An analytical solution for Kepler’s problem

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ABSTRACT
In this paper, we present a framework which provides an analytical (i.e. infinitely differentiable) transformation between spatial coordinates and orbital elements for the solution of the gravitational two-body problem. The formalism omits all singular variables which otherwise would yield discontinuities. This method is based on two simple real functions for which the derivative rules are only required to be known, all other applications – e.g. calculating the orbital velocities, obtaining the partial derivatives of radial velocity curves with respect to the orbital elements – are thereafter straightforward. As it is shown, the presented formalism can be applied to find optimal instants for radial velocity measurements in transiting explanatory systems to constrain the orbital eccentricity as well as to detect secular variations in the eccentricity or in the longitude of periastron.

Key words: methods: analytical – techniques: radial velocities – celestial mechanics – ephemerides.

1 INTRODUCTION
In recent years, precise radial velocity (RV) observation of stars and careful analysis of RV data have become relevant in astrophysics since the vast majority of the known extrasolar planets have been discovered using this method (Mayor & Queloz 1995) or have been confirmed this way when the planet itself was first detected as a transiting object around its host star (Konacki et al. 2003). In the cases where there are no observable transits, the characterization of extrasolar planets relies on the RV observations alone.1 Moreover, observations of photometric transits in addition to RV measurements constrain the mass of the planet (instead of yielding only a lower limit), and provide more precise information on the epoch and period; see e.g. the case of the low-mass transiting planet HAT-P-11b (Bakos et al. 2009), where the uncertainty in the epoch would be ~500 times larger if the analysis had relied only on RV measurements. Thus, incorporating constraints given by transit timings reduce the uncertainties in the RV amplitude and the orbital parameters (e.g. semimajor axis, eccentricity).

The aim of this paper is to present a set of analytic relations (based on a few smooth functions defined in a closed form) which provides a straightforward solution of Kepler’s problem, and consequently, time series of RV data and RV model functions. Due to the analytic property, the partial derivatives can also be obtained directly and therefore can be utilized in various fitting and data analysis methods, including, for instance, the Fisher analysis of covariances, uncertainties and correlations. The functions presented here are nearly as simple to manage as trigonometric functions.

As an application, we give a detailed description about scheduling RV measurements in the case of transiting extrasolar planets in order to derive accurate orbital eccentricity and/or detect the variations in the eccentricity. Discussions related to this problem in the case of extrasolar planets with no additional constraints on their orbital motions other than RV data can be found in Loredo & Chernoff (2003), Ford (2008) or Baluev (2008). Photometric measurements for transits constrain the orbital period and epoch much more precisely than pure RV observations. Thus, for a given fixed eccentricity and argument of pericentre, the optimal time instances for RV observations depend only on the orbital phase. In the case of transiting planets, the long-term variations in the orbital eccentricity e and longitude of pericentre σ are quite relevant. The presented strategy focuses on the precise measurement of the Lagrangian orbital elements k = e cos σ and h = e sin σ (including the cases when the circular property is intended to be confirmed at high significance).

In Section 2, the basics of the mathematical formalism are presented, including the rules for calculating partial derivatives. In Section 3, the solution of the spatial problem is shown, supplemented with the inverse problem, still using infinitely differentiable functions. The spatial case discusses how the inclination and the argument of node should be incorporated in the formalism without losing the analytic property while the inverse problem describes how the orbital elements are derived from the coordinates and velocities. In Section 4, we demonstrate how this formalism can be used for transiting extrasolar planets to efficiently schedule the phase

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1 Of course, with the exception of planets discovered by microlensing or direct imaging. The characterization scheme of planets around pulsars is highly similar to the analysis of RV data.
of RV observations in order to minimize uncertainties in the Lagrangian orbital elements. The results are summarized in the last section.

2 MATHEMATICAL FORMALISM

The solution for the time evolution of Kepler’s problem can be derived in the standard way as given in various textbooks (see e.g. Murray & Dermott 1999). The restricted two-body problem itself is an integrable ordinary differential equation. In the planar case, three independent integrals of motion exist and one variable has uniform monotonicity. The integrals are related to the well-known orbital elements, which are used to characterize the orbit. These are the semimajor axis \( a \), the eccentricity \( e \) and the longitude of pericentre \( \omega \). The fourth quantity is the mean anomaly \( M = nt \), where \( n = \sqrt{\mu/a^3} = 2\pi/P \), the mean motion, which is zero at pericentre passage \(^2\) and \( t \) is the elapsed time since the pericentre passage. The solution to Kepler’s problem can be given in terms of the mean anomaly \( M \) defined as

