An infinite family of self-consistent models for axisymmetric flat galaxies

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ABSTRACT

We present the formulation of a new infinite family of self-consistent stellar models, designed to describe axisymmetric flat galaxies. The corresponding density–potential pair is obtained as a superposition of members belonging to the generalized Kalnajs family, by imposing the condition that the density can be expressed as a regular function of the gravitational potential, in order to derive analytically the corresponding equilibrium distribution functions (DFs). The resulting models are characterized by a well-behaved surface density, as in the case of generalized Kalnajs discs. Then, we present a study of the kinematical behaviour which reveals, in some particular cases, a very satisfactory behaviour of the rotational curves (without the assumption of a dark matter halo). We also analyse the equatorial orbit’s stability, and Poincaré surfaces of section are performed for the three-dimensional orbits. Finally, we obtain the corresponding equilibrium DFs, using the approaches introduced by Kalnajs and Dejonghe.

Key words: stellar dynamics – galaxies: kinematics and dynamics.

1 INTRODUCTION

The obtention of density–potential pairs (PDP) corresponding to idealized thin discs is a problem of great astrophysical relevance, motivated by the fact that the main part of the mass in many galaxies is concentrated in an stellar flat distribution, usually assumed as axisymmetric (Binney & Tremaine 2008). Once the potential–density pair (PDP) is formulated as a model for a galaxy, usually the next step is to find the corresponding distribution function (DF). This is one of the fundamental quantities in galactic dynamics specifying the distribution of the stars in the phase space of positions and velocities. Although the DF generally cannot be measured directly, there are some observationally accessible quantities that are closely related to the DF: the projected density and the light-of-sight velocity, provided by photometric and kinematic observations, are the examples of DF’s moments. Thus, the formulation of a PDP with its corresponding equilibrium DF establishes a self-consistent stellar model that can be corroborated by astronomical observations.

Now then there is a variety of PDPs for such flat stellar models, e.g. Wyse & Mayall (1942), Kuzmin (1956), Schnith (1956), Brandt & Belton (1962) and Kalnajs (1972). An interesting model, describing a disc of finite extent which has a constant-velocity profile, was introduced by Mestel (1963). However, the Mestel disc has a mass density distribution becoming singular at the galaxy centre. Toomre (1963, 1964) formulated a generalized family of models whose first member is the one introduced by Kuzmin (1956). This family represents a set of discs of infinite extension, derived by solving the Laplace equation in cylindrical coordinates subject to appropriated boundary conditions on the discs and at infinity. In a more recent work, Evans & de Zeeuw (1992) found a set of elementary PDPs that generalize Toomre–Mestel discs.

Analogously, González & Reina (2006) obtained a family of finite thin discs (generalized Kalnajs discs) whose first member corresponds, precisely, to the well-known model derived by Kalnajs (1972). Such family was derived by using the Hunter’s method (1963), which is based on the obtention of the solutions of Laplace equation in terms of oblate spheroidal coordinates, by imposing some appropriate conditions on the surface density. So, by requiring that the surface density behaves as a monotonously decreasing function of the radius, with a maximum at the centre of the disc and vanishing at the edge, the detailed expressions for the gravitational potential and the rotational velocity were obtained as a series of elementary functions. Also, some two-integral DFs for the first four members of this family were recently obtained by Pedraza, Ramos-Caro & González (2008). Now then, as the generalized Kalnajs models correspond to discs of finite extension, one can consider that they describe the mass distribution of a flattened galaxy more accurately than Toomre’s family. However, it have been shown that discs of finite extent are susceptible to edge modes and hence unstable (Erickson 1974; Toomre 1981).

In this paper, we formulate a new infinite set of finite thin discs, obtained by superposing the members of the generalized Kalnajs family in such a way that the resulting density surface can be expressed as a well-behaved function of the gravitational potential. As it was pointed out by some authors, this is a fundamental requirement for the searching of equilibrium DFs describing such

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axisymmetric systems (see e.g. Fricke 1952; Hunter & Quian 1993; Jiang & Ossipkov 2007). Thus, the new family formulated here has the advantage of easily providing the corresponding two-integral DFs.

Furthermore, the models have two additional advantages. On one hand, the mass surface density is well behaved as in the case of generalized Kalnajs discs, having a maximum at the centre and vanishing at the edge. Moreover, the mass distribution of the higher members of the family is more concentrated at the centre. On the other hand, the rotation curves are better behaved than in the Kalnajs discs. We found that, in some cases, the circular velocity increases from a value of zero at the centre of the disc, then reaches a maximum at some critical radius and, after that, remains approximately constant. As it is known, such behaviour has been observed in many disc-like galaxies.

Now, apart from the circular velocity, there are two important quantities concerning to the interior kinematics of the models: the epicyclic and vertical frequencies, which describe the stability against radial and vertical perturbations of particles in quasi-circular orbits. We found that the models formulated here are radially stable whereas vertically unstable, which is a characteristic inherited from the generalized Kalnajs family (Ramos-Caro, López-Suspeúz & González 2008). Moreover, since we are dealing with discoidal matter distributions with sharp edges, the models suffer from bending instabilities (see Hunter & Toomre 1969). On the other hand, when we deal with three-dimensional orbits, there are also a common feature between the new models and the Kalnajs family: the phase-space structure of disc-crossing orbits, that can be viewed through the Poincaré surfaces of section, is composed by shape-of-figuring Kolmogorov–Arnold–Mosser (KAM) curves and prominent chaotic zones. However, there are certain situations in which the chaoticity disappears and the three-dimensional motion of test particles is completely regular. In such cases, one can suggest the existence of a third integral of motion, as in the case of Stückel and Kuzmin potentials.

