Transit timing effects due to an exomoon

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ABSTRACT
As the number of known exoplanets continues to grow, the question as to whether such bodies harbour satellite systems has become one of increasing interest. In this paper, we explore the transit timing effects that should be detectable due to an exomoon and predict a new observable. We first consider transit time variation (TTV), where we update the model to include the effects of orbital eccentricity. We draw two key conclusions.

(i) In order to maintain Hill stability, the orbital frequency of the exomoon will always be higher than the sampling frequency. Therefore, the period of the exomoon cannot be reliably determined from TTV, only a set of harmonic frequencies.

(ii) The TTV amplitude is \( \propto M_S a_S \) where \( M_S \) is the exomoon mass and \( a_S \) is the semimajor axis of the moon's orbit. Therefore, \( M_S \) and \( a_S \) cannot be separately determined.

We go on to predict a new observable due to exomoons – transit duration variation (TDV). We derive the TDV amplitude and conclude that its amplitude is not only detectable, but the TDV signal will also provide two robust advantages.

(i) The TDV amplitude is \( \propto M_S a_S^{-1/2} \) and therefore the ratio of TDV to TTV allows for \( M_S \) and \( a_S \) to be separately determined.

(ii) TDV has a \( \pi/2 \) phase difference to the TTV signal, making it an excellent complementary technique.

Key words: techniques: photometric – methods: analytical – occultations – planets and satellites: general – planetary systems.

1 INTRODUCTION
Over 300 exoplanets have been discovered to date with detection rates escalating (see http://exoplanet.eu by J. Schneider). The detections have been so far biased towards large bodies and the smallest transiting planet to date is still Neptune-sized, for Gliese 436b (Gillon et al. 2008). Current instruments cannot yet detect transiting Earths but transit time variation (TTV) could offer a way to bring sensitivity down to sub Earth–mass level (Holman & Murray 2005; Agol et al. 2005). Given the large number of moons in our own Solar system, it is reasonable to postulate that satellites are common around exoplanets.

Another profound motivation for looking for exomoons is that they are likely to be terrestrial in nature, based upon our own Solar system, and hence one would propose that exomoons could be more habitable environments than the host of extrasolar giant planets (EGPs) so far discovered.

Photometric detection of a moon is likely to be exceptionally challenging. An exomoon is likely to be sub-Earth sized, based upon our Solar system, and so even a specialized transit space-based telescope like COROT will struggle to spot the signature (Sartoretti & Schneider 1999). An additional problem lies in the fact that much of the time the exomoon will not appear to be its orbital distance from the planet, but some fraction of it, depending on the orbital phase of the exomoon during transit. The moon can effectively hide behind the planet or in front of it. This makes disentangling the photometric signature exceptionally difficult and was outlined by Sartoretti & Schneider (1999) and Cabrera & Schneider (2007).

Previously, several different authors (Szabó et al. 2006; Sartoretti & Schneider 1999 and Simon, Szatmáry & Szabó 2007) have noted that TTVs could be used to indirectly detect the presence of such exomoons. In the first half of this paper, we update the model for the TTV effect by including orbital eccentricity. We compare our formulation to the previously proposed mathematical treatment and demonstrate our model reduces to the original analytic equations for zero eccentricity.

However, the crucial problem with TTV is that the amplitude of the signal is proportional to both the exomoon mass, \( M_S \), and orbital separation, \( a_S \), Ford & Holman (2007) referred to this obstacle as the ‘inverse problem’, in regard to transit timing effects. In Section 3.1, we show that the exomoon’s period cannot be reliably determined from TTV and hence \( a_S \) remains unknown. Ergo, one cannot
establish the mass of the exomoon without assuming a value for the orbital separation. It is therefore clear that a strong desideratum is a secondary method which can complement TTV and remove this degeneracy and this constitutes the focus of the second part of our paper.

By considering the transit duration, we predict a new observable timing effect due to an exomoon, which we label as TDV. We find that the amplitude of this timing signal is of the same order of magnitude to the TTV signal and indeed often larger. The effect is also predicted to be $\propto M_0 a^{-1/2}$. Hence, the ratio of TDV to TTV allows for the mass of the exomoon to be found without assuming an orbital distance. In addition, TDV is $\pi/2$ out-of-phase with TTV, making it an ideal complementary method for exomoon detection.

2 TTV AMPLITUDE DUE TO AN EXOMOON

2.1 Outline of the model

In the first half of this paper, we aim to update the model for the TTV signal due to a transiting exoplanet with a single satellite of mass $M_0$ and non-zero orbital eccentricity. In this work, we consider the variation of the mid-transit point of the planetary transit, $T_{\text{MID}}$. This is in contrast to Simon et al. (2007) who consider the planet and moon combined transit. Throughout our discussion, we also make the assumption that planet–moon orbital plane is co-aligned with the planet–star orbital plane.

In our case, the planet orbits the barycentre of the planet–moon system with a semimajor axis of $a_\text{W}$, where $W$ denotes wobble. The fact that $a_\text{W} > 0$ means that the time between the planet being at the mid-transit point and the barycentre being at the mid-transit point is, in general, non-zero, and this is the origin of the TTV effect.

Consider the projected distance between the planet and the planet–moon barycentre to be given by $\tilde{x}_2$, as illustrated in Fig. 1. The TTV effect will be given by $\tilde{x}_2$ divided by the $\tilde{x}_2$-direction component of the barycentre’s orbital velocity around the star, given by $v_{B,L}$. Since $\tilde{x}_2$ is a function of the planet’s true anomaly around the planet–moon barycentre, $f_\text{W}$, so too is the TTV effect:

$$TTV(f_\text{W}) = \frac{\tilde{x}_2(f_\text{W})}{v_{B,L}}. \quad (1)$$

In the case of a circular orbit, we expect the TTV signal to have a sinusoidal nature, and infact the peak-to-peak amplitude of the wave may be unambiguously defined as $2a_\text{W}/v_{B,L}$. In the case of eccentric orbits, we expect non-sinusoidal waveforms, due to Kepler’s equation. Peak-to-peak amplitudes can become ambiguous in such cases, and so we choose to use the rms amplitude definition, which is valid for all waveforms. For a simple circular orbit, the rms amplitude of the TTV signal, $\delta_{TTV}$, will be given by

$$\delta_{TTV}(\text{circular}) = \frac{a_\text{W}}{\sqrt{2}v_{B,L}}, \quad (2)$$

where the distance $a_\text{W}$ is small enough that we may assume $v_{B,L}$ is a constant over the time-scale of the TTV effect.

Let us now extend the analysis to include the effect of exomoon orbital eccentricity, $e_\text{S}$ and position of pericentre, $\sigma_{B,S}$, as well as planetary eccentricity $e_\text{P}$ and position of pericentre, $\sigma_{B,P}$. In Appendix A, we derive the general equation for the rms amplitude of the TTV effect due to an exomoon to be given by

$$\delta_{TTV} = \frac{1}{\sqrt{2}} \frac{a_\text{W}^{1/2} a_\text{S} M_0 M_{\text{PRV}}}{\sqrt{G(M_0 + M_{\text{PRV}})}} \frac{\zeta_1(e_\text{S}, \sigma_{B,S})}{\Upsilon(e_\text{P}, \sigma_{B,P})}, \quad (3)$$

where

$$\zeta_1 = \frac{(1 - e_\text{S}^2)^{1/2}}{e_\text{S}} \sqrt{e_\text{S}^2 + \cos(2\sigma_\text{S})[2(1 - e_\text{S}^2)]^{1/2} - 2 + 3e_\text{S}^2}, \quad (4)$$

$$\Upsilon = \cos \left[ \arctan \left( \frac{-e_\text{P}\cos\sigma_{B,P}}{1 + e_\text{P}\sin\sigma_{B,P}} \right) \right] \sqrt{\frac{2(1 + e_\text{S}\sin\sigma_{B,S})}{(1 - e_\text{S}^2)}} - 1, \quad (5)$$

where $a_\text{S}$ is the semimajor axis of the exoplanet’s orbit around the central star, $e_\text{S}$ is the semimajor axis of the exomoon’s orbit around the exoplanet, $G$ is the gravitational constant, $M_0$ is the mass of the host star and $M_{\text{PRV}} = M_0 + M_{\text{P}}$, i.e. the combined mass of the exomoon and exoplanet, respectively.\(^1\)

In the case of $e_\text{S} \to 0$, we have $\zeta_1 \to 1$ and similarly for $e_\text{P} \to 0$ we have $\Upsilon \to 1$. Thus, we may compare our equation to that derived by previous authors, who assumed circular orbits and expect to produce the same result. Sartoretti & Schneider (1999) predicted the peak-to-peak amplitude of an exomoon around an exoplanet with orbital period $P_\text{P}$ to be given by equation (6):

$$\Delta t_{\text{PRV}} \sim 2a_\text{W} M_0 M_{\text{P}}^{-1} \times P_\text{P}(2\pi a_\text{P})^{-1}. \quad (6)$$

Using Kepler’s third law, it is trivial to show that this is equivalent to our expression, for circular orbits and $M_P \gg M_0$, except for a factor of $2\sqrt{2}$, which comes from the fact we use an rms definition of amplitude, rather than a peak-to-peak definition. Thus, we have confirmed that our general expression for the TTV effect due to an exomoon is equivalent to the circular equations first derived by Sartoretti & Schneider (1999). For completion, we note that the TTV amplitude may be written purely in terms of the masses of the three bodies and the period of the exoplanet (see Appendix A, equation A28 and A29).

The effect of eccentricity is discussed in more detail in Section 3.3 and typical waveforms produced are shown later in Fig. 3, Section 4.2. In Section 4.2, Table 1 gives a list of the expected TTV rms amplitudes due to a $1 M_\text{E}$ exomoon around some of the most promising exoplanet candidates.

\(^1\) $M_{\text{PRV}}$ is the mass of the planet as measured by radial velocity surveys, and thus includes the mass of satellites.

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\(\frac{\text{R}}{\text{A}}\)
Table 1. Predicted TTV and TDV rms amplitudes due to a 1 $M_\oplus$ exomoon, for a selection of the best candidate transiting planets. System parameters are taken from various references, which are shown.

<table>
<thead>
<tr>
<th>Planet</th>
<th>$\delta_{\text{TTV}}$ s$^{-1}$</th>
<th>$\delta_{\text{TDV}}$ s$^{-1}$</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>GL436b</td>
<td>13.71</td>
<td>12.65</td>
<td>Alonso et al. (2008)</td>
</tr>
<tr>
<td>CoRoT-Exo-4b</td>
<td>7.72</td>
<td>9.28</td>
<td>Aigrain et al. (2008)</td>
</tr>
<tr>
<td>HAT-P-1b</td>
<td>4.58</td>
<td>6.78</td>
<td>Johnson et al. (2008)</td>
</tr>
<tr>
<td>OGLE-TR-111b</td>
<td>4.63</td>
<td>7.27</td>
<td>Diaz et al. (2008)</td>
</tr>
<tr>
<td>HD 199026b</td>
<td>3.61</td>
<td>9.68</td>
<td>Winn et al. (2007)</td>
</tr>
<tr>
<td>Lupus-TR-3b</td>
<td>3.28</td>
<td>5.17</td>
<td>Weldrake et al. (2008)</td>
</tr>
<tr>
<td>WASP-7b</td>
<td>3.26</td>
<td>5.86</td>
<td>Hellier et al. (2008)</td>
</tr>
<tr>
<td>TrES-1b</td>
<td>3.07</td>
<td>5.97</td>
<td>Winn, Holman &amp; Rousseanov (2007)</td>
</tr>
<tr>
<td>HD 17156b</td>
<td>3.07</td>
<td>1.06</td>
<td>Barbieri et al. (2007)</td>
</tr>
<tr>
<td>HD 209458b</td>
<td>2.97</td>
<td>5.92</td>
<td>Kipping (2008)</td>
</tr>
<tr>
<td>XO−5b</td>
<td>2.65</td>
<td>4.68</td>
<td>Burke et al. (2008)</td>
</tr>
<tr>
<td>HAT-P-4b</td>
<td>2.54</td>
<td>8.30</td>
<td>Kovacs et al. (2007)</td>
</tr>
<tr>
<td>HD 189733b</td>
<td>1.51</td>
<td>2.94</td>
<td>Winn et al. (2007)</td>
</tr>
<tr>
<td>XO−3b</td>
<td>0.41</td>
<td>0.91</td>
<td>Winn et al. (2008)</td>
</tr>
</tbody>
</table>

2.2 Permitted range for $a_\delta$

The orbital radius of any satellite around a planet must lie somewhere between the Hill radius, $d_{\text{max}}$, and the Roche limit, $d_{\text{min}}$, to maintain stability.\(^2\) We express $a_\delta$ by assuming that it is equal to some fraction, $\chi$, of the Hill radius, $d_{\text{max}}$:

$$a_\delta = \chi d_{\text{max}}.$$

$$d_{\text{max}} = a_p \left( \frac{M_P}{3M_*} \right)^{1/3} = a_p \left( \frac{M_{\text{PRV}} - M_S}{3M_*} \right)^{1/3},$$

where $a_p$ is the semimajor axis of the planet around the host star, $R_p$ is the planetary radius, $M_*$ is the mass of the central star, and $\rho$ denotes density.

$\chi$ may be further constrained by noting that Barnes & O’Brien (2002) estimated $\chi \lesssim 1/3$ and Domingos, Winter & Yokoyama (2006) estimated $\chi \lesssim 1/2$. This is because the Hill sphere is just an approximation, and in reality other effects, like radiation pressure or the Yarkovsky effect, can perturb a body outside of the sphere. We choose to use the conservative choice of $\chi \lesssim 1/3$ and combining this limit with the Roche limit, and rewriting in terms of planetary period, $P_p$ for $M_\star \gg M_p$, we can estimate

$$\left( \frac{18\pi}{G P_p^2 \rho S} \right)^{1/3} \lesssim \chi \lesssim \frac{1}{3}.$$

3 IMPLICATIONS

3.1 The high frequency nature of an exomoon’s TTV

If we make the approximation that $M_\star \gg M_{\text{PRV}}$ and $M_{\text{PRV}} \gg M_S$, and employ Kepler’s third law, we may provide an estimate for the ratio of the exomoon to planet orbital period, which can be shown (see Appendix B) to be

$$\frac{P_s}{P_p} \simeq \sqrt{\frac{\chi}{3}}.$$

Since $\chi \lesssim 1$ we will always be in the regime where $P_s < P_p$. Taking $\chi \sim 1/3$ gives a rough estimate of $P_s/P_p \sim 1/9$ and so the frequency of the signal is $\tau_k \sim 9P_p^{-1}$. However, the Nyquist frequency will be given by one-half of the sampling rate ($0.5P_p^{-1}$), which represents the maximum frequency we can resolve without aliasing. The usual technique for detecting signals within data is to employ a periodogram, but this method will suffer from aliasing in the exomoon case. Therefore, we can only derive a set of harmonic frequencies which the exomoon’s orbit could exhibit. We therefore conclude $P_S$ cannot be reliably determined from the TTV signal.

3.2 The limitation of TTV in determining $M_\star$

As originally pointed out by Sartoretti & Schneider (1999), the major problem with TTV is that one cannot determine the mass of the exomoon without making an assumption on the distance at which the moon orbits the planet. Using equation (3), it is possible to write the TTV amplitude as a function of purely the exomoon properties:

$$\delta_{\text{TTV}} \propto M_\ast a_\delta.$$

Therefore, we can effectively only determine the moment of the exomoon. If one knows the period of the exomoon, then it is trivial to derive $a_\delta$ using Kepler’s third law, but as seen in Section 3.1, $P_p$ cannot be reliably ascertained.

This crucial limitation of exomoon TTV makes mass estimation unfeasible and the best we can ever do is merely provide evidence for an exomoon within a large mass range. This severe limitation will be resolved later in this paper.

\(^2\) Note we use the rigid body Roche limit for simplicity.
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Figure 2. Behaviour of $\gamma^{-1}$ as a function of $e_P$ and $\sigma_P$, which is effectively the factor by which the TTV is enhanced by $e_P > 0$. Positions of pericentre near 270° produce consistently enhanced TTV amplitudes. The flat line corresponds to $e_P = 0$ and each progressively larger amplitude wave is for $e_P = 0.3, 0.6$ and 0.9, respectively.

3.3 The effect of eccentricity

The effects of orbital eccentricity on the TTV amplitude are all absorbed into the two parameters, $\zeta_T(e_S, \sigma_S)$ and $\Upsilon(e_P, \sigma_P)$. If we increase $e_S$ from zero to unity, regardless of what value $\sigma_S$ takes, $\zeta_T$ decreases below 1 and hence the TTV amplitude will always decrease. This implies exomoon detections are biased towards satellites on circular orbits.

The situation is more complicated for $e_P$, where a non-zero $e_P$ makes the TTV amplitude significantly increased or decreased depending on $\sigma_P$. In Fig. 2, we plot $\gamma^{-1}$ as a function of $\sigma_P$ for several different eccentricities and find that for $e_P > 0$, the most favourable position of pericentre is $\sigma_P \sim 3\pi/2$. However, we point out a recent study by Kane & von Braun (2008) which predicts such exoplanets to possess a low transit probability.

For the eccentric transiting exoplanets GJ436b, XO-3b, HAT-P-2b and HD 17156b, $\gamma^{-1}$ takes values 1.01, 1.03, 0.94 and 0.47, respectively. However, the stability of satellites around eccentric exoplanets remains unclear.

4 TRANSIT DURATION VARIATION (TDV)

4.1 The TDV due to an exomoon

TDV is the periodic change in the duration of a transit ($t_T$) over many measurements, where we define $t_T$ as the time between the first and fourth contact points. It has previously been discussed as a possible test of general relativity by Pál & Kocsis (2008). In this discussion, we consider the TDV due to an exomoon and conclude that it should produce a detectable signal.

We first consider that the duration of a transit is inversely proportional to the projected velocity of the planet across the star, $v_P \perp$, and make the following assumptions.

(i) The orbital inclination of the planet and exomoon does not vary from orbit to orbit.
(ii) We do not consider additional perturbing bodies in the system:

\[ t_T \propto 1/v_P \perp. \]

In this case, any TDV must be solely due to the variation of the velocity. For a single companion exomoon, the velocity of the planet will have two components:

\[ v_P \perp = v_B \perp + v_W \perp, \]

where $v_B$ is the velocity of the planet–moon barycentre around the host star and $v_W$ is the velocity of the planet around the planet–moon barycentre, i.e. the wobble of the planet due its companion satellite.

It is clear that $v_W \perp$ will be significantly different for each transit unless $P_P/P_B$ is some low-order integer. Sometimes $v_W \perp$ will be additive to the barycentre's velocity and sometimes subtractive resulting in shorter and longer transit durations, respectively; an effect we label TDV.

The factor by which $t_T$ will vary must be equal to the ratio of the velocities $R = v_W \perp/v_B \perp$ and from this starting point the TDV amplitude may be shown (see Appendix C2) to be equal to

\[ \delta_{TDV} = \frac{v_P \perp}{a_S} = \frac{M_S^2}{M_{PRV}(M_{PRV} + M_*)} \sqrt{2/\gamma} \frac{t_T}{\zeta_D(e_S, \sigma_S)}, \]

where

\[ \zeta_D(e_S, \sigma_S) = \frac{1 + e_S^2 - e_S^2 \cos(2\sigma_S)}{1 - e_S^2}. \]

In an analogous way to the TTV amplitude, this may be rewritten in terms of simply the masses in the system, and this equation may be found in the Appendix C2, equations (C28) and (C29). We also point out that this effect has the following proportionality:

\[ \delta_{TDV} \propto M_S a_S^{-1/2}. \]

As a final note, we point out that in Section 3.3 we discussed how increasing $e_S$ tends to decrease the TTV amplitude. For the TDV signal, the opposite is true, increasing $e_S$ tends to increase the TDV signal.

4.2 TTV and TDV as complementary methods

From equations (13) and (18), it is clear that the ratio of TDV to TTV should be able to eliminate $M_S$ and we can directly measure the orbital separation of the exomoon $a_S$ and hence $M_S$. The introduction of TDV allows for the precise measurement of $M_S$ without any assumption on the orbital separation. Using Kepler’s third law, it is then possible to derive the exomoon’s orbital period, which was shown in Section 3.1 to be unattainable from TTV alone. In Appendix D, it is shown that if we assume $e_S \simeq 0$ (but $e_P$ can take any value between zero and unity), then $\eta = \delta_{TDV}/\delta_{TTV}$ is approximately given by

\[ \eta = \frac{\delta_{TDV}}{\delta_{TTV}} \approx \frac{2\pi t_T}{P_P} \frac{\sqrt{3}}{\chi^{1/2}}. \]

Therefore, one may determine $\chi$ and hence $a_S$ (and $M_S$) if one assumes the exomoon has zero eccentricity. This assumption is supported based on a study by Domingos et al. (2006) and the pattern observed amongst large ($>500$ km) moons in our own Solar system.

Another major advantage of TDV is that the signal should lag TTV by a $\pi/2$ phase difference, originating from the fact that TTV is a spatial effect whereas TDV is a velocity effect. Unfortunately, the direction and value of the phase shift remain unchanged between prograde and retrograde satellites.

Combining TTV and TDV should allow for a much more significant confirmation of a potential exomoon than just using TTV alone. The phase difference can be seen in the example waveforms in Fig. 3, where we plot the TTV and TDV waveforms due to a hypothetical exomoon of 1 M$_{\oplus}$ around GJ436b. We show (a) the effects of exomoon eccentricity: the solid line gives $e_S = 0$, the dashed line $e_S = 0.3$ and the dotted line $e_S = 0.6$; (b) TTV leads TDV by a $\pi/2$ phase difference; (c) the high-frequency nature of both waves (as discussed in Section 3.1) is apparent and (d) increasing $e_S$ tends to decrease the TTV amplitude and increase the TDV amplitude (as discussed in Sections 3.3 and 4.2).

5 DISCUSSION AND CONCLUSIONS

We have presented an updated model for the TTV signal due to an exomoon to include the effects of orbital eccentricity in both the exoplanet and the exomoon. From the updated TTV model, we draw the following conclusions.

(i) TTV is degenerated in that it can only determine $M_S \times a_S$, where $M_S$ is the exomoon mass and $a_S$ is the exomoon’s orbital radius.

(ii) The TTV due to an exomoon can be significantly enhanced for exoplanets of $e_P > 0$ and $\sigma_P \sim 270^\circ$. However, it remains unclear how dynamically stable such exomoons would be.

(iii) The TTV frequency will always be greater than the sampling frequency of once every transit, implying we can only determine a set of possible harmonic frequencies for the exomoon’s period, $P_S$.

An exomoon is predicted to have another detectable timing effect on a transit in the form of TDV. We have derived an equation for predicting the TDV amplitude and drawn the following conclusions.

(i) The ratio of TDV to TTV allows for the separate determination of $M_S$ and $a_S$; thus solving the ‘inverse problem’ for exomoons.

(ii) The TDV signal is of a similar order of magnitude to the TTV signal.

(iii) the TDV signal is also enhanced for $e_P > 0$ and $\sigma_P \sim 270^\circ$.

(iv) TDV lags TTV by a $90^\circ$ phase difference, making it an excellent complementary technique for exomoon detection.

We also find that current ground-based telescopes could detect a 1 M$_{\oplus}$ exomoon in the habitable zone around a Neptune-like exoplanet. The author would therefore encourage observers to produce not only their mid-transit times, but also transit durations for each transit, rather than composite light curve durations. This will allow constraints to be placed on the presence of exomoons around such planets.

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REFERENCES


APPENDIX A: TTV RMS AMPLITUDE

The rms amplitude of a waveform with displacement TTV ($f_w$) as a function of some variable $f_w$ may be written as

$$
\delta_{TTV} = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} \langle [TTV(f_w)]^2 \rangle df_w},
$$

(A1)

where $f_w$ is true anomaly of the planet during its orbit around the planet–moon barycentre and the only variable we integrate over and thus we are assuming.

(i) $e_p$ and $\sigma_p$ do not change over the measurement time-scale of the TTV effect.

(ii) There are no secular changes in the planet’s orbit.

(iii) $e_s$ does not change over the measurement time-scale of the TTV effect.

(iv) $\sigma_s$ takes the same value when sampled once every planetary orbital period, but $f_w$ do not.

In our case, the function $TTV(f_w)$ is given by the projected distance between the planet and the planet–moon barycentre divided by the projected velocity of the barycentre, as a function of the planet’s true anomaly around the planet–moon barycentre, $f_w$:

$$
TTV(f_w) = \frac{\tilde{x}_2(f_w, e_s, \sigma_s)}{v_B(e_p, \sigma_p)},
$$

(A2)

where $\tilde{x}_2(f_w, e_s, \sigma_s)$ is the projected displacement of the planet away from the planet–moon barycentre (see Fig. 1), as a function of $f_w$.

To find $\tilde{x}_2(f_w, e_s, \sigma_s)$, we take a similar approach to that of Kipping (2008), where the author considered an initial frame $S_1$ and then made a series of transformations. If we start out with the same setup as the cited author, we have an ellipse located with the centre at the origin of an $\tilde{x} - \tilde{y}$ plot and the barycentre located at $(\omega_w e_w, 0)$. We account for the position of periapsis (defined in Fig. 1) by rotating the ellipse counterclockwise by an angle $\sigma_w$ about the $\tilde{x}$-axis. This gives us the $S_1$ frame:

$$
\tilde{x}_1 = \tilde{x}\cos\sigma_w - \tilde{y}\sin\sigma_w,
$$

(A3)

$$
\tilde{y}_1 = \tilde{x}\sin\sigma_w + \tilde{y}\cos\sigma_w.
$$

(A4)

To match the figure, we require a translation to place the planet–moon barycentre at the origin. After applying this translation, we have found the desired $S_1$ frame:

$$
\tilde{x}_2 = \tilde{x}_1 - a_w e_w \cos\sigma_w,
$$

(A5)

$$
\tilde{y}_2 = \tilde{y}_1 - a_w e_w \sin\sigma_w.
$$

(A6)

Note that $e_w$, the eccentricity of the planet’s orbit around the planet–moon barycentre, must be equal to $e_s$. However, $\sigma_w$ will have a phase difference of $\pi$ relative to $\sigma_s$. In the defined ellipse, the distance between the barycentre and the centre of the planet, $r_w$, as a function of true anomaly, $f_w$, is given by

$$
r_w(f_w) = \frac{a_w (1 - e_w^2)}{1 + e_w \cos f_w}.
$$

(A7)

With this equation, we may define the $\tilde{x}$ and $\tilde{y}$ positions of the centre of the planet as a function of $f_w$:

$$
\tilde{x} = a_w e_w + r_w(f_w) \cos \phi_w.
$$

(A8)

$$
\tilde{y} = r_w(f_w) \sin \phi_w.
$$

(A9)

Having defined $\tilde{x}(f_w)$ and $\tilde{y}(f_w)$, we have also acquired $\tilde{x}_2(f_w)$, using equation (A5). Putting $\tilde{x}_2(f_w)$ into equation (A2) and integrating over the limits 0 and $2\pi$, as dictated by equation (A1), we find

$$
\delta_{TTV} = \frac{a_w \xi_T(e_s, \sigma_s)}{\sqrt{2 v_B(e_p, \sigma_p)}}.
$$

(A10)

where we define $\xi_T(e_s, \sigma_s)$ by

$$
\xi_T = \frac{(1 - e_s^2)^{1/4}}{e_s} \sqrt{e_s^2 \cos(2\sigma_s) [2(1 - e_s^2)^{1/2} - 2 + 3 e_s^2]}.
$$

(A11)

Note that we have replaced $e_w$ with $e_s$, since they are equivalent, and $\sigma_w$ with $\sigma_s$ since the $\pi$ phase difference between them does not affect this expression.

$a_w$ may be rewritten as

$$
a_w = a_s \frac{M_s}{M_{PRV}}
$$

(A12)

and $a_s$ may be expressed in terms of the Hill radius, $d_{max}$:

$$
a_s = \chi d_{max} = \chi a_p \left( \frac{M_{PRV} - M_s}{3 M_s} \right)^{1/3}.
$$

(A13)

In this expression, $a_p$ may also be rewritten using Kepler’s third law:

$$
a_p = \frac{G(M_{PRV} + M_s) P_{p}^2}{4 \pi^2}.
$$

(A14)

Finally giving

$$
a_w = \chi \frac{M_s}{M_{PRV}} \left( \frac{G(M_{PRV} + M_s) (M_{PRV} - M_s) P_{p}^2}{12 \pi^2 M_s} \right)^{1/3}.
$$

(A15)

Let us now turn our attention to the expression for $v_B(e_p, \sigma_p)$. If we assume the time it takes for the planet to cross a distance $a_w$ is small compared to the period of the orbit, then we may assume $v_B$ does not vary. However, it is only the perpendicular component of $v_B$, which we label as $v_{B\perp}$, which dictates the TTV effect. In the case of a circular orbit, it is easy to see that $v_{B\perp} = v_B$, but for eccentric orbits, this is not the case.

Consider the orbit of the planet–moon barycentre around the host star. To find the perpendicular component of the barycentre’s
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Thus,
\[ v_{\text{Bl}} = \Upsilon(e_P, \sigma_P) \sqrt{\frac{G(M_{\text{PRV}} + M_*)}{a_P}}. \]  
(A25)

The final expression may be written as
\[ \delta_{\text{TTV}} = \frac{a_P \sqrt{e_P}}{\sqrt{2GM_{\text{PRV}} + M_*}} \frac{\zeta(e_P, \sigma_P)}{\Upsilon(e_P, \sigma_P)}, \]  
(A26)

where \( a_P \) is given by equation (A15), \( e_P \) is given by equation (A14), \( \zeta(e_P, \sigma_P) \) is given by equation (A11) and \( \Upsilon(e_P, \sigma_P) \) is given by equation (A27). An alternative expression of (A26) is given by
\[ \delta_{\text{TTV}} = \frac{1}{2} \frac{a_P^{1/2} M_5 M_{\text{PRV}}^{1/2}}{\sqrt{2GM_5 + M_{\text{PRV}}}} \frac{\zeta(e_P, \sigma_P)}{\Upsilon(e_P, \sigma_P)}. \]  
(A27)

It is therefore clear that our derived equation is equivalent to that of previous authors in the case of circular orbits. It is also clear that \( \delta_{\text{TTV}} \propto M_5 a_5 \).

For completion and scaling purposes, we also derive the TTV amplitude in terms of just masses and periods by using Kepler's third law:
\[ \delta_{\text{TTV}} = \frac{P_P}{2\pi} \frac{\chi}{3^{1/2}} \frac{Z_T(M_*, M_{\text{PRV}}, M_3)}{\zeta(e_P, \sigma_P)}, \]  
(A28)

where \( Z_T(M_*, M_{\text{PRV}}, M_3) \) is the TTV mass function given by
\[ Z_T = \left( M_5^2(M_{\text{PRV}} - M_3)^2 \right)^{1/6} \approx \left( \frac{M_5^2}{M_5^2 M_{\text{PRV}}^2} \right)^{1/6}, \]  
(A29)

where the approximation is based on \( M_{\text{PRV}} \gg M_5 \).

APPENDIX B: THE \( P_5/P_0 \) RATIO

From Kepler's third law, we may write
\[ P_S = \frac{4\pi^2 a_5^3}{GM_{\text{PRV}}}, \]  
(B1)

where \( a_5 \) is some fraction, \( \chi \), of the Hill radius:
\[ a_5 = \chi a_P \left( \frac{M_{\text{PRV}} - M_3}{3M_*} \right)^{1/3} \approx \chi a_P \left( \frac{M_{\text{PRV}}}{3M_*} \right)^{1/3}. \]  
(B2)

Putting equations (B1) and (B2) together
\[ P_S = \left( \frac{4\pi^2 \chi a_P^3}{3GM_*} \right)^{1/2}. \]  
(B3)

In comparison, the orbital period of the planet around the host star is given by
\[ P_P = \left( \frac{4\pi^2 a_P^3}{GM_* + M_{\text{PRV}}} \right)^{1/2} \approx \left( \frac{4\pi^2 a_P^3}{GM_*} \right)^{1/2}. \]  
(B4)

Therefore, we may write
\[ \frac{P_S}{P_P} \approx \sqrt{\frac{\chi}{3}}. \]  
(B5)

APPENDIX C: DERIVATION OF THE TDV AMPLITUDE

C1 Derivation of the circular form

We will follow the same definition for TTV as for TTV, where TTV is equal to the observed-calculated (OC) transit duration. The
calculated (or expected) transit duration, \( \tau \), is given by

\[
\tau = \frac{X}{v_{BL}},
\]

where \( X \) is the distance the planet has to cross in order to complete the transit and \( v_B \) is the velocity of the planet during transit, as given by Kepler’s third law.

In the case of an additional moon orbiting the planet, the velocity of the planet now has two components, the velocity of the planet–moon barycentre (\( v_B \)) and the wobble velocity due to the perturbation of the moon:

\[
\tau_O = \frac{X}{v_{BL} + v_{WL}}. \tag{C2}
\]

Therefore, TDV is defined as

\[
TDV = \tau_O - \tau = \left( \frac{v_{BL}}{v_{BL} + v_{WL}} - 1 \right) \tau. \tag{C3}
\]

If \( v_{BL} \gg v_{WL} \), then we may write

\[
TDV \simeq -\frac{v_{WL}}{v_{BL}} \tau. \tag{C4}
\]

If we assume a circular orbit, then the velocity of the planet around the barycentre of the planet–moon system, the wobble velocity, is given by

\[
v_W = \frac{2\pi a_W}{P_S} = \frac{2\pi M_S a_S}{M_{PRV}} \frac{1}{P_S}. \tag{C5}
\]

Using equation (B1), we may write

\[
v_W = \sqrt{\frac{GM_S^2}{a_S M_{PRV}}}. \tag{C6}
\]

However, we want the perpendicular component of this velocity, \( v_{WL} \), as a function of true anomaly, \( f_W \). For a circular orbit, it is trivial to show that the variation will be sinusoidal and hence the rms amplitude of \( v_{WL} \) is given by

\[
|v_{WL}| = \frac{1}{\sqrt{2}} \left( \frac{GM_S^2}{a_S M_{PRV}} \right)^{1/2}. \tag{C7}
\]

Let us now consider the velocity of the planet–moon barycentre around the host star. For a circular orbit, the velocity of the barycentre is given by

\[
v_B = v_{BL} = \frac{2\pi a_P}{P_P} \simeq \left( \frac{2\pi G M_*}{P_P} \right)^{1/3}. \tag{C8}
\]

Once again, due to the circular orbit, the perpendicular component of the velocity is the same as the absolute value during transit. The ratio of \( v_{WL} / v_{BL} \) multiplied by the transit duration will be the TDV signal:

\[
TDV = \frac{v_{WL}}{v_{BL}} t_T = R t_T. \tag{C9}
\]

In the case of circular orbits, \( R \) is therefore given by

\[
R = \frac{1}{\sqrt{2}} \left( \frac{GM_P^2}{a_S M_{PRV}} \right)^{1/2} \left( \frac{2\pi G M_*}{P_P} \right)^{-1/3}. \tag{C10}
\]

From equation (C5), we may instantly infer that the TDV effect is \( \propto M_S a_S^{1/2} \), in contrast to the TTV effect which is \( \propto M_S a_S \). In the case of \( M_* \gg M_{PRV} \) and \( M_{PRV} \gg M_S a_S \), this is given by

\[
\delta S \simeq x \left( \frac{GM_{PRV} P_P^5}{12\pi^2} \right)^{1/3}, \tag{C11}
\]

and feeding this into the equation for \( R \) we get:

\[
R = \frac{1}{\sqrt{2}} \left( \frac{3M_M^6}{M_{PRV}^2 M_{PRV}^4} \right)^{1/6}. \tag{C12}
\]

Therefore, the TDV amplitude for a planet, on a circular orbit, with a single moon, which is also on a circular orbit, is given by

\[
\delta_{TDV} \simeq \frac{\tau_T}{\sqrt{2}} \left( \frac{3M_M^6}{M_{PRV}^2 M_{PRV}^4} \right)^{1/6} Z_D(M_*, M_{PRV}, M_S), \tag{C13}
\]

where the TDV mass function is defined by

\[
Z_D(M_*, M_{PRV}, M_S) \simeq \left( \frac{M_M^6}{M_{PRV}^2 M_{PRV}^4} \right)^{1/6}. \tag{C14}
\]

### C2 Derivation of the eccentric form

Here, we derive the TDV rms amplitude in the case of eccentric orbits. We use the definition of TDV given by equation (C4). The orbital velocity of the moon around the barycentre of the planet–moon system for a circular orbit is given by

\[
v_B = \sqrt{\frac{GM_S^2}{a_S M_{PRV}}} = \sqrt{\frac{GM_*^2}{a_{PRV}}} = \frac{\mu}{a_W}. \tag{C15}
\]

In the case of an eccentric orbit, this velocity becomes a function of true anomaly, \( f_W \):

\[
v_W(f_W) = \frac{\mu}{r_W} \left( \frac{2}{r_W} - \frac{1}{a_W} \right)^{1/2}. \tag{C16}
\]

The TDV signal has a waveform governed by the ratio of the perpendicular component of the planet’s wobble velocity to the perpendicular component of the velocity of the planet–moon barycentre around the host star, multiplied by the duration of the transit:

\[
TDV(f_W) = \frac{t_T}{v_{BL}} v_{WL}(f_W). \tag{C17}
\]

In order to find the rms amplitude of this signal, we need to derive \( v_{WL}(f_W) \), which is not the same as \( v_W(f_W) \). To do this, we need to find the gradient of the tangent of the planet’s position along our transformed ellipse. To find the gradient of the tangent to the ellipse at any true anomaly, \( f_W \), we need to implicitly differentiate the equation for the ellipse in the \( \tilde{x} \) frame. Equations (A5) and (A6) may be rewritten making \( \tilde{x} \) and \( \tilde{y} \) the subject:

\[
\tilde{x} = -\tilde{x}_2 \cos \sigma_W - \tilde{y}_2 \sin \sigma_W - a_W e_W, \tag{C18}
\]

\[
\tilde{y} = \tilde{x}_2 \sin \sigma_W - \tilde{y}_2 \cos \sigma_W. \tag{C19}
\]

These two equations must satisfy the standard equation of the ellipse:

\[
\frac{\tilde{x}^2}{a_W^2} + \frac{\tilde{y}^2}{a_W^2(1-e_W^2)} = 1 \tag{C20}
\]

We substitute equations (C18) and (C19) into equation (C20) and then differentiate implicitly with respect to \( \tilde{x}_2 \). We arrange the
equation to make $\frac{d\tilde{y}_2}{d\tilde{x}_2}$ the subject. Finally, we replace the $\tilde{x}_2$ and $\tilde{y}_2$ terms using equations (A5) and (A6) and find

$$\frac{d\tilde{y}_2}{d\tilde{x}_2} = \frac{\tilde{y} \sin \sigma_w - (1 - e^2) \tilde{y} \cos \sigma_w}{\cos \sigma_w + (1 - e^2) \tilde{y} \sin \sigma_w}. \quad \text{(C21)}$$

$\tilde{x}(f_w)$ and $\tilde{y}(f_w)$ are known and so we have found the gradient of our tangent. The angle of the tangent is given by

$$\tilde{\theta}(f_w) = \arctan \left( \frac{d\tilde{y}_2}{d\tilde{x}_2} \right). \quad \text{(C22)}$$

and finally, we may express $v_{w,\perp}(f_w)$ as

$$v_{w,\perp}(f_w) = v_w(f_w) \cos[\tilde{\theta}(f_w)]. \quad \text{(C23)}$$

We now take the rms of this waveform in the way detailed in Appendix A and find

$$\delta \text{TDV} = \frac{t_T}{v_{B,\perp}} \sqrt{\frac{\mu_w}{2\sigma_w}} \zeta_d(e_3, \sigma_3), \quad \text{(C24)}$$

where

$$\zeta_d(e_3, \sigma_3) = \sqrt{\frac{1 + e_3^2 - e_3^2 \cos(2\sigma_3)}{1 - e_3^2}}. \quad \text{(C25)}$$

Note that we have replaced $e_w$ by $e_3$ since they are equivalent and similarly for $\cos 2\sigma_w$ and $\cos 2\sigma_3$. Taking the limit of this expression as $e_3 \to 0$ gives the expected result of $t_T \sqrt{\mu_w/v_w} \sqrt{2\sigma_w} v_{B,\perp}$.

Using this and equation (C9), we may rewrite $\delta \text{TDV}$ as

$$\delta \text{TDV} = \sqrt{\frac{a_p}{a_\ast}} \sqrt{\frac{M_\ast^2}{M_{PRV}(M_{PRV} + M_\ast)}} \frac{t_T}{\sqrt{2}} \tilde{\zeta}_d(e_3, \sigma_3). \quad \text{(C27)}$$

This once again demonstrates the effect’s proportionality of $\propto M_\ast a_\ast^{-1/2}$. Using equation (A13), the fraction $a_p/a_\ast$ may be substituted for and we find

$$\delta \text{TDV} = \frac{t_T}{\sqrt{2}} \frac{3^{1/6}}{\sqrt{2}} Z_D(M_\ast, M_{PRV}, M_\ast) \frac{\tilde{\zeta}_d(e_3, \sigma_3)}{T(e_p, \sigma_p)}. \quad \text{(C28)}$$

where

$$Z_D(M_\ast, M_{PRV}, M_\ast) = \left[ \frac{M_\ast M_{PRV}^2}{M_{PRV}^2(M_{PRV} + M_\ast)^3} \right]^{1/6}. \quad \text{(C29)}$$

This general expression can be shown to be equivalent to our approximate form with the same approximations made.

**APPENDIX D: RATIO OF TDV TO TTV AMPLITUDE**

In this section, we derive the ratio of the TDV amplitude to the TTV amplitude. This will allow us to analytically quickly see which signal is stronger for any new system we come across and solve for the exomoon mass exactly. We have

$$\frac{\delta \text{TTV}}{\delta \text{TDV}} = \frac{P_e}{2\pi} \frac{\chi}{3^{1/6} \sqrt{2}} Z_D(M_\ast, M_{PRV}, M_\ast) \frac{\tilde{\zeta}_d(e_3, \sigma_3)}{\tilde{\zeta}_d(e_p, \sigma_p)} \cdot \text{(D1)}$$

$$\frac{\delta \text{TDV}}{\delta \text{TTV}} = \frac{t_T}{\sqrt{2}} \frac{3^{1/6}}{\sqrt{2}} Z_D(M_\ast, M_{PRV}, M_\ast) \frac{\tilde{\zeta}_d(e_3, \sigma_3)}{T(e_p, \sigma_p)}. \quad \text{(D2)}$$

We make the assumptions that $M_\ast \gg M_{PRV}$ and $M_{PRV} \gg M_\ast$. In this case, the TTV mass function, $Z_\ast$, and the TDV mass function, $Z_D$, are equal. We also make the assumption that moon is on a circular orbit and so $\zeta_T = \zeta_D = 1$. Furthermore, the planet’s eccentricity parameter, $\chi$, will cancel out and so we need not make the assumption $e_p = 0$. This gives us the ratio of TDV to TTV, $\eta$ to be

$$\eta = \frac{\delta \text{TDV}}{\delta \text{TTV}} \approx \frac{2\pi t_T}{P_e} \frac{\sqrt{3}}{\chi^{1/2}}. \quad \text{(D3)}$$

The masses have completely cancelled out and we are able to solve the equation for $\chi$:

$$\chi \approx \left( \frac{2\pi t_T}{P_e} \frac{\sqrt{3}}{\eta} \right)^{2/3}. \quad \text{(D4)}$$

If we know $\chi$ then we have found the orbital distance the moon orbits the planet at $a_\ast$, in a completely rigorous way.

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