\[
E - e \sin E = M, \tag{1}
\]

where \( E \) is the eccentric anomaly. The planar coordinates are

\[
\xi = \xi_0 \cos \sigma - \eta_0 \sin \sigma, \tag{2}
\]

\[
\eta = \xi_0 \sin \sigma + \eta_0 \cos \sigma, \tag{3}
\]

where

\[
\xi_0 = a (\cos E - e), \tag{4}
\]

\[
\eta_0 = a \sqrt{1 - e^2} \sin E; \tag{5}
\]

see also Murray & Dermott (1999, section 2.4) for the derivation of these equations. Since for circular orbits the longitude of pericentre and pericentre passage cannot be defined, and for nearly circular orbits, these can only be badly constrained; in these cases, it is useful to define a new variable, the mean longitude as \( \lambda = M + \sigma \) to use instead of \( M \). Since \( \sigma \) is an integral of the motion, \( \dot{\lambda} = \dot{M} = \eta \). Therefore for circular orbits \( \sigma = 0 \) and equations (4) and (5) should be replaced by

\[
\xi_0 = a \cos \lambda, \tag{6}
\]

\[
\eta_0 = a \sin \lambda. \tag{7}
\]

To obtain an analytical solution to the problem, i.e. which is infinitely differentiable with respect to all of the orbital elements and the mean longitude, first let us define the Lagrangian orbital elements \( k = e \cos \sigma \) and \( h = e \sin \sigma \). Substituting equations (4) and (5) into equations (2) and (3) gives

\[
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix} = a \begin{pmatrix}
c \\
s
\end{pmatrix} + \frac{e \sin E}{2 - \ell} \begin{pmatrix}
+h \\
-k
\end{pmatrix} - \begin{pmatrix}
k \\
h
\end{pmatrix}, \tag{8}
\]

where \( c = \cos (\lambda + e \sin E) \), \( s = \sin (\lambda + e \sin E) \) and \( \ell = 1 - \sqrt{1 - e^2} \), the oblateness of the orbit. The derivation of the above equation is straightforward, one should only keep in mind

\(^2\) In two dimensions, the argument of pericentre is always equal to the longitude of pericentre, i.e. \( \sigma \equiv \omega \).

\(^3\) Throughout this paper, the mass parameter of Kepler’s problem is denoted by \( \mu = \mathcal{G}(m_1 + m_2) \), where \( m_1 \) and \( m_2 \) are the masses of the two orbiting bodies and \( \mathcal{G} \) is the Newtonian gravitational constant. The orbital period is denoted by \( P \).

that \( E + \sigma = \lambda + e \sin E \). In the first part of this section, we prove that the quantities

\[
p(\lambda, k, h) = \begin{cases} 0 & \text{if } k = 0 \text{ and } h = 0 \\ e \sin E & \text{otherwise} \end{cases} \tag{9}
\]

and

\[
q(\lambda, k, h) = \begin{cases} 0 & \text{if } k = 0 \text{ and } h = 0 \\ e \cos E & \text{otherwise} \end{cases} \tag{10}
\]

are analytic – infinitely differentiable – functions of \( \lambda, k \) and \( h \) for all real values of \( \lambda \) and for all \( k^2 + h^2 = e^2 < 1 \). In the following parts, we utilize the partial derivatives of these analytic functions to obtain the orbital velocities and we derive other relations. In this section, we only deal with planar orbits, the three-dimensional case is discussed in the next section.

2.1 Partial derivatives and the analytic property

A real function is analytic when all of its partial derivatives exist, the partial derivatives are continuous functions and only depend on other analytic functions. It is proven in Appendix A that the partial derivatives of \( q = q(\lambda, k, h) \) and \( p = p(\lambda, k, h) \) are the following for \( (k, h) \neq (0, 0) \):

\[
\frac{\partial q}{\partial \lambda} = -\frac{p}{1 - q}, \tag{11}
\]

\[
\frac{\partial q}{\partial k} = \frac{c - k}{1 - q} = \frac{\cos(\lambda + p) - k}{1 - q}, \tag{12}
\]

\[
\frac{\partial q}{\partial h} = \frac{s - h}{1 - q} = \frac{\sin(\lambda + p) - h}{1 - q}, \tag{13}
\]

and

\[
\frac{\partial p}{\partial \lambda} = \frac{q}{1 - q}, \tag{14}
\]

\[
\frac{\partial p}{\partial k} = \frac{s + \sin(\lambda + p)}{1 - q}, \tag{15}
\]

\[
\frac{\partial p}{\partial h} = \frac{-c - \cos(\lambda + p)}{1 - q}. \tag{16}
\]

Since, for all \( k^2 + h^2 < 1, q < 1 \) and therefore \( 1 - q > 0 \), all of the above functions are continuous on their domains. Since the \( \sin (\cdot) \) and \( \cos (\cdot) \) functions are analytic, therefore one can conclude that the functions \( q(\cdot, \cdot, \cdot) \) and \( p(\cdot, \cdot, \cdot) \) are also analytic.

Substituting the definition of \( p = p(\lambda, k, h) \) into equation (8), one can write

\[
\begin{pmatrix}
\xi \\
\eta
\end{pmatrix} = a \begin{pmatrix}
(c + e \sin E)/2 - \ell & +h \\
\sin(\lambda + p) & +k
\end{pmatrix} - \begin{pmatrix}
k \\
h
\end{pmatrix}, \tag{17}
\]

while the radial distance of the orbiting particle from the centre

\[
\sqrt{k^2 + h^2} = r = a(1 - q). \tag{18}
\]

For small eccentricities in equation (17), third term \((k, h)\) is negligible as compared to the first term \((\cos, \sin)\) while the second term \((h, -k)\) is negligible as compared to the third term. Therefore, for \( e \ll 1, p \) is proportional to the difference between the true anomaly and the mean longitude and \( q \) is proportional to the distance offset relative to a circular orbit; both caused by the non-zero orbital eccentricity.

Since equation (17) is a combination of purely analytic functions, the solution of Kepler’s problem is analytic with respect to the orbital elements \( a, (k, h) \), and to the mean longitude \( \dot{\lambda} \) in the
domain $a > 0$ and $k^2 + h^2 < 1$. We note here that this formalism omits the parabolic or hyperbolic solutions. The formalism based on the Stumpff functions (see Stiefel & Scheifele 1971) provides a continuous set of formulae for the elliptic, parabolic and hyperbolic orbits, but this parametrization is still singular in the $e \rightarrow 0$ limit.

### 2.2 Orbital velocities

Assuming a non-perturbed orbit, i.e. when $(\dot{k}, \dot{h}) = 0$ and $a = 0$ and when the mean motion $n = \dot{\lambda}$ is constant, the orbital velocities can be directly obtained by calculating the partial derivative of equation (17) with respect to $\lambda$ and applying the chain rule since

$$\frac{\partial \{\xi\}}{\partial t} = \left[ \frac{\partial \{\xi\}}{\partial \lambda} \right] \frac{\partial \lambda}{\partial t} = n \frac{\partial \{\xi\}}{\partial \lambda}. \tag{18}$$

Substituting the partial derivative equation (14) into the expansion of $\delta \xi / \partial \lambda$ and $\delta \xi / \partial \eta$, one gets

$$\left(\frac{\xi}{\eta}\right) = \frac{1}{1 - q} \left[ \left( -\sin(\lambda + p) \right) + \frac{q}{2 - \ell} \left( +h -k \right) \right]. \tag{19}$$

Note that equation (19) is also a combination of purely analytic functions, the components of the orbital velocity are analytic with respect to the orbital elements $a, k, h$, and to the mean longitude $\lambda$.

It is also evident that the time derivative of equation (19) is

$$\frac{\partial \{\xi\}}{\partial \eta} = -\frac{an^2}{(1 - q)^2} \left[ \left( \cos(\lambda + p) \right) \sin(\lambda + p) \right] \right.$$

$$+ \frac{p}{2 - \ell} \left( -h -k \right) \right). \tag{20}$$

Obviously, equation (20) can be written as

$$\left(\frac{\xi}{\eta}\right) = -\frac{n^2}{(1 - q)^2} \left( \frac{\xi}{\eta} \right), \tag{21}$$

which is equivalent to the equations of motion since $\mu = n^2a^3$ and $\sqrt{\xi^2 + \eta^2} = r = a(1 - q)$.

### 2.3 Other properties

In this section, we summarize some other properties of the functions $p = p(\lambda, k, h)$ and $q = q(\lambda, k, h)$ which can also be helpful in some derivations or during numerical evaluation.

