -- the two synchrotron cases of $\alpha = -(p - 1)/2$ and 1/3 and the SSC cases of $\alpha < 0$ and $> 0$. These numerical results are in rough agreement with our analytical estimations.

In summary, IC scattering of prompt $\gamma$-ray photons, when the GRB emission is produced via the synchrotron process, gives rise to a spectrum that peaks at $\sim \text{TeV}$, the flux is of the order of $10^{-8}$ erg cm$^{-2}$ s$^{-1}$ (Fig. 13), the spectrum below the peak is between $v_f^{1/3}$ or $v_f^{-0.6}$, and the spectrum above the peak is expected to be sharply cut-off due to pair production. On the other hand, if the GRB emission is produced via the SSC process then the spectrum of second IC scattered photons should peak at $\sim \text{GeV}$, with a flux of $\sim 10^{-6}$ erg cm$^{-2}$ s$^{-1}$ (Fig. 13), and the spectrum above the peak should be $f_c \propto v^{-1.5}$. This signal should be detected by GLAST. If GRBs are not detected in the GLAST band that would suggest that GRB-prompt emission is not generated via the SSC process. We note that the GLAST band flux cannot be reduced in the case of repeated acceleration of electrons during a single GRB pulse.

### 6 COMPARISON WITH PRIOR WORK ON $\gamma$-RAY GENERATION MECHANISM

We provide a brief comparison with published work on $\gamma$-ray generation in the internal shock model (Paczynski & Xu 1994; Rees & Meszaros 1994; Papathanassiou & Meszaros 1996; Meszaros & Rees 1997b; Sari & Piran 1997b) and the formalism/results of this paper. There is

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Modelling $\gamma$-ray burst prompt emission

Figure 13. Numerical results for the IC scattering of prompt $\gamma$-ray photons in the source. The four panels show the peak of IC spectrum ($\nu_G$) and the flux ($\nu f_\nu$) at the peak when the $\gamma$-ray emission is produced via the synchrotron process (top two panels) and via the SSC process (bottom two panels) for $\alpha > 0$ and $\alpha < 0$ cases; for all of these cases we took the underlying $\gamma$-ray spectrum with $\nu_{\gamma} = 100$ keV and $f_{\gamma} = 0.1$ mJy. The IC signal for many of the synchrotron solutions are affected by photon–photon pair production within the GRB source, and that is included in these numerical calculations; however, the conversion of $\gtrsim 1$ TeV photons to $e^\pm$ due to collision with infrared photons in the intergalactic medium is not shown in the figure (no photon above 1 TeV can reach us from a GRB source at cosmological distances). The effect of pair production is very small in the SSC cases. The Klein–Nishina cross-section has been used in these calculations, and it significantly affects the IC peak flux when $\gamma$-rays are generated via the synchrotron process.

a fairly extensive literature on this topic and this is not the place to provide a general review. What we wish to do is to describe the main difference between previous approaches and the work presented here.

We should note that $\gamma$-ray generation in the external shock model has also been looked at by a number of people e.g. Rees & Meszaros (1992), Meszaros & Rees (1993), Piran, Shemi & Narayan (1993), Dermer, Chiang & Botcher (1999), Dermer & Mitman (1999), McMahon, Kumar & Panaitescu (2004), Ramirez-Ruiz & Granot (2006); the issue of variability in external shocks is discussed in Sari & Piran (1997a), Dermer & Mitman (1999) and Nakar & Granot (2007). We do not have anything particularly enlightening to say regarding the external shock model that has not already been mentioned by one of these authors; the general problem with shocks is discussed in Sections 3 and 4.

The main difference between previous works and our approach is that previous works considered the forward problem i.e. starting with a parameterization of the properties of colliding shells and resulting shocks the emergent radiation field was calculated, whereas our approach is to start with the minimum number of physical parameters needed (five) to calculate the observed flux and spectrum – at one instant in time or for one pulse in a multipulse GRB LC – and determine these using the observed data. For synchrotron and IC radiations the parameters needed are $\gamma_i, \Gamma, B, N$ and $\tau$, which are determined by the observational data $\nu_{\gamma}, f_{\gamma}, t_{\gamma}$ and $\alpha$ for a pulse in GRB LC. The five parameters in turn are used to provide constraint on the nature of GRB source. The old forward approach is wedded to a particular model – either internal or external shock – whereas the method used in this work is relatively model independent.