Finally, in order to formulate the new family as a set of self-consistent stellar models, we will deal with the problem of obtaining the corresponding equilibrium DFs. By Jeans’s theorem, they are the functions of the isolating integrals of motion that are conserved in each orbit. Some authors have shown that, if the density can be written as a function of the gravitational potential, it is possible to find such kind of two-integral DFs (see Eddington 1916; Fricke 1952; Kalnajs 1976; Jiang & Ossipkov 2007). In this paper, we will adopt the approach introduced by Kalnajs (1976), which fits quite well to axisymmetric disc-like systems. Then, starting from the DFs derived from this method, a second kind of DFs is obtained by using the formulae introduced by Dejonghe (1986) that takes into account the principle of maximum entropy. These DFs describe stellar systems with a preferred rotational state.

Accordingly, the paper is organized as follows. First, in Section 2, we obtain the PDPs for the new family of thin disc models. Then, in Section 3, we study the motion of test particles around these new galactic models and, in Section 4, we derive the DFs associated with the models. Finally, in Section 5, we summarize our main results.

2 A NEW FAMILY OF THIN DISC MODELS

2.1 The generalized Kalnajs discs

In this section, we summarize the principal features of the generalized Kalnajs family, introduced by González & Reina (2006), an infinite family of axially symmetric finite thin discs. The mass surface density of each disc (labelled with the positive integer \( n \)) is given by

\[
\Sigma_n(R) = \Sigma_n^{(0)} \left(1 - \frac{R^2}{a^2}\right)^{-n/2},
\]

where \( M \) is the total mass, \( a \) is the disc radius and \( \Sigma_n^{(0)} \) are constants given by

\[
\Sigma_n^{(0)} = \frac{(2n + 1)M}{2\pi a^2}.
\]

Such mass distribution generates an axially symmetric gravitational potential that can be written as

\[
\Phi_n(\xi, \eta) = -\sum_{j=0}^{n} C_{2j} q_{2j}(\xi) P_{2j}(\eta).
\]

Here, \( P_{2j}(\eta) \) and \( q_{2j}(\xi) = i^{j+1} Q_{2j}(i\xi) \) are the usual Legendre polynomials and the Legendre functions of the second kind, respectively, \(-1 \leq \xi \leq 1\) and \( 0 \leq \eta < \infty \) are spheroidal oblate coordinates related to the usual cylindrical coordinates \((R, z)\) through the relations

\[
R^2 = a^2(1 + \xi^2)(1 - \eta^2), \quad z = a\eta\xi
\]

in such a way that the discs are located at \( \xi = 0 \) and \( \eta = \sqrt{1 - R^2/a^2} \). Finally, the \( C_{2j} \) are constants given by

\[
C_{2j} = \frac{MG2^{2j+1}(2j + 1)(2n + 1)!}{a2^{2n+1}(2j + 1)(n - j)!\Gamma(n + j + \frac{1}{2}) q_{2j+1}(0)},
\]

where \( G \) is the gravitational constant.

As was shown by González & Reina (2006), in these models, the surface density is a monotonously decreasing function of the radius with a maximum at the centre and vanishing at the edge, being the mass distribution of the higher members of the family more concentrated at the centre. On the other hand, the corresponding rotation curves behave as follows: for \( n = 1 \), the circular velocity is proportional to the radius, whereas for the other members of the family it increases from a value of zero at the centre of the discs then reaches a maximum at a critical radius and, finally, decreases to a finite value at the edge. Besides, the critical radius decreases as the value of \( n \) increases.

2.2 Formulation of the new family

Now, we will show that it is possible to formulate a new family of stellar models by performing a linear combination of the generalized Kalnajs discs, in such a way that the new surface densities can be written as the polynomials of the new potentials. As it was quoted, this is a basic requirement for the derivation of equilibrium DFs through the Kalnajs formalism sketched above. In particular, we are interested on to derive a simple relation between the relative potential on the disc and the surface density.

At first note that \( \Phi_n(0, \eta) \), given by equation (3), can be rewritten as

\[
\Phi_n(0, \eta) = -\sum_{j=0}^{n} A_{n, j} \eta^{2n-2j},
\]

where \( A_{n, j} \) are constants defined as

\[
A_{n, j} = \frac{(-1)^j (4s - 2r) ! C_{2j} q_{2j}(0)}{2^{2j} r ! (2s - 2r) ! (2s - r) !},
\]

and the relation (equation 5) was derived by introducing the identity (Arfken 2005)

\[
P_{2j}(\eta) = \sum_{j=0}^{n} \frac{(-1)^j (4s - 2r) !}{2^{2j} r ! (2s - 2r) ! (2s - r) !} \eta^{2n-2j}.
\]

From equation (5), we note that the maximum value of the gravitational potential on the n-th disc is \( \Phi_n(0,0) \). Therefore, we define the relative potential on such a disc as

\[
\Psi_m(\eta) = \Phi_m(0,0) - \Phi_m(0,\eta).
\]  

(8)

Now, suppose that we can choose a linear combination of those \( \Psi_m \) leading to a new relative potential \( \tilde{\Psi}_m \) of the form

\[
\tilde{\Psi}_m = \sum_{n=1}^{2m} B_n \Psi_n = A_m \eta^{2m},
\]  

(9)

where \( B_n \) are constants that can be determined from \( A_m \) and \( C_n \) (see Section 2.3).