One of the most important properties is the rotational invariance. This is a direct consequence of the relation $M = \lambda - \sigma$, i.e. $q$ and $p$ would not change if the mean longitude is increased by an arbitrary angle of $\Omega$ and simultaneously the vector $(k, h)$ is rotated with the same angle. Therefore,

$$q = p(\lambda - \Omega, k \cos \Omega + h \sin \Omega, -k \sin \Omega + h \cos \Omega), \tag{22}$$

$$p = p(\lambda - \Omega, k \cos \Omega + h \sin \Omega, -k \sin \Omega + h \cos \Omega). \tag{23}$$

This property also results that

$$q = q(k \cos \lambda + h \sin \lambda, -k \sin \lambda + h \cos \lambda), \tag{24}$$

$$p = p(k \cos \lambda + h \sin \lambda, -k \sin \lambda + h \cos \lambda). \tag{25}$$

The terms $c = \cos(\lambda + p)$ and $s = \sin(\lambda + p)$ appear frequently in the expressions for both the coordinates and the velocities. These are related to the $(q, p)$ functions and the orbital elements $(k, h)$ as

$$k = qc + ps, \tag{26}$$

$$p = kc - hs, \tag{27}$$

or similarly

$$q = kc + hs, \tag{28}$$

$$p = ks - hc. \tag{29}$$

The partial derivatives of $c$ and $s$ with respect to the mean longitude and the Lagrangian orbital elements $k$ and $h$ are

$$\frac{\partial \{c\}}{\partial \lambda} = \frac{1}{1 - q} \left( -s \right) \tag{30}$$

and

$$\frac{\partial \{s\}}{\partial \lambda} = -\frac{1}{1 - q} \left( -c \right). \tag{31}$$

### 3 THREE-DIMENSIONAL CASE AND THE INVERSE PROBLEM

For an eccentric and inclined orbit, the spatial coordinates of an orbiting body can be obtained in the similar manner as it is done in Murray & Dermott (1999, section 2.8). Without going into the details, here we present the results of the basic calculations. The spatial coordinates $r = (x, y, z)$ are

$$r = P_1 P_2 P_3 r_0, \tag{32}$$

where $r_0 = (\xi_0, \eta_0, 0)$ (see also equations 4 and 5) and $P_1, P_2$ and $P_3$ are the rotational matrices with respect to the argument of pericentre, $\omega$, the inclination, $i$, and the argument of the ascending node, $\Omega$. By substituting the equations for $(\xi_0, \eta_0)$ into equation (32) and using the rotational invariance of $q(\cdot, \cdot)$ and $p(\cdot, \cdot)$, i.e. equations (22) and (23), one can derive the spatial coordinates of the orbiting body in a similar manner as it was done in the planar case. The result can be written in a compact form using some other auxiliary quantities. First, define the Lagrangian orbital elements $i_x = 2 \sin(i/2) \cos \Omega, i_y = 2 \sin(i/2) \sin \Omega, i_z = \sqrt{4 - i_x^2 - i_y^2}$ and the quantity $W = \eta i_x - \xi i_z$, where $(\xi, \eta)$ is defined in equation (8). The spatial coordinates are then

$$x = \xi + \frac{1}{2} i_y W, \tag{33}$$

$$y = \eta - \frac{1}{2} i_x W, \tag{34}$$

$$z = i_z W. \tag{35}$$

Since $W$ is a linear combination of $\xi$ and $\eta$, the orbital velocities are linear combinations of $\xi$ and $\eta$, with the same coefficients, i.e.

$$x = \xi + \frac{1}{2} i_y W, \tag{36}$$

$$y = \eta - \frac{1}{2} i_x W, \tag{37}$$

$$z = i_z W. \tag{38}$$

where $W = \eta i_x - \xi i_y$.

3.1 The inverse problem

To compute the orbital elements \((a, \lambda, k, h, i, \omega)\) from the spatial coordinates \((x, y, z)\) and velocities \((\dot{x}, \dot{y}, \dot{z})\), first define

\[
\begin{align*}
    r^2 &= x^2 + y^2 + z^2, \\
    v^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2, \\
    c_i &= y\dot{z} - z\dot{y}, \\
    c_y &= z\dot{x} - x\dot{z}, \\
    c_x &= x\dot{y} - y\dot{x}, \\
    C^2 &= c_x^2 + c_y^2 + c_z^2, \\
    \hat{c} &= x\dot{y} + y\dot{x} + z\dot{z}.
\end{align*}
\]

The Lagrangian orbital elements for the inclination and the argument of the ascending node are then

\[
\begin{align*}
    i_x &= -\frac{\sqrt{2}}{\sqrt{1 + c / C}} c_x, \\
    i_y &= +\frac{\sqrt{2}}{\sqrt{1 + c / C}} c_y.
\end{align*}
\]