The parameters of the internal shock model can be mapped into the five parameters ($\gamma_i, \Gamma, B, N, \tau$) in a straightforward manner – this in fact is done implicitly in all the forward approach papers in order to calculate the emergent radiation. The converse of this is not true; however, since the internal shock model has more than five independent parameters (Appendix A describes how to go from the five parameters to providing a limit on some of the internal shock parameters such as the initial LFs of colliding shells and their comoving densities).

Papathanassiou & Meszaros (1996) and Sari & Piran (1997b) carried out a fairly detailed analysis of prompt emission in the internal shock model. These authors addressed a set of questions such as the ability of synchrotron/SSC in internal shocks to produce a spectral peak near 100 keV, the observed flux in the $\gamma$-ray band, and short time-scale variability. Papathanassiou & Meszaros (1996) and Sari & Piran (1997b) realized that the cooling time-scale for electrons (compared to the dynamical time) for internal shocks is short and although Sari & Piran (1997b) do not explicitly say, their work applies to GRBs with $\alpha = -1/2$. Papathanassiou & Meszaros (1996) look at composite
synchrotron/SSC spectra; however, there are many free parameters and little comparison to observed properties of GRBs. Had these authors investigated the self-consistency of synchrotron internal-shock model for the case of $\alpha = 1/3$, they would have discovered the problem reported in this paper using their forward modelling approach.

Ghisellini, Celotti & Lazzati (2000) did in fact worry about synchrotron solutions when $\alpha > 0$ and concluded that it is impossible to account for it in shock based models (this is paraphrasing their actual wordings). They pointed out that electron cooling time, during prompt emission, is much smaller than the dynamical time [as was reported in Papathanassiou & Meszaros (1996) and Sari & Piran (1997b)] and therefore the GRB spectrum below the peak should be always $\nu^{-1/2}$ if the radiation is produced via synchrotron process. Ghisellini et al. (2000) considered the possibility that a lower strength magnetic field would avoid the excessive cooling of electrons, and rejected it based on the argument that IC emission would dominate in this case, i.e. $Y \gg 1$, and that IC spectrum too would be falling of as $\nu^{-1/2}$ or steeper due to IC cooling. We find that a smaller magnetic field can avoid excessive cooling, so that $\alpha = 1/3$, and at the same time Compton $Y \sim 1$. The reason for these different conclusions is that we do not impose any restriction on the source distance that forward calculations based on internal shocks do. The most serious problem with synchrotron $\alpha = 1/3$ case is that $R_\gamma > R_1$ unless $n_0 \ll 1\text{cm}^{-3}$ (see Section 3.2).

Ghisellini et al. (2000) correctly pointed out that re-acceleration of electrons, in shock based models, would not work because it requires too much energy; one has a continuous stream of electrons crossing the shock front – and to accelerate all of these electrons to their original energy distribution, so that $\alpha = 1/3$, while they are rapidly losing energy to radiation will indeed require much more energy than is available in the shock. The re-acceleration invoked in this work is not shock based. It in fact requires abandoning shock models and considering a scenario where particles are NOT being added to the ‘system’ – the source for $\gamma$-ray photons – continuously (as in shock based models) but where the source has a fixed number of particles that are being continuously accelerated; there is no excessive energy problem in this scenario.

7 DISCUSSION

In this paper we have investigated the generation of $\gamma$-rays in GRBs via synchrotron or synchrotron-self-inverse-Compton (SSC) emissions in a relativistic outflow. The SSC radiation from a relativistic source can be fully described by a set of five parameters ($\gamma_\text{r}$, $\Gamma$, $B$, $N$, $\tau$); see Table 1 for definition of symbols used in the paper. For each possible low-energy spectral index, we have analytically and numerically determined the region of the 5D parameter space for the synchrotron process. Allowing for the possibility of multiple electron acceleration episodes, i.e. $t_1 \ll t_2$, makes it possible to have smaller $\gamma_\text{r}$ in a relativistic outflow. The SSC radiation from a relativistic source can be fully described by a set of five parameters ($\gamma_\text{r}$, $\Gamma$, $B$, $N$, $\tau$). In this paper we have investigated the generation of $\gamma$-rays in GRBs via synchrotron or synchrotron-self-inverse-Compton (SSC) emissions in a relativistic outflow.

The SSC solution space for $\alpha = 1/3$ case is that $R_\gamma > R_1$ unless $n_0 \ll 1\text{cm}^{-3}$ (see Section 3.2). The SSC solution space for $\alpha = 1/3$ case is that $R_\gamma > R_1$ unless $n_0 \ll 1\text{cm}^{-3}$ (see Section 3.2). The SSC solution space for $\alpha = 1/3$ case is that $R_\gamma > R_1$ unless $n_0 \ll 1\text{cm}^{-3}$ (see Section 3.2). The SSC solution space for $\alpha = 1/3$ case is that $R_\gamma > R_1$ unless $n_0 \ll 1\text{cm}^{-3}$ (see Section 3.2). The SSC solution space for $\alpha = 1/3$ case is that $R_\gamma > R_1$ unless $n_0 \ll 1\text{cm}^{-3}$ (see Section 3.2).

We find that if $\gamma$-ray emission were to be produced via the synchrotron process, the required set of parameters and burst radius have extreme values that are not internally consistent and are in conflict with afterglow data. In particular, when the low-energy spectrum is $f_\nu \propto \nu^{1/3}$ or $\nu^{-(p-1)/2}$, the LF of the source is required to be larger than $10^3$, in disagreement with afterglow modelling, and the source distance ($R_\gamma$) is larger than the deceleration radius even when the density of the medium is as small as $\sim 0.1\text{cm}^{-3}$. The requirement on the magnetic field strength is also very stringent; the comoving field strength is required to lie in a very narrow range of about $10-30\text{G}$ in order to explain the radiation for a typical burst as synchrotron emission. Allowing for the possibility of multiple electron acceleration episodes, i.e. $t_1 \ll t_2$, alleviates the problem of large $R_\gamma$ and $\Gamma$; in this scenario the synchrotron process could account for the prompt $\gamma$-ray radiation for GRBs (see Sections 3.1 and 3.2) although $R_\gamma$ is still larger than the source distance determined for a subset of bursts detected by Swift (Kumar et al. 2007).

The reason that synchrotron solutions with $\alpha = -(p - 1)/2$ and $1/3$ require large $R_\gamma$ and $\Gamma$ is easy to understand. The number of electrons needed to produce the observed flux via the synchrotron process is $\sim 10^{53} f_\gamma/\Gamma(B^t)$. And in order to keep the Compton $Y$ parameter ($Y \sim \tau Y/\gamma_\text{r}$) less than $\sim 10 - 10$ otherwise most of the energy will come out in IC-scattered photons at $\nu \gg 1\text{GeV}$ – the source must have small $\Gamma$ or large $R_\gamma$. Moreover, since $t_\gamma \sim R_\gamma/(4c\Gamma^2)$, large $R_\gamma$ solutions also have large $\Gamma$ for a given GRB pulse width of $t_\gamma$. The reason that $t_\gamma \ll t_\gamma$ offers a way out of this problem is also easy to understand. Frequent re-acceleration of charge particles makes it possible to have larger magnetic field while keeping $\nu_\gamma \gtrsim 100\text{keV}$. This decreases the number of particles required to produce the observed $\gamma$ flux ($f_\gamma$), and that in turn makes it possible to have a smaller $R_\gamma$.

For $f_\gamma \propto \nu^{1/2}$, the allowed region of the 5D parameter space for the synchrotron process is quite reasonable. However, interpreting these solutions in terms of the internal shock model requires the ratio of LFs of the two colliding shells to be rather large ($\gtrsim 10-20$) when the ratio of magnetic energy to electron energy is $\lesssim 10$, i.e. if we want the energy fraction in electrons to be not too small – otherwise $\gamma$-ray production would be very inefficient (see Section 3.3).

The SSC process provides viable solutions for the prompt emission of a large fraction of GRBs. We have considered almost all different possibilities of the low-energy spectrum for GRBs: $f_\nu \propto \nu^p$ below the peak of $\nu f_\nu$ with $-1 \lesssim \alpha \lesssim 1$. The solution space (a hypersurface in the 5D parameter space) is quite large for $\alpha < 0$ and $0.5 \lesssim \alpha \lesssim 1$. However, there are no SSC solutions when $\alpha \sim 1/3$; the reason is that the synchrotron characteristic and cooling frequencies should be equal ($\nu_i \sim \nu_\gamma$) in order that $\alpha = 1/3$, and in that case the synchrotron-self-absorption frequency is shown to be roughly equal to $\nu_i$ as well (see Section 4.1.1), and therefore the low-energy spectral index is $\sim 1$ and NOT $\sim 1/3$ as desired.

The SSC solution space for $\alpha < 0$ and $0.5 \lesssim \alpha \lesssim 1$ has source LF of the order of $100$, the minimum electron energy $\lesssim 10^3 m_e c^2$ (characteristic of mildly relativistic shocks), and $10^{14} \lesssim R_\gamma \lesssim 10^{16}\text{cm}$ is smaller than the deceleration radius. These solutions are accompanied by bright optical synchrotron flux of $\sim 10\text{mJy}$ (14 mag) to several hundred mJy for bursts at $z \sim 2$ – brightest for bursts with $\alpha \sim 1$ and those...
bursts with pulse duration of the order of $\gtrsim 1$ s. Moreover, the optical flux is correlated with $R_p$ – the flux is larger for larger $R_p$ – and for $R_p \gtrsim 2 \times 10^{15}$ (cf. Kumar et al. 2007), the optical flux is $\gtrsim 100$ mJy ($< 12$ mag). Bright optical flux contemporaneous with $\gamma$-rays is a prediction of the SSC model that is in conflict with prompt optical follow-up observations of a large number of bursts detected by Swift (Roming et al. 2008). We note that the large optical flux accompanying $\gamma$-rays can be reduced only if each radiating electron is accelerated numerous times ($\gtrsim 100$) in time period of the order of the duration of a pulse in the GRB LC, i.e. $t_s \lesssim t_p/10^2$. GRB models based on converting kinetic energy to radiation via shocks have $t_s \sim t_p$, where particles are accelerated at the shock-front and not downstream, but continuous particle acceleration might work for some alternate scenarios such as magnetic field reconnection/dissipation. Magnetic outflow model for GRBs has been proposed/investigated by a number of people cf. Usov (1992, 1994), Thompson (1994), Katz (1997), Meszaros & Rees (1997a), Wheeler et al. (2000) and Wheeler, Meier & Wilson (2002), Vlahakis & Konigl (2001), Spruit, Daigne & Drenkhahn (2001), Lyutikov & Blandford (2003) and Thompson (2006). However, the model has not been developed to the extend where it can be tested with GRB observations.

The data from the upcoming high-energy mission GLAST should be able to settle the question whether GRBs are produced via synchrotron or the SSC process (see Section 5); see Gupta & Zhang (2007), (Granot, Cohen-Tanugi & do Couto e Silva 2007) (and references therein) for recent work on how GLAST would help our understanding of GRBs.

To summarize our main conclusions, we find that the synchrotron process has serious difficulty accounting for the prompt emission in GRBs. The SSC offers reasonable solutions for all GRBs except those with spectral index of $\sim 1/3$ below the peak. SSC solutions predict very bright optical emission ($> 10$ mJy or 14 mag for $z \sim 2$) accompanying $\gamma$-ray LCs which is in conflict with a number of well-observed bursts. A possible solution to this problem might be to drop the assumption that $t_s \sim t_p$. The assumption of one shot acceleration of electrons, i.e. $t_s \sim t_p$, is motivated by shock based physics for GRBs and it may have to be replaced with an alternate scenario in which all electrons that radiate in the $\gamma$-ray band are accelerated continuously throughout the duration of a $\gamma$-ray pulse. In that scenario there are viable synchrotron solutions when $\alpha \sim 1/3$ – a case that otherwise cannot be explained by the SSC mechanism.

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$^7$ The simultaneous optical and $\gamma$-ray observations for a few bursts show bright optical flares. For example, GRB 041219a was observed in optical and IR simultaneously with $\gamma$-rays (Blake et al. 2005; Vestrand et al. 2005); the optical flux peaked at 13.7 mag at approximately the same time as the first main pulse of the GRB LC (as expected for the SSC model), and IR measurements Blake et al. (2005) show evidence of rapid variability with the last spikes in the GRB LC. GRB 990123, with a positive $\alpha$, had a peak optical flux during the burst of $\sim 1$ Jy (Akerlof et al. 1999); however, the optical LC was not correlated with the $\gamma$-ray LC.
APPENDIX A: DETERMINATION OF THE RELATIVE LORENTZ FACTOR OF COLLIDING SHELLS

We can relate solutions \((\gamma, \Gamma, B, N, \tau)\) we find for a set of observables to parameters for any model for GRBs. In this appendix we relate our solutions to the parameters for the internal shock model where \(\gamma\)-rays are produced by collision of two shells, as shown in Fig. A1. Internal...
The LF of unshocked inner shell, which is moving faster and lies a bit closer to the centre of explosion than the outer shell, can be determined from the addition of LFs $\Gamma_{\text{sh}}$ and $\Gamma$:

$$\Gamma_1 = \Gamma_{\text{sh}} \Gamma (1 + v_{\text{sh}}/v),$$

(A1)

provided that the $\gamma$-ray emission we observe is produced in the inner shell when it is shock heated ($v$ and $v_{\text{sh}}$ are speeds corresponding to LFs $\Gamma$ – the LF of the $\gamma$-ray source – and $\Gamma_{\text{sh}}$). On the other hand if it is the outer shell that produces the observed $\gamma$-ray emission then its LF factor, before it was shocked, is given by

$$\Gamma_2 = \Gamma_{\text{sh}} \Gamma (1 - v_{\text{sh}}/v).$$

(A2)

The LF of the shell that does not contribute significantly to the observed $\gamma$-ray emission cannot be determined uniquely. The problem is that in the absence of any observed emission that can be identified with this shell we cannot say anything directly about the LF of the shock front moving into this shell. However, we can still constrain its LF by requiring that the comoving density of this shell is such that the efficiency for thermal energy produced in the collision is no less than $\sim 10$ per cent (most GRBs for which we have good afterglow data and we can determine the kinetic energy of the ejecta show that the efficiency for converting kinetic energy to $\gamma$-rays is about 10 per cent or more).

The efficiency of $\gamma$-ray production is used to constrain $n_1/m_2$, and that in turn provides a lower limit for the relative LF $\Gamma_{\text{rel}}$. The fraction of kinetic energy converted to thermal energy when two shells of mass $m_1$ and $m_2$ collide with a relative LF of $\Gamma_{\text{rel}}$ is (Kumar 1999; Piran 1999)

$$f_t = 1 - \left[ 1 + \frac{2(m_2/m_1)(\Gamma_{\text{rel}} - 1)}{[1 + m_2/m_1]^2} \right]^{-1/2}.$$  

(A3)

For equal-mass shells a $\Gamma_{\text{rel}} = 1.5$ collision converts 10 per cent of the kinetic energy of shells to thermal energy, and for $\Gamma_{\text{rel}} > 5$ the conversion efficiency is more than 40 per cent. We need, however, to figure out the fraction of energy produced in a collision that is radiated in the typical $\gamma$-ray observing band of $15–400$ keV during the time interval $t_p$, in two shells that are not of equal mass.

The ratio of the mass of the two shells is the mass swept up by the two shocks in the time equal to the shock transit time for the shell that produces the observed $\gamma$-ray emission. This is given by

$$\frac{m_1}{m_2} = \frac{n_1(4\Gamma_{\text{sh}} + 3)v_{\text{sh}}}{n_2(4\Gamma_{\text{sh}} + 3)v_{\text{sh}}},$$

(A4)

where $n_1$ and $n_2$ are the densities of shells 1 and 2, $\Gamma_{\text{sh}}$ and $\Gamma_{\text{sh}}$ are the shock LFs in the frame of each unshocked shell (if the GRB emission is predominantly from shell 1, then $\Gamma_{\text{sh}} = \Gamma_{\text{sh}}$), and $v_{\text{sh}}$ and $v_{\text{sh}}$ are the speeds of the shock fronts with respect to each unshocked shell.

The shock front speeds are determined from the following cubic equation obtained from the continuity of energy, momentum and particle number fluxes across the shock front (Landau & Lifshitz 1980):

$$\Gamma_{\text{sh}}^4 + \left( 1 - \frac{2}{\alpha} \right) \Gamma_{\text{sh}} \Gamma_{\text{sh}}^2 - \left( 1 - \frac{1}{\alpha^2} \right) \Gamma_{\text{sh}} - \Gamma_{\text{sh}} \left( 1 - \frac{1}{\alpha} \right)^2 = 0,$$

(A5)

where $\Gamma_{\text{sh}}$ and $\Gamma_{\text{sh}}$ are the LF of the shock front with respect to the unshocked and shocked fluid, respectively, and $\alpha$ depends on the equation of state of the shocked gas and is approximated by

$$\alpha = \frac{4\Gamma_{\text{sh}} + 1}{\Gamma_{\text{sh}} + 1},$$

(A6)

which provides a smooth interpolation between the subrelativistic value for $\alpha = 5/2$ and the highly relativistic value of 4.

The shock LF with respect to the unshocked fluid in the shell not dominating the GRB emission ($\Gamma_{s2}$, if we assume that the GRB is being produced in shell 1) can be determined from the condition of pressure balance across the surface of discontinuity separating the shocked fluids in the two shells:

\[ n_1(4\Gamma_{s1}^4 + 3)(\Gamma_{s1} - 1) = n_2(4\Gamma_{s2}^4 + 3)(\Gamma_{s2} - 1). \] (A7)

For a given density ratio $n_1/n_2$ and $\Gamma_{s1}$ we can solve the above equation to determine $\Gamma_{s2}$, which in turn is used to determine shock front speed with respect to the unshocked fluid for the outer/inner shell using

\[ \Gamma_{s2} = \Gamma_{s1}/\left[1 - \frac{n_{s1} m_{s2}}{n_{s2} m_{s1}}\right], \] (A8)

and the cubic equation (A5) for $\Gamma_{s1}$. These pieces together give us $m_1/m_2$ and $\Gamma_{rel}$, which are used to calculate $f_1$ – the fraction of the kinetic energy of the two shells converted to thermal energy in shell collision (equation A3).

The ratio of the internal energy of the shocked gas in these two shells is

\[ \frac{E_1}{E_2} = \frac{n_1(4\Gamma_{s1}^4 + 3)(\Gamma_{s1} - 1)v_{s1}}{n_2(4\Gamma_{s2}^4 + 3)(\Gamma_{s2} - 1)v_{s2}}. \] (A9)

The fraction of the total internal energy of the shocked gas in shell ‘1’ is then $f_1 = \frac{E_1}{E_{GRB}}$. We assume that the majority of this energy is indeed being radiated in the GRB band. For shell ‘2’, $f_2 = 1 - f_1$, and we find the fraction of the total radiation contributing to the GRB band, $f_{GRB}$; $f_{GRB}$ is found from the ratio $v_{i1} f_{i1} (v_{i1}/v_{i2})/v_{i2} f_{i2} (v_{i2})$, where $v_{i1}/v_{i2} = (\gamma_{s1}/\gamma_{s2})^2$ (assuming that the magnetic field is equal in both shells). The radiation efficiency in the GRB band for the shell collision is then $f_{GRB} f_1 + f_2 f_{GRB}$.

When considering synchrotron radiation as the primary source of emission in the $\gamma$-ray band we need to take into account the energy fraction that is lost to very high-energy photons ($\nu > 200$ keV) produced via the IC process. The fraction of energy radiated via the synchrotron emission is $\nu(1 + \gamma)/\nu$, therefore, the total efficiency for energy production in shell collisions must be larger than the desired 10 per cent by a factor of $1 + Y$ – this effectively restricts solutions to $Y \lesssim 10$. $Y \gtrsim 0.1$ if IC emission is the main source for the observed $\gamma$-ray emission and also $Y$ must not be greater than $\sim 10^3$ otherwise most of the radiative energy will be in the second IC photons of much higher energy. In the same sense, we need to make sure the ratio of magnetic energy to that in electrons is $\lesssim 1$, in order for the electrons to radiate efficiently.

For highly relativistic forward and reverse shocks $v_{s1} \approx v_{s2} \approx 1/3$, $\Gamma_{s1} \approx \Gamma_{s2}(n_1/n_2)^{1/2}$, $\Gamma_{rel} \approx \Gamma_{s1} \Gamma_{s2}$, $m_1/m_2 \approx (n_1/n_2)^{1/2}$, and the ratio of the characteristic synchrotron frequency in shell 1 to shell 2 (assuming the same magnetic field strength) is $\sim n_2/m_1$. We note that the assumption of highly relativistic shocks is not valid for many solutions we find for the prompt $\gamma$-ray emissions, and that all the results reported in paper are obtained by numerically solving the appropriate equations. The numerically solved relationships of the ratio of the masses and injection frequencies in the two shells are shown in Fig. A2, and compared to the analytical estimates for highly relativistic shocks.
In summary, for a given $\Gamma_{\text{sh}}(\gamma_i)$ we vary $n_1/n_2$ and determine the mass ratio and the relative LF of the shell collision ($\Gamma_{\text{sh}}$) so that the $\gamma$-ray production efficiency is above a certain desired value (10 per cent). All of the numerical results we show for $\Gamma_{\text{sh}}$ were calculated using these steps. Using our upper limit on $\Gamma_{\text{sh}}$, we also calculate the expected emission from the shell 2 in the X-ray and optical bands, and include this in our analysis.