The new relative potential \( \tilde{\Psi}_m \) is generated by a new mass distribution described by a surface density \( \tilde{\Sigma}_m \) that is also a linear combination of the generalized Kalnajs discs \( \Sigma_m \). That is

\[
\tilde{\Sigma}_m(R) = \sum_{n=1}^{2m} B_n \Sigma_n^{(m)} \eta^{2n-1}.
\]  

(10)

From this relation and equation (9), we can note that \( \tilde{\Sigma}_m \) can be rewritten as

\[
\tilde{\Sigma}_m(R) = \sum_{n=1}^{2m} B_n \Sigma_n^{(m)} \left( \frac{\Psi_m}{A_m} \right)^{(2n-1)/(2m)}.
\]  

(11)

Thus, we can see that this new family of discs is characterized by the fact that the surface density can be split as a combination of powers of the relative potential. This important fact makes viable the further derivation of two-integral DFs for the whole family (see Section 4) that can be considered as a set of self-consistent galactic models. Now, the above statements are only true if we can determine the constants \( B_n \) introduced in equation (9). In the next section, we show a procedure that, by using the orthogonality properties of \( P_n \), leads to a recurrence relation expressing \( B_n \) in terms of \( A_m \) and \( C_n \).

### 2.3 Calculation of \( B_n \)

According to the definitions (equations 3 and 8), the relative potential associated to the generalized Kalnajs discs can be written as

\[
\Psi_m(\eta) = \sum_{n=0}^{m} \tilde{C}_n P_{2n}(\eta),
\]  

(12)

where \( \tilde{C}_n \) are constants defined by

\[
\tilde{C}_n = C_{2n} P_{2n}(0) - \delta_{n0} \sum_{i=0}^{m} C_{2i} q_{2i}(0) P_{2i}(0).
\]  

(13)

So, by introducing equation (12) into equation (9), \( \tilde{\Psi}_m \) can also be written in terms of Legendre polynomials as

\[
\tilde{\Psi}_m = \sum_{n=0}^{m} \sum_{i=0}^{m} B_n \tilde{C}_i P_{2i}(\eta) = \sum_{n=0}^{m} D_n P_{2n}(\eta),
\]  

(14)

where \( D_n \) are constants to be determined. Here, it is important to note that these \( D_n \) are such that

\[
D_m = B_m \tilde{C}_m,
\]  

(15a)

\[
D_{m-1} = B_m \tilde{C}_{m-1} + B_{m-1} \tilde{C}_{m-1},
\]  

(15b)

\[
D_{m-2} = B_m \tilde{C}_{m-2} + B_{m-1} \tilde{C}_{m-2} + B_{m-2} \tilde{C}_{m-2},
\]  

(15c)

and so on. In general, we have

\[
D_n = A_m \eta^{2m} \int_{-1}^{1} \eta^{2m} P_{2n}(\eta) d\eta
\]  

that reduces to

\[
D_n = A_m \left[ \frac{\pi^{3/2}(4n+1)\Gamma(2m+1)}{2^{2m+1}\Gamma(1+m-n)^{m+n+3/2}} \right]
\]  

(18)

in such a way that, by using equation (16), we can determine the constants \( B_n \). In order to do this, note that equations (15a)–(15d) give us recurrence relations: from equation (15a), we obtain \( B_m \); from equation (15b), we obtain \( B_{m-1} \); and so on. In general, we have

\[
B_m = \frac{D_m}{\tilde{C}_m} = 1,
\]  

(19a)

\[
B_{m-1} = \frac{D_{m-1}}{\tilde{C}_{m-1}} - B_m,
\]  

(19b)

\[
B_{m-2} = \frac{D_{m-2}}{\tilde{C}_{m-2}} - B_m - B_{m-1},
\]  

(19c)

\[
B_{m-k} = \frac{D_{m-k}}{\tilde{C}_{m-k}} - \sum_{i=0}^{k-1} B_{m-i}, \quad \text{for } k \geq 1
\]  

(19d)

relations that can be summarized as

\[
B_n = \frac{D_n}{\tilde{C}_n} - \sum_{i=0}^{n-1} B_{n-i}, \quad \text{for } n \leq m - 1.
\]  

(20)

In Table 1, we show the values of \( B_n \) for the first four models, i.e. \( m = 2, 3, 4 \) and 5. Although we have solved the problem of finding these constants, there is another inconvenience. If one introduces such coefficients in equation (10), the corresponding surface density is negative for certain ranges of \( R \), as it is shown in Fig. 1, where we plot the resulting \( \tilde{\Sigma} \) for the first four models. However, this problem can be solved by correcting \( B_1 \), i.e. the coefficient corresponding to

<table>
<thead>
<tr>
<th>( m )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_1 )</td>
<td>-5/8</td>
<td>-35/192</td>
<td>-105/1024</td>
<td>-1155/16384</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>1</td>
<td>-7/12</td>
<td>-21/128</td>
<td>-231/2560</td>
</tr>
<tr>
<td>( B_3 )</td>
<td>1</td>
<td>1</td>
<td>-9/16</td>
<td>-99/640</td>
</tr>
<tr>
<td>( B_4 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-11/20</td>
</tr>
<tr>
<td>( B_5 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Corrected constants $R$ is static and $B_0$, in the range $0 \leq B \leq 200$, as an arbitrary parameter that can be determined by equation (20). Since the resulting densities are negative and the dominant term in equation (10). In the next section, we show that it is possible to define minimum values for $B_1$, corresponding to each model, in such a way that all of them described by positive surface densities.

### 2.4 Correction of $B_1$

From here on, we consider $B_1$ as an arbitrary parameter that can be chosen in such a way that $\Sigma \geq 0$, in the range $0 \leq R \leq a$. For each model, we expect that $B_1$ has a lower limit but does not have an upper limit. The reason is that according to equation (1), we have

$$\lim_{R \to a} \frac{d\Sigma_m}{dR} = \left\{ \begin{array}{ll} -\infty & \text{for } m = 1, \\ 0 & \text{for } m \geq 2. \end{array} \right. \quad (21)$$

This means the behaviour of $\Sigma$, for $R \to a$, differs from the remaining density surfaces, characterized by a rate of change tending asymptotically to zero at the disc edge. Thus, it is evident that one always can find a minimum value $B_{1\text{min}}$, such that the product $B_{1\text{min}} \Sigma$ is larger than any linear combination of the generalized Kalnajs discs, can be cast as

$$\Phi_m(\xi, \eta) = -\sum_{n=1}^{m} \sum_{k=0}^{n} B_k C_{2n} q_{2k}(\xi) P_{2k}(\eta), \quad (25)$$

where $B_k$ are given by equation (20) for $n \geq 2$, and $B_1$ is an arbitrary parameter with lower bound determined by equations (23) and (24).

## 3 KINEMATICS OF THE NEW FAMILY

In this section, we study the motion of test particles around the galactic models formulated above. Since each $\Phi_m$ is static and axially symmetric, the specific energy $E$ and the specific axial angular momentum $\ell$ are conserved along the particle motion. This fact restricts such motion to a three-dimensional subspace of the $(R, z, V_R, V_z)$ phase space. By defining an effective potential $\Phi_e^* as

$$\Phi_e^* = \Phi_m + \frac{\ell^2}{2R^2}, \quad (26)$$

the motion will be determined by the equations (Binney & Tremaine 2008):

$$\dot{R} = V_R, \quad (27a)$$

$$\dot{z} = V_z, \quad (27b)$$

$$V_R = \frac{\partial \Phi_e^*}{\partial R}, \quad (27c)$$

$$V_z = -\frac{\partial \Phi_e^*}{\partial z} \quad (27d)$$

together with

$$E = \frac{1}{2} (V_R^2 + V_z^2) + \Phi_e^*, \quad (28)$$

which gives the total energy of the particle.

Relations (equation 27a–28) are the basic equations that determine the motion of a particle with specific axial angular momentum $\ell$ and energy $E$, in cylindrical coordinates. At first, we restrict our attention on particles belonging the disc, i.e. the interior kinematics, in order to describe rotation curves (for circular orbits) and the stability of nearly circular orbits (by deriving the epicyclic and vertical frequencies). Then, we will focus on three-dimensional motion, in particular, the case of disc-crossing orbits. In such case, we use $z = 0$ surfaces of section in order to illustrate the regularity or chaoticity characterizing those orbits.

### Table 2. Corrected constants $B_k$ for the first four models: $m = 2, 3, 4$ and 5.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$B_{1\text{min}}$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>3</td>
<td>0.101273</td>
<td>-7/12</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.128914</td>
<td>-21/128</td>
<td>-9/16</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.143207</td>
<td>-231/2560</td>
<td>-99/640</td>
<td>-11/20</td>
<td>1</td>
</tr>
</tbody>
</table>

Figure 1. We plot the mass surface density $\Sigma_2$, $\Sigma_3$, $\Sigma_4$, and $\Sigma_5$ (from bottom to top in the left-hand edge), given by equation (22), where the parameters $B_k$ are determined by equation (20). Since the resulting densities are negative in some ranges, we must correct $B_1$. The relation (equation 23) imposes the condition that the surface density can be split as

$$\Sigma_m(R, B_1) = B_1 \Sigma^{(1)}(1 - \frac{R^2}{a^2})^{1/2} + \sum_{n=2}^{m} B_n \Sigma^{(n)}(1 - \frac{R^2}{a^2})^{n-1/2}, \quad (22)$$

where the coefficients $B_n$, for $n \geq 2$, are given by equation (20). A simple way to find $B_{1\text{min}}$ is by demanding that $\Sigma_m$ has a minimum (equal to 0) at $B_1 = B_{1\text{min}}$ and $R = R_{\text{min}}$. That is, we demand that the following two equations hold

$$\left. \frac{d\Sigma_m(R, B_{1\text{min}})}{dR} \right|_{R=R_{\text{min}}} = 0, \quad (23)$$

$$\Sigma(R_{\text{min}}, B_{1\text{min}}) = 0. \quad (24)$$

The relation (equation 23) imposes the condition that the surface density has a minimum at $B_1 = B_{1\text{min}}$ and $R = R_{\text{min}}$, while through the relation (equation 24) we demand that its value at such critical point is 0.

We find numerically the lower bounds of $B_1$ for the first four members of the new family and they are showed in Table 2. The corresponding surface densities, for different values of $B_1 > B_{1\text{min}}$, are plotted in Fig. 2. We note that, in all of these cases, the mass concentration starts from 0 for $R = a$ and increases towards the disc bulge. Moreover, such concentration increases with $m$. Finally, the gravitational potential corresponding to the new family, derived as a linear combination of the generalized Kalnajs discs, can be cast as

$$\Phi_m(\xi, \eta) = -\sum_{n=1}^{m} \sum_{k=0}^{n} B_k C_{2n} q_{2k}(\xi) P_{2k}(\eta), \quad (25)$$

where $B_k$ are given by equation (20) for $n \geq 2$, and $B_1$ is an arbitrary parameter with lower bound determined by equations (23) and (24).

An infinite family of self-consistent models

3.1 Interior kinematics

We start by pointing out that the system (equation 27) has equilibrium points at \( V_r = V_z = z = 0 \) and \( R = R_c \), where \( R_c \) must satisfy the equation

\[
\left( \frac{\partial \Phi_m^*}{\partial R} \right)_{(R_0, 0)} = \frac{\ell_c^2}{R_c^3} + \left( \frac{\partial \Phi_m}{\partial R} \right)_{(R_0, 0)} = 0,
\]

i.e. the condition for a circular orbit in the plane \( z = 0 \). In other words, the equilibrium points of equation (27a) occur when the test particle describes equatorial circular orbits of radius \( R_c \), specific axial angular momentum given by

\[
\ell_c = \pm \sqrt{R_c^3 \left( \frac{\partial \Phi_m}{\partial R} \right)_{(R_0, 0)}},
\]

and specific energy

\[
E = \Phi_m^*(R_c, 0).
\]

The subscript \( c \) in \( \ell_c \) indicates that we are dealing with circular orbits.

Now, a feature of special interest is the circular velocity \( V_c = \ell_c R_c \) (sometimes denoted by \( V_c \)), which can be directly compared with astronomical observations. Then, it is convenient to express \( V_c \) as a function of the radius. From equation (30), we can write the magnitude of the circular velocity as

\[
V_c^2 = R \left( \frac{\partial \Phi_m}{\partial R} \right)_{z=0}.
\]

In Fig. 3, we plot \( V_c(R) \) corresponding to the first four models of the new family of discs. For \( B_1 = B_{1\text{min}} \), the rotation curve has a maximum value and then decreases to a constant value, in a similar fashion the behaviour of the generalized Kalnajs discs (González & Reina 2006). Moreover, the maximum of the rotation curve is at

\[
\Phi_m^*(R_c, 0).
\]
an even smaller radius when the value of \( m \) increases. Now, in the case in which \( B_1 \) is very large, the curve approximates to a straight line, as a consequence of the dominance of term associated with the usual Kalnajs disc. On the other hand, for intermediate values of \( B_1 \), we note an interesting feature: when the rotation curve reaches its maximum, it remains nearly constant, as it is the case in many real galaxies.

In order to study the stability of these trajectories under small radial and vertical (\( z \)-direction) perturbations, we analyse the nature of quasi-circular orbits. They are characterized by an epicycle frequency \( \kappa \) and a vertical frequency \( \nu \) given by (Binney & Tremaine 2008)

\[
\kappa^2 = \left( \frac{\partial^2 \Phi_m^*}{\partial R^2} \right)_{(R_c, 0)}, \quad \nu^2 = \left( \frac{\partial^2 \Phi_m^*}{\partial z^2} \right)_{(R_c, 0)}.
\]

This means that by introducing equation (30) in the second derivatives of \( \Phi_m^* \), we obtain \( \kappa^2 \) and \( \nu^2 \) as the functions of the radius. Values of \( R \) such that \( \kappa^2 > 0 \) and (or) \( \nu^2 > 0 \) correspond to stable circular orbits under small radial and (or) vertical perturbations, respectively. In Figs 4 and 5, we show the behaviour of the epicycle and vertical frequency, respectively, for the first four models and using the same values of \( B_1 \) as in Fig. 2. We note that these models

![Figure 4](https://example.com/fig4.png)

**Figure 4.** Quadratic epicyclic frequency as a function of \( R \) for the cases (a) \( m = 2 \), (b) \( m = 3 \), (c) \( m = 4 \) and (d) \( m = 5 \). We use the same values of parameter \( B_1 \) as in Fig. 2. The lower curves correspond to \( B_{1\text{min}} \) and the upper curves correspond to larger values of \( B_1 \). In all of these cases, we note a prominent range of radial stability. Such range increases with \( B_1 \).

![Figure 5](https://example.com/fig5.png)

**Figure 5.** Quadratic vertical frequency as a function of \( R \) for the cases (a) \( m = 2 \), (b) \( m = 3 \), (c) \( m = 4 \) and (d) \( m = 5 \). We use the same values of parameter \( B_1 \) as in Fig. 2. The lower curves correspond to \( B_{1\text{min}} \) and the upper curves correspond to larger values of \( B_1 \). The range of vertical stability is small and increases with \( B_1 \).
are characterized by a prominent range of stability under radial perturbations. In particular, for \( B_1 > B_{1\text{min}} \), there will be radially stable orbits with radius in the range \( 0 \leq R \leq a \). In contrast, there are prominent ranges of vertical instability. Such ranges tend to decrease when \( m \) and \( B_1 \) increase. Thus, we can say that the stability under vertical perturbations, in quasi-circular orbits, improves in models with a large \( m \).

### 3.2 Exterior kinematics: disc-crossing orbits

In this section, we study the behaviour of three-dimensional motion, i.e. orbits outside the equatorial plane (except when they cross the plane \( z = 0 \)). As it was mentioned above, the motion is determined by equation (27a) and can be described in an effective phase space with three dimensions (there are two integrals of motion: \( E \) and \( \ell \)). An adequate tool to investigate such orbits is the Poincaré surface of section, in order to find the chaotic and (or) regular regions characterizing the structure of the phase space.

In particular, we present numerical solutions of equations (27a)–(27d) for the case of bounded disc-crossing orbits. There are certain values of \( E \) and \( \ell \) for which they are confined to regions that contain the disc and will cross back and forth through it. As it was shown by Hunter & Quian (1993), this fact usually gives rise to many chaotic orbits due to the discontinuity in the \( z \)-component of the gravitational field, producing a fairly abrupt change in their curvatures. There is an important exceptional case of this behaviour: the Kuzmin’s disc characterized by an integrable potential of the form \( \Phi = -G M [R^2 + (a + z^2)^{-1/2}] \) with \( a > 0 \). However, the so-called Kuzmin-like potentials, characterized by \( \Phi(\varepsilon) \) where \( \varepsilon = [R^2 + (a + z^2)^{1/2}] \), are non-integrable and present the behaviour mentioned above. The family of models formulated here presents a very similar structure and we can expect an analogous dynamics. Each potential \( \Phi_m(\xi, \eta) \) can be cast in a Kuzmin-like form if we take into account that, according to equation (4), \( \xi = (R_+ + R_-)/2a \) and \( \eta = (R_+ - R_-)/2ai \), where \( R_+ = [R^2 + (a + i\eta)^2]^{1/2} \) and \( R_- = [R^2 + (a - i\eta)^2]^{1/2} \). Moreover, they are characterized by a \( z \)-derivative discontinuity in the disc, given by

\[
\left( \frac{\partial \Phi_m}{\partial z} \right)_{z=0^+} - \left( \frac{\partial \Phi_m}{\partial z} \right)_{z=0^-} = 2\pi G \Sigma_m(R).
\]  

(33)

Despite the above relation makes the KAM theorem inapplicable, we also found a large variety of regular disc-crossing orbits.

**Figure 6.** Surface of section for some orbits with \( \ell = 0.2, E = -1.245 \), around the model \( m = 2 \) with \( B_1 = 0.1 \).

**Figure 7.** Surface of section for some orbits with \( \ell = 0.2, E = -1.245 \), around the model \( m = 2 \) with \( B_1 = 0.5 \).

In Fig. 6, we show the \( z = 0 \) surface of section corresponding to some orbits with \( E = -1.245 \) and \( \ell = 0.2 \) (these same values are used to perform the next three surfaces of section), and corresponding to a test particle moving around the model \( m = 2 \) with \( B_1 = 0.1 \). As it was expected, from our experience with the generalized Kalnajs discs (Ramos-Caro et al. 2008), this plot exhibits a variety of regular and chaotic trajectories. There is a regular central region confined by two kinds of KAM curves: the central rings made by box orbits and a set of resonant islands chain (made by loop orbits) enclosing the rings. Moreover, there are two lateral regular zones of loop orbits that, as well as the central region, are enclosed by a sea of chaos. In Fig. 7, we show the effect of increasing \( B_1 \) in model \( m = 2 \). Then, for the same initial conditions with \( E = -1.245 \) and \( \ell = 0.2 \), the surface of section reveals an increase in the chaotic region along with a distortion of the KAM curves in regular zones (e.g. now the central torus are made only by box orbits).

One could expect similar chaotic surfaces of section for models \( m = 3, \ldots \) but what really happens is that for certain values of \( B_1 \) all the orbits are regular. This is the case illustrated in Fig. 8, where the Poincaré section, corresponding to a particle moving around the model \( m = 3 \), reveals completely regular motion. This is a surprising fact, since in this kind of potentials chaos is the rule. If one increases \( B_1 \) (e.g. from 0.2 to 1), as in the case of Fig. 9, we find

**Figure 8.** Surface of section for some orbits with \( \ell = 0.2, E = -1.245 \), around the model \( m = 3 \) with \( B_1 = 0.2 \). In this case, we have only regular orbits.
and by choosing properly the constants, the relative effective potential will be
\[ \Psi_{m,n} = A_{m0} \eta^{2n} + \frac{1}{2} \left( B_{1} - B_{1}^{*} \right) \Omega_{0}^{2} \alpha^{2} \eta^{2} - \frac{1}{2} \Omega_{0}^{2} \alpha^{2} \eta^{2}. \] (37)

Note that if one chooses \( \Omega \) as
\[ \Omega = \pm \Omega_{0} \sqrt{B_{1} - B_{1}^{*}}, \]
the term with \( \eta^{2} \) vanishes and equation (37) reduces to
\[ \Psi_{m,n} = A_{m0} \eta^{2n}. \] (39)

Therefore, the relation between the density and the relative effective potential can be written as
\[ \Sigma_{m}(R) = \sum_{n=1}^{m} B_{n} \Sigma_{c}^{(n)} \left( \frac{\Psi_{m,n}}{A_{0}} \right)^{(2n-1)/(2m)}. \] (40)

Finally, by using the Kalnajs method (Kalnajs 1976), we obtain that the DF corresponding to the \( m \) model is given by
\[ f_{m}(\varepsilon, L_{c}) = \frac{1}{4 \pi m} \sum_{n=1}^{m} \frac{B_{n} \Sigma_{c}^{(n)}(2n - 1)A_{m0}^{(2n-1)/(2m)}}{(\varepsilon + \Omega L_{c} - \frac{1}{2} \Omega_{0}^{2} \alpha^{2})^{(2n-1)/(2m)}}. \]

The explicit DFs for the first four models and the associated \( \Omega \), given by equation (38), are
\[ f_{2} = \frac{3 \Sigma_{c}^{(2)} A_{m0}^{3/4}}{4 J^{3/4}} + \frac{B_{1} \Sigma_{c}^{(1)} A_{m0}^{1/4}}{4 J^{1/4}}, \] (42a)
\[ f_{3} = \frac{5 \Sigma_{c}^{(3)} A_{m0}^{5/6}}{6 J^{5/6}} - \frac{7 \Sigma_{c}^{(2)} A_{m0}^{3/2}}{24 J^{3/2}} + \frac{B_{1} \Sigma_{c}^{(1)} A_{m0}^{1/2}}{6 J^{1/2}}, \] (42b)
\[ f_{4} = \frac{7 \Sigma_{c}^{(4)} A_{m0}^{7/8}}{8 J^{7/8}} - \frac{45 \Sigma_{c}^{(3)} A_{m0}^{5/8}}{128 J^{5/8}} \] \[ - \frac{63 \Sigma_{c}^{(2)} A_{m0}^{3/4}}{1024 J^{3/4}} + \frac{B_{1} \Sigma_{c}^{(1)} A_{m0}^{1/4}}{8 J^{1/4}}, \] (42c)
\[ f_{5} = \frac{9 \Sigma_{c}^{(5)} A_{m0}^{9/10}}{10 J^{9/10}} - \frac{77 \Sigma_{c}^{(4)} A_{m0}^{7/10}}{200 J^{7/10}} \] \[ - \frac{99 \Sigma_{c}^{(3)} A_{m0}^{5/10}}{1280 J^{5/10}} + \frac{693 \Sigma_{c}^{(2)} A_{m0}^{3/10}}{25600 J^{3/10}} + \frac{B_{1} \Sigma_{c}^{(1)} A_{m0}^{1/10}}{10 J^{1/10}}, \] (42d)

where \( J = \varepsilon + \Omega L_{c} - \frac{1}{2} \Omega_{0}^{2} \alpha^{2} \) is the Jacobi’s integral and
\[ A_{20} = \frac{45 GM \pi}{128 \alpha}, \] \[ \Omega_{2} = \frac{3 \pi GM(5 + 8 B_{1})}{32 \alpha^{3}}, \] (43a)
\[ A_{30} = \frac{175 GM \pi}{512 \alpha}, \] \[ \Omega_{3} = \frac{3 \pi GM(35 + 192 B_{1})}{768 \alpha^{3}}, \] (43b)
\[ A_{40} = \frac{11025 GM \pi}{32768 \alpha}, \] \[ \Omega_{4} = \frac{3 \pi GM(105 + 1024 B_{1})}{4096 \alpha^{3}}, \] (43c)
\[ A_{50} = \frac{43659 GM \pi}{131072 \alpha}, \] \[ \Omega_{5} = \frac{3 \pi GM(1155 + 16384 B_{1})}{65536 \alpha^{3}}. \] (43d)

It is easy to see that the DFs obtained by equation (41), for the cases \( m \geq 3 \), could be negative in some regions of the phase space corresponding to the physical domain. To avoid this inconvenience, it is necessary to impose a stronger condition for the constants \( B_{1} \), in order to obtain well-defined DFs. To do this, we formulate the following equations (in a similar fashion as in Section 2.4):
\[ \frac{df_{m}(J, B_{i_{\text{max}}})}{dR} \bigg|_{J=J_{\text{max}}} = 0, \] (44)
$f_w(J_{\text{min}}, B_{1\text{min}}') = 0$. 

Relation (equation 44) imposes the condition that the DF has a minimum at $B_1 = B_{1\text{min}}'$ and $R = R_{\text{min}}$, while through the relation (equation 45), we demand that its value at such critical point vanishes. The numeric solution to these equations gives us the values shown in Table 3, for the models with $m = 2, 3$ and $5$. So, taking these values as a lower limit for $B_1$, the DFs given by equation (41) become positive defined in the physical domain of the phase space. Fig. 10 shows the graphics of the DFs as the functions of the Jacobi’s integral, with different values of $B_1$. In general, we can observe that the probability is maximum for small values of $J$, and tends to a constant as $J$ increases. Moreover, in the cases $m \geq 3$, we can see that for values of $B_1$ near to $B_{1\text{min}}'$, the probability has a minimum at $J \approx J_{\text{min}}$, and it is zero for $B_1 = B_{1\text{min}}'$ at $J = J_{\text{min}}$, in agreement with equations (44) and (45).

On the other hand, it is convenient to derive a new kind of DFs, corresponding to more probable rotational states. As it was shown by Dejonghe (1986), it is possible to obtain DFs obeying the maximum entropy principle, through the equation

$$f_w(J, L_z) = \frac{2f_{w+}(e, L_z)}{1 + e^{-dL_z}},$$

where $f_{w+}$ is the even part of equation (41). Fig. 11 shows the behaviour of the DFs given by equation (46). In Figs 11(a) and (b), the contours corresponding to the model $m = 2$ are plotted, with different values of parameter $\alpha$. As it can be seen, $\alpha$ determines a particular rotational state in the stellar system. As $\alpha$ increases, the probability to find a star with positive $L_z$ increases as well. A similar result can be obtained for $\alpha < 0$, when the probability to find a star with negative $L_z$ decreases as $\alpha$ decreases, and the corresponding plots would be analogous to Fig. 11, after a reflection about $L_z = 0$.

In Figs 11(c) and (d), the contours of model $m = 3$ are plotted with $B_1 \approx B_{1\text{min}}'$ for the same values of $\alpha$. The behaviour of the DFs for the remaining cases is pretty similar to the shown in these figures: when $B_1 \gg B_{1\text{min}}'$, the contours are similar to Figs 11(a) and (b), while if $B_1 \approx B_{1\text{min}}'$, the contours are similar to Figs 11(c) and (d), in agreement with Fig. 10.

<table>
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<th>$m$</th>
<th>$B_{1\text{min}}'$</th>
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<tr>
<td>2</td>
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<tr>
<td>3</td>
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<td>4</td>
<td>0.287827</td>
</tr>
<tr>
<td>5</td>
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Table 3. Coefficients $B_{1\text{min}}'$ for the first four models: $m = 2, 3, 4$ and 5.

5 CONCLUDING REMARKS

We have obtained a set of models for axisymmetric flat galaxies by superposing members belonging to the generalized Kalnajs discs family. The mass distribution of each model (labelled through the parameter $m = 2, 3, \ldots$), described by equation (10), is maximum at the centre and vanishes at the edge, in concordance with a great variety of galaxies. Moreover, the mass density can be expressed as a function of the gravitational potential (see equation 22), which makes possible to derive, analytically, the equilibrium DFs describing the statistical features of the models.

These models have also interesting features concerning with the interior kinematical behaviour. On one hand, we showed that for some values of $B_1$, the circular velocity has a behaviour very similar to that seen in many discoidal galaxies. This is a very relevant fact, which suggests that it is not always necessary to introduce the hypothesis of dark matter haloes (or Modified Newtonian Dynamics theories) in order to describe adequately a variety of rotational curves.

On the other hand, the analysis of epicyclic and vertical frequencies, associated to quasi-circular orbits, reveals that the models are stable under radial perturbations but unstable under vertical disturbances, apart from bending instabilities due to the presence of sharp edges. It may be that these limitations will be reduced by the

Figure 10. We plot the DFs as functions of the Jacobi’s integral for (a) $m = 2$, (b) $m = 3$, (c) $m = 4$ and (d) $m = 5$, for different values of the parameter $B_1$. The lower curves correspond to $B_{1\text{min}}'$ and the upper curves correspond to larger values of $B_1$. © 2008 The Authors. Journal compilation © 2008 RAS, MNRAS 390, 1587–1597
superposition of a dark halo. With regard to the motion of test particles around the models formulated here, we found that the behaviour of disc-crossing orbits is similar to that seen in the generalized Kalnajs family. However, for certain values of the parameter $B_1$, the Poincaré surface of section reveals that one can suggest the existence of a (non-analytical) third integral of motion.

Now, it is worth to mention that a finite thin disc with a completely flat rotation curve, obtained without the hypothesis of a dark matter halo, was already obtained in 1963 by Mestel. However, the problem with the disc of Mestel is not only that it is unstable in the absence of the dark halo, but also that it has a mass density that becomes singular at the centre. So, apart from the stability problems, the disc models here presented have the advantage that their mass densities are finite and well behaved always on the disc.

Furthermore, we find two kinds of equilibrium DFs for the models. Such two-integral DFs can be formulated, at first, as functionals of the Jacobi’s integral, as it was sketched in the formalism developed by Kalnajs (1976). This class of DFs essentially describes systems with rotational state, in average, behave as a rigid body. Then, we use the procedure introduced by Dejonghe (1986), obtaining DFs which represent systems with a mean rotational state consistent with the maximum entropy principle and, therefore, more probable than the first ones. The statements exposed above suggest that the family presented here, apart from the stability problems, can be considered as a set of realistic models that describe satisfactorily a great variety of galaxies.

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