A systematic description of shocks in gamma-ray bursts – I. Formulation

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Accepted 2009 April 16. Received 2009 April 14; in original form 2008 December 26

ABSTRACT
Since the suggestion of relativistic shocks as the origin of gamma-ray bursts (GRBs) in the early 1990s, the mathematical formulation of this process has stayed at a phenomenological level. One of the reasons for the slow development of theoretical works has been the simple power-law behaviour of the afterglows hours or days after the prompt gamma-ray emission. It was believed that they could be explained with these formulations. Nowadays, with the launch of the Swift satellite and implementation of robotic ground follow-ups, GRBs and their afterglow can be observed at multi-wavelengths from a few tens of seconds after trigger onwards. These observations have led to the discovery of features unexplainable by the simple formulation of the shocks and emission processes used up to now. Some of these features can be inherent in the nature and activities of the GRBs’ central engines which are not yet well understood. On the other hand, the devil is in the detail and others may be explained with a more detailed formulation of these phenomena and without ad hoc addition of new processes. Such a formulation is the goal of this work. We present a consistent formulation of the kinematics and dynamics of the collision between two spherical relativistic shells, their energy dissipation and their coalescence. It can be applied to both internal and external shocks. Notably, we propose two phenomenological models for the evolution of the emitting region during the collision. One of these models is more suitable for the prompt/internal shocks and late external shocks, and the other for the afterglow/external collisions as well as the onset of internal shocks. We calculate a number of observables such as flux, lag between energy bands and hardness ratios. One of our aims has been a formulation complex enough to include the essential processes, but simple enough such that the data can be directly compared with the theory to extract the value and evolution of physical quantities. To accomplish this goal, we also suggest a procedure for extracting parameters of the model from data. In a companion paper, we numerically calculate the evolution of some simulated models and compare their features with the properties of the observed GRBs.

Key words: shock waves – gamma-rays: bursts.

1 INTRODUCTION
The Swift (Gehrels et al. 2004) observations of more than 200 gamma-ray bursts (GRBs) and their follow-ups have been a revolution in our knowledge and understanding of these elusive phenomena. The rapid slew of the Swift X-ray and UV/optical telescopes – respectively XRT (Burrows et al. 2005) and UVOT (Roming et al. 2005) – as well as ground-based robotic telescopes have permitted the observation of GRBs and their afterglow at multi-wavelengths from a few tens of seconds after the prompt or precursor gamma-ray emission is detected by the Burst Alert Telescope (BAT) (Barthelmy et al. 2005), up to days after trigger. These observations show that the emission can be essentially divided into three regimes. (1) The prompt gamma-ray emission which can be very short, a few tenths of milliseconds, or long, up to a few hundreds of seconds. (2) A tail emission in X-rays which is observed for more than 90 per cent of bursts. For some bursts, this tail is also detected as a soft faint continuum in gamma-rays. In about 40 per cent of bursts, this early emission has been detected in optical and infrared too. In this regime, for many bursts flares have been observed mainly in X-rays. Sometimes, the counterparts of flares have also been observed in gamma-rays and/or optical/IR. In many bursts, the early steep slope of the X-ray emission at the beginning of this regime becomes much shallower and somehow harder at the end. (3) The late emission can be considered as the epoch after the break of the shallow regime in which the emission is usually a continuum and no or little flaring activity is observed [but there are exceptions such as GRB 070110

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On the one hand, it seems that the idea of synchrotron emission from accelerated particles in a relativistic shock as the origin of the prompt emission (Rees & Mészáros 1994, 1998) is essentially correct. On the other hand, the early observations of what is usually called the afterglow – the emission in lower energy bands usually observed from $\lesssim 100$ s after trigger onwards – have been the source of surprises and raised a number of questions about many issues: the activity (Fan & Wei 2005) and the nature of the engine (Petrovic et al. 2005; Fryer, Rockefeller & Young 2006; Gal-Yam et al. 2008); the concept of prompt/internal-afterglow/external shocks (Ramirez-Ruiz & Granot 2007); the efficiency of energy transfer from the outflow – the fireball – to synchrotron radiation (Zhang et al. 2006); the collimation and jet break (Covino et al. 2006; Panaitescu et al. 2006); the behaviour of X-ray and optical light curves (Panaitescu & Vestrand 2008), etc. Many predictions such as the existence of a significant high-latitude emission with a strict relation between the light curve time evolution slope and the spectrum index, and an achromatic jet break have not been observed. Moreover, the origin of unexpected behaviours such as a very steep decline in low-energy bands after the prompt emission (Zhang, Liang & Zhang 2007b) and a very shallow regime which lasts for thousands of seconds is not well understood. Other unexpected observations are: chromatic multiple breaks in the X-ray light curves, flares in X-ray and optical bands hundreds of seconds after the prompt even in some short bursts [e.g. GRB 060313 (Pagnani et al. 2006; Roming et al. 2006), GRB 070724A (Ziaeepour et al. 2007a), a tail emission in short bursts [e.g. GRB 070714B (Racusin et al. 2007; Bradley 2008), GRB 080426 (Ziaeepour et al. 2008b), GRB 080503 (Mao et al. 2008)], and very short, hard and high-amplitude spikes in long bursts that could lead to the classification of the burst as short if the instrument was not sensitive enough to detect the rest of the prompt emission [e.g. GRB 060614 (Mangano et al. 2007; Gehrels et al. 2006), GRB 061006 (Schady et al. 2006a)]. This makes the classification of bursts as short and long much more ambiguous (Zhang et al. 2007a).