APPENDIX B: JITTER RADIATION PROCESS AND GRBS

A radiation mechanism, called jitter radiation, has been suggested by Medvedev (2000) as the process for $\gamma$-ray emission for those cases where the low-energy spectrum rises more steeply than $v^{1/3}$ expected of the synchrotron radiation (such spectra are said to lie ‘above the line of death’ because a non-self-absorbed synchrotron spectrum cannot have this steep of a rise). The jitter radiation is produced when the coherence length scale for magnetic field is short and electron trajectory is perturbed before it has travelled a distance of a Larmor radius. This is an attractive idea for explaining a class of GRBs lying above the line of death, and we explore its applicability to GRBs in this appendix.

The peak jitter frequency in lab frame (as seen by an observer at rest in the host galaxy) is

$$v_j = \sqrt{16\pi q^2 \Gamma_{\text{sh}} n_e / (m_e c^2) Y_p} \gamma_i^2 \Gamma,$$

where $q$ is electron charge, $\Gamma$ is the bulk LF of the source, $\gamma_i = \min\{\gamma_i, \gamma_c\}$, $\gamma_c$ is the thermal LF of electrons that cool on a dynamical time, $\gamma_i$ is the minimum thermal LF of electrons behind the shock front (note that $\gamma_i = \epsilon_i (m_i / m_e) \Gamma_{\text{sh}} (p - 2) / (p - 1)$, where $\Gamma_{\text{sh}}$ is the LF of the shock front with respect to the unshocked shell), $n_e$ is the comoving electron density in the unshocked shell and $\gamma_c \sim 3-4$ is the initial effective thermal LF of the streaming electrons.

In order to get the GRB spectrum below the peak to be proportional to $\sim v^4$, we want $v_j \sim 10^5$ eV (the peak of the GRB spectrum is of the order of 100 keV). Therefore, from the above equation we find the following condition on comoving electron density in the unshocked shell:

$$n_e \approx 4 \times 10^{38} \Gamma^2 \gamma_i^4 / \Gamma_{\text{sh}}^4 (\gamma_c / \Gamma_{\text{sh}})$$

or the optical depth of the source to Thomson scattering is

$$\tau \approx 4 n_e \Gamma_{\text{sh}} R_e / \sigma_{\text{T}} \Gamma \approx 10^{16} \gamma_c t_v / \Gamma_{\text{sh}}^2,$$

where $R_e$ is the distance of the source from the centre of the explosion, and $t_v \approx R_e / (4 \Gamma_{\text{sh}} c)$ is the GRB variability time-scale.

For internal shock $\Gamma_{\text{sh}}$ is of the order of a few, and therefore, $\gamma_i \approx 10^3$. In this case we find the optical depth of the source to be

$$\tau \approx 10^{13} \gamma_c t_v / \Gamma.$$  

Or $\tau \sim 1$ for $t_v \sim 10^{-2}$ s, and $\Gamma \sim 10^3$. The next step is to estimate the LF of cooling electrons ($\gamma_c$), which can be shown to be

$$\gamma_c \approx 6 \pi \tilde{m}_e c (1 + z) \sqrt{B^2 / \sigma_T^2 \Gamma (1 + Y)},$$

where $B$ is the magnetic field in the source comoving frame, and $Y$ is the Compton $Y$ parameter; $Y \approx \tau \gamma_c \gamma_i$ (for $p \approx 2$). If $\gamma_c$ is not much less than $\gamma_i$, then $\gamma$ is very large (of the order of a million), and most of the radiation energy will come out as SSC photons at $> 10^9$ GeV. And moreover, electrons cool very rapidly resulting in the synchrotron cooling frequency ($\nu_c$) to be much less than 100 keV, and therefore the spectrum below the peak will be $\nu^{-1/2}$ and not $\propto \nu^4$. Even if we take $\gamma_c$ to be of the order of unity, we still run into similar problems.

We, therefore, do not include the jitter process in our analysis of GRB-prompt radiation mechanism.

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