For the eccentricity and the longitude of pericentre, one gets

\[
\begin{align*}
    \left(\begin{array}{l} k \\ h \end{array}\right) &= \frac{C}{\mu} \left(\begin{array}{l} \dot{y} y + \dot{z} z \\ \dot{x} x + \dot{z} z \end{array}\right) - \frac{1}{r} \left(\begin{array}{l} x - c / C + c_x \\ y - c / C + c_y \end{array}\right). \\
\end{align*}
\]

The semimajor axis satisfies the well-known relation (Murray & Dermott 1999)

\[
a = \frac{C^2}{\mu(1 - k^2 - h^2)} = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{1/2}.
\]

The mean longitude is then

\[
\lambda = \arg \left(\begin{array}{l} k \\ h \end{array}\right) = \frac{C}{\mu} \left(\begin{array}{l} \dot{y} y + \dot{z} z \\ \dot{x} x + \dot{z} z \end{array}\right) - \frac{1}{r} \left(\begin{array}{l} x (1 - \ell) \\
\end{array}\right)
\]

where \(\ell = 1 - \sqrt{1 - k^2 - h^2}\). It can be shown that in the planar case, i.e. when \(z = 0\) and \(\dot{z} = 0\), equations (48) and (50) do reduce to

\[
\left(\begin{array}{l} k \\ h \end{array}\right) = \frac{C}{\mu} \left(\begin{array}{l} \dot{y} y + \dot{z} z \\ \dot{x} x + \dot{z} z \end{array}\right) - \frac{1}{r} \left(\begin{array}{l} x \\
\end{array}\right)
\]

and

\[
\lambda = \arg \left(\begin{array}{l} k \\ h \end{array}\right) = \frac{C}{\mu} \left(\begin{array}{l} \dot{y} y + \dot{z} z \\ \dot{x} x + \dot{z} z \end{array}\right) - \frac{\hat{c}}{C} (1 - \ell),
\]

respectively. Here, \(\arg(\cdot, \cdot)\) is defined as \(\arg(x, y) = \arctan(y/x)\) if \(x \geq 0\) and \(\pi + \arctan(y/x)\) otherwise.

4 APPLICATIONS

The utilization of the presented formalism can cover various aspects of RV curve analysis. Due to the conventions and the definitions of the orbital elements used in the astrophysics of stellar binaries, the evaluation of equation (19) simply yields the actual RV as the \(\eta\) component.\(^4\) The partial derivatives in equations (11)–(16) and in equations (30) and (31) can easily be used to calculate the parametric derivatives of the RV curves and therefore to support various fitting algorithms (e.g. Levenberg–Marquardt; see Press et al. 1992). Moreover, the secular variations in the radial velocities due to the secular changes in the orbital elements can also be estimated in an analytic way.

In the follow-up observations of planets discovered by transits in photometric data series, the detection of systematic variations in the RV signal is one of the most relevant steps, either to rule out transits of late-type dwarf stars, and/or blends, or to characterize the mass of the planet and the orbital parameters. Since transit timing constrains the epoch and orbital period much more precisely than RV alone, these two can be assumed to be fixed in the analysis of the RV data. However, this constraint also includes an additional feature. Using equation (1) in Pál & Kocsis (2008), the mean longitude at transits can be calculated as

\[
\lambda_{tr} = \arg \left(\begin{array}{l} k \\ h \end{array}\right) = \frac{1}{1 + h - \frac{k^2}{2 - \ell}} - \frac{1}{1 + h}.
\]

therefore the mean longitude at the orbital phase \(\varphi\) becomes \(\lambda = \lambda_{tr} + 2\pi \varphi\). Consequently, the partial derivatives of the \(\eta\) RV component \(v = \eta(\lambda_{tr} + 2\pi \varphi, k, h)\) with respect to the orbital elements \(k\) and \(h\) are

\[
\begin{align*}
    \frac{\partial v}{\partial k} &= \frac{\partial \eta}{\partial \lambda_{tr}} \frac{\partial \lambda_{tr}}{\partial k} + \frac{\partial \eta}{\partial k}, \\
    \frac{\partial v}{\partial h} &= \frac{\partial \eta}{\partial \lambda_{tr}} \frac{\partial \lambda_{tr}}{\partial h} + \frac{\partial \eta}{\partial h}.
\end{align*}
\]

A RV curve of a star, caused by the perturbation of a single companion, can be parametrized by six quantities: the semi-amplitude of RV variations, \(K\), the zero point, \(\gamma\), the Lagrangian orbital elements, \((k, h)\), the epoch, \(T_0\) (or equivalently the phase at an arbitrary fixed time instant) and the period \(P\). In the cases of transiting planets, the later two are known since the transit photometric observations constrain both with exceeding precision (relative to the precision attainable purely by the RV data). Therefore, one has to fit only four quantities, i.e. \(a = (K, \gamma, \lambda_{tr}, h)\). The vast majority of the known transiting planets orbit their host stars on a tight orbit and these tight orbits are expected to be circular (however, there are few known exceptions\(^5\)).

Now, we show how the optimal phases of observations can be determined to confirm the orbital circularity, i.e. to yield the smallest uncertainty in the orbital elements \(k\) and \(h\). To obtain the uncertainty of a fitted parameter and/or the correlation between the parameters, the Fisher matrix method can be utilized (Finn 1992; Baluuev 2008). This method gives the covariance matrix as

\[
\Gamma_{\eta\eta} = \sum_i \frac{\partial \eta(\{a, \gamma, \lambda_{tr}, h\})}{\partial a_i} \frac{\partial \eta(\{a, \gamma, \lambda_{tr}, h\})}{\partial a_i} \sigma_i^{-2}.
\]

Here, \(f(a, t)\) is the model function which depends on its adjusted parameters \(a = (a_1, \ldots, a_N)\), and \(t\) represents the independent variable(s) (in the case of time series, there is one independent variable, the time itself). Since in our case there are four unknowns, one

\(^4\) In this case, \(an = K / \sqrt{1 - e^2}\), where \(K\) is the semi-amplitude of the RV curve.

\(^5\) See e.g. http://www.exoplanet.eu for up to date information.
Figure 1. RV variations for a transiting planet orbiting its star on a circular orbit. The transit occurs at zero (or unity) phase. The four dots represent the phases $\varphi = 0.1292, 0.4138, 0.5862$ and $0.8708$ when RV measurements should be obtained to achieve the largest significance of the orbital circularity.

has to have at least four data points, in order to completely determine the parameters. Nevertheless, equation (57) can be formally evaluated providing both the uncertainties and correlations. To determine the best four phases ($\varphi_1, \varphi_2, \varphi_3$ and $\varphi_4$) which minimizes the uncertainty in the eccentricity, one has to minimize the volume of the covariance ellipsoid of the ($k$, $h$) parameters (also known as generalized $D$-optimality; see Baluška 2008). It can be shown that this volume is

$$U = \left[ \det \left( \begin{array}{cc} \langle \delta k^2 \rangle & \langle \delta k \delta h \rangle \\ \langle \delta h \delta k \rangle & \langle \delta h^2 \rangle \end{array} \right) \right]^{1/2} = \sqrt{\langle \delta k^2 \rangle \langle \delta h^2 \rangle - \langle \delta k \delta h \rangle^2}, \quad (58)$$

where the covariances can be calculated using equation (56), i.e. $\langle \delta k^2 \rangle = (\Gamma^{-1})_{33}$, $\langle \delta k \delta h \rangle = (\Gamma^{-1})_{34}$ and $\langle \delta h^2 \rangle = (\Gamma^{-1})_{44}$ if the parameters are $a = (a_1, a_2, a_3, a_4) = (K, G, k, h)$.

We minimized $U$ as the function of the phases $\varphi_1, \ldots, \varphi_4$ using the Markov chain Monte-Carlo (MCMC) method (see e.g. Ford 2004). Multiple chains were initiated from four random phases (namely, all of them are chosen uniformly from the interval $[0, 1]$). We found that all of the chains converge to a single set of phases that represent the minimum of the function $U(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$. Therefore, the set of optimal phases is unique: we found the optimal phases are $\varphi = 0.1292, 0.4138, 0.5862$ and $0.8708$. In Fig. 1, these phases are marked on a hypothetical RV curve. We note that this volume of the covariance ellipsoid of the ($k$, $h$) parameters is approximately 12 times smaller on the average if the phases were chosen randomly, and 2.5 times smaller if the phases were chosen to be closer with a factor of 2 to the phases 0.25 and 0.75 (i.e. $\varphi' = 0.1896, 0.3319$ and 0.6681, 0.8104, respectively).

Of course, the same kind of calculation of the phases that yields the smallest combined uncertainty in the ($k$, $h$) parameters can be performed for arbitrary orbital eccentricity. For some certain values of ($k$, $h$), these optimal phases are shown in Table 1. Thus, if some initial values for the orbital elements $k$ and $h$ are known, further observations can be planned accordingly.

Similarly, the optimal phases can be derived for arbitrary number of observations ($N_{\text{obs}}$). However, it turns out that for $N_{\text{obs}} \geq 5$ the phase volume $U$ have more than one local minima. In order to find the global minimum, we initiated several hundreds or thousands of individual initial conditions and applied both the previously discussed MCMC method and the downhill simplex algorithm (Press et al. 1992) in order to find the global minimum of $U$. For $N_{\text{obs}} = 5$, we have two local minima and the global minimum is at the phases $\varphi = (0.1318, 0.3978, 0.5, 0.6022, 0.8682)$. For $N_{\text{obs}} > 5$, one or more of the phases will be degenerated. In other words, one should take RV measurements at the same phases in order to minimize the uncertainty in the orbital eccentricity. For instance, the global minimum is at $\varphi = (0.1376, 0.4204, 0.4204, 0.5796, 0.5796, 0.8624)$ for $N_{\text{obs}} = 6$ or $\varphi = (0.1405, 0.4315, 0.4315, 0.5965, 0.5965, 0.8746, 0.8746)$ for $N_{\text{obs}} = 7$. It can be shown that for larger $N_{\text{obs}}$ which are multiple of 4, the optimal phases will be at exactly the same location as in the case of $N_{\text{obs}} = 4$ (see Table 1), and at each phase one should acquire the same number of measurements, namely $N_{\text{obs}}/4$.

Due to the limitations in the telescope time and in the observation conditions (including day/night variations or the visibility of the target object), RV observations cannot be scheduled at the optimal phases. In practice, we have a series of RV data points and then the upcoming measurements are intended to be acquired to yield the smallest uncertainty in the orbital eccentricity (or any other parameters that are the points of interest). Since this set of phases yields enormous amount of free parameters on which the phases of the upcoming measurements depend, these phases have to derived independently by hand for each case. The \textsc{optfn} code\footnote{http://szofi.elte.hu/~apal/utils/astro/eof/} is intended to derive these optimal phases\footnote{This code was also used to derive the previously discussed orbital phases as well as the values found in Table 1.} for arbitrary set of fixed observations.

Table 1. Optimal phases of RV measurements in order to obtain the orbital eccentricity as precise as possible. The eccentricity and the alignment of the orbit are quantified by the Lagrangian orbital elements $k$ and $h$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$h$</th>
<th>$\varphi_1$</th>
<th>$\varphi_2$</th>
<th>$\varphi_3$</th>
<th>$\varphi_4$</th>
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<tr>
<td>-0.4</td>
<td>-0.4</td>
<td>0.1305</td>
<td>0.2064</td>
<td>0.2519</td>
<td>0.6943</td>
</tr>
<tr>
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<td>0.1060</td>
<td>0.2048</td>
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<td>0.1879</td>
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<td>0.1584</td>
<td>0.3398</td>
<td>0.9197</td>
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<tr>
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<td>0.0316</td>
<td>0.1180</td>
<td>0.3701</td>
<td>0.9555</td>
</tr>
<tr>
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<td>0.3307</td>
<td>0.3910</td>
<td>0.7027</td>
</tr>
<tr>
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<td>0.7943</td>
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<td>0.8146</td>
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<td>0.7481</td>
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<td>0.8695</td>
</tr>
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<td>0.6299</td>
<td>0.8820</td>
<td>0.9684</td>
</tr>
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5 SUMMARY

Although the Newtonian gravitational two-body problem (also known as Kepler’s problem) is integrable, its solution requires transcendental equations. Moreover, the parameter of these equations has a finite discontinuity in the limit of circular orbits, and this ill-behaved property leads to numerical disadvantages. For instance, the calculation of the optimal phases, as presented in Section 4, cannot be performed if the orbit was parametrized by \( (e, \sigma) \) instead of \((k, h)\) since the partial derivative \( \frac{\partial \nu}{\partial e_{\nu,0}} \), which is required to be known to obtain the Fisher matrix, does not exist.

In this paper, a new framework has been presented to describe the time evolution of Kepler’s problem using analytic functions which omit singular parameterization. This formalism is then used to construct orbital solution for three-dimensional orbits in a similar manner. Using this analytic solution, it is straightforward to derive the RV function, which is one of the most relevant observable quantities in the physics of stellar binaries and extrasolar planets. The functions \( p(, . , .) \) and \( q(, . , .) \) can be handled as an analytic function, just as simply as if they were a trigonometric function in practice. It is also straightforward to carry out a Fisher analysis of RV data to estimate the expected uncertainty of the physical parameters. The implementation of the functions \( p(, . , .) \) and \( q(, . , .) \) are simple in any kind of programming languages, a demonstration code is provided in the gnuplot language.\(^8\)

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Mayor M., Queloz D., 1995, Nat, 378, 355

\(^8\) http://szofi.elte.hu/~apal/utils/astro/eof/eof.gnuplot

APPENDIX A: THE PARTIAL DERIVATIVES OF THE ECCENTRIC OFFSET FUNCTIONS

We know that the eccentric offsets \( q \) and \( p \) are continuous functions of \( \lambda, k \) and \( h \). Now, the partial derivatives of \( \lambda \equiv q(\lambda, k, h) \) and \( p \equiv p(\lambda, k, h) \) are calculated. If all of the partial derivatives are non-singular around zero (where \( k = h = 0 \)), one could conclude that the eccentric offsets are not only continuous but smooth functions. For the derivation of the partial derivatives

\[
\frac{\partial (q, p)}{\partial (\lambda, k, h)} = \left( \begin{array}{ccc} \frac{\partial q}{\partial \lambda} & \frac{\partial q}{\partial k} & \frac{\partial q}{\partial h} \\ \frac{\partial p}{\partial \lambda} & \frac{\partial p}{\partial k} & \frac{\partial p}{\partial h} \end{array} \right),
\]

(A1)

we will use implicit function theorem. Let us define

\[
F_i(q, p; \lambda, k, h) = +q \cos p + p \sin p - (k \cos \lambda + h \sin \lambda),
\]

(A2)

\[
F_i(q, p; \lambda, k, h) = -q \sin p + p \cos p - (k \sin \lambda - h \cos \lambda).
\]

(A3)

For a fixed value of \( \lambda, k \) and \( h \), Kepler’s equation is equivalent with

\[
F_i(q, p; \lambda, k, h) = 0.
\]

(A4)

According to the implicit function theorem, the required partial derivatives are

\[
\left( \begin{array}{ccc} \frac{\partial q}{\partial \lambda} & \frac{\partial q}{\partial k} & \frac{\partial q}{\partial h} \\ \frac{\partial p}{\partial \lambda} & \frac{\partial p}{\partial k} & \frac{\partial p}{\partial h} \end{array} \right) = - \left( \begin{array}{ccc} \frac{\partial F_i}{\partial q} & \frac{\partial F_i}{\partial p} & \frac{\partial F_i}{\partial \lambda} \\ \frac{\partial F_i}{\partial q} & \frac{\partial F_i}{\partial p} & \frac{\partial F_i}{\partial k} \\ \frac{\partial F_i}{\partial q} & \frac{\partial F_i}{\partial p} & \frac{\partial F_i}{\partial h} \end{array} \right)^{-1} \times \left( \begin{array}{ccc} \frac{\partial F_i}{\partial q} & \frac{\partial F_i}{\partial p} & \frac{\partial F_i}{\partial \lambda} \\ \frac{\partial F_i}{\partial q} & \frac{\partial F_i}{\partial p} & \frac{\partial F_i}{\partial k} \\ \frac{\partial F_i}{\partial q} & \frac{\partial F_i}{\partial p} & \frac{\partial F_i}{\partial h} \end{array} \right).
\]

(A5)

The partial derivatives on the right-hand side of equation (A5) are

\[
\left( \begin{array}{ccc} \frac{\partial F_i}{\partial q} & \frac{\partial F_i}{\partial p} & \frac{\partial F_i}{\partial \lambda} \\ \frac{\partial F_i}{\partial q} & \frac{\partial F_i}{\partial p} & \frac{\partial F_i}{\partial k} \\ \frac{\partial F_i}{\partial q} & \frac{\partial F_i}{\partial p} & \frac{\partial F_i}{\partial h} \end{array} \right) = 
\left( \begin{array}{ccc} +k \sin \lambda - h \cos \lambda & -\cos \lambda & -\sin \lambda \\ -k \cos \lambda - h \sin \lambda & -\sin \lambda & +\cos \lambda \end{array} \right).
\]

(A7)

Therefore, expanding the above partial derivatives, one can obtain that

\[
\left( \begin{array}{ccc} \frac{\partial q}{\partial \lambda} & \frac{\partial q}{\partial k} & \frac{\partial q}{\partial h} \\ \frac{\partial p}{\partial \lambda} & \frac{\partial p}{\partial k} & \frac{\partial p}{\partial h} \end{array} \right) = 
\left( \begin{array}{ccc} +k \sin \lambda - h \cos \lambda & -\cos \lambda & -\sin \lambda \\ -k \cos \lambda - h \sin \lambda & -\sin \lambda & +\cos \lambda \end{array} \right) \left( \begin{array}{ccc} -p \cos(\lambda + p) - k & \sin(\lambda + p) - h \\ q & +\sin(\lambda + p) & -\cos(\lambda + p) \end{array} \right).
\]

(A8)

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