One conclusion that has been made from these observations is that the central engine can be active for up to thousands of seconds after the prompt emission (Fan & Wei 2005). But the nature of the fireball and its source of energy are not yet well understood, and we cannot yet verify this interpretation or relate it to any specific process in the engine. It seems, however, that whatever the origin of the fireball, it must be baryon dominated, otherwise it could not make long-term effects correlated to the prompt emission. In this case, the internal and external shock models as the origin of the prompt emission and afterglow are good candidates. None the less, the lack of a simple explanation for the observed complexities has encouraged authors to consider other possibilities, for instance associating both the prompt gamma-ray and the afterglows to external shocks and fast variations to abrupt density variation of the surrounding material (Ramirez-Ruiz & Granot 2007). However, it has been shown that in such models it is not possible to explain the fast variations of the prompt emission even in the presence of a bubbly environment or pulse-like density change (Nakar & Granot 2007; Ramirez-Ruiz & Granot 2007).

Here, we suggest that at least some of the features of early afterglows can be related to complex shock physics and/or features in the fireball/jet. In fact, simulations of the acceleration of electrons and positrons by the first and second Fermi processes show that the evolution of electric and magnetic fields as well as the energy distribution of accelerated particles are quite complex (Bednarz & Ostrowski 1996; Dieckmann et al. 2006; Rieger, Bosch-Ramon & Duffy 2007). Plasma instabilities lead to the formation of coherent electric and magnetic fields and acceleration of particles (Yang, Arnos & Langdon 1994; Silva et al. 2002; Wiersma & Achterberg 2004; Reville, Kirk & Duffy 2006). Their time evolution in relativistic shocks can significantly affect the behaviour of the prompt emission and the afterglow of GRBs. If the number density of particles in the ejecta is significant and the shock is collisional, the state of matter in the jet can be also an important factor in determining the behaviour of the fields. Many aspects of these processes are not well understood; however realistic interpretations of observations should consider these complexities at least phenomenologically. For instance, simple distributions such as a power-law distribution for the Lorentz factor of electrons, or a constant magnetic field for the whole duration of prompt emission and afterglow, can be quite unrealistic. Ideally, these quantities should come from the simulation of Fermi processes and plasma instabilities such as the Weibel instability (Yang et al. 1994; Wiersma & Achterberg 2004) that produce the coherent transverse magnetic field. However, these phenomena are complex and their simulations are very time and CPU consuming. For these reasons, they cannot yet explore the parameter space of the phenomena and are mostly useful for demonstrating the concepts and how they work. Therefore, we are obliged to use simple analytical approximations for quantities related to the physics of relativistic shocks. In this situation, a compromise between complex non-analytical expressions and too simplistic and too simplified but unrealistic analytical behaviour of the physical quantities can be the consideration of intervals in which a simple analytical function can be a good approximation. Then, by adding together these intervals – regimes – one can reconstruct the entire evolution of a burst and its afterglow.

Even with a simplified presentation of the physical processes, one would not be able to explain GRB data without a model including both microphysics and dynamics of the fireball. The majority of works on the modelling of shocks and synchrotron emissions either deal with the emission (Sari & Piran 1995; Sari, Narayan & Piran 1996; Sari, Piran & Narayan 1998; Wijers & Galama 1999; Nakar & Piran 2004; Zhang & Kobayashi 2005; Pe'er & Zhang 2006) or deal with the kinematics of the shock (Blandford & McKee 1977; Piran, Shemi & Narayan 1993; Fenimore, Madras & Nayakshin 1996), or both but in a phenomenological way (Rhoads 1999). A few works (Daigne & Mochkovitch 1998, 2003) have tried to include both these aspects in a consistent model, but either they have not been very successful – their predictions especially for quantities such as lags in different bands were far from observed values and additional parametrization was necessary – or the formulation is too abstract to be compared directly with data (Vietri 2003; Blasi & Vietri 2005).

With these issues in mind, in Section 2 we present a simplified shock model that includes both the kinematics of the ejecta and the dynamics of the synchrotron emission. The microphysics are included by means of a simple parametrization. We calculate a number of observables such as flux, hardness ratios and lags between different energy bands. In this paper and Paper II (Ziaeepour 2009) in which we...
simulate part of the prompt and afterglow regimes of GRBs in some time intervals, we show that this model can explain many aspects of bursts as long as we divide the data into separate regimes. The reason is that the simple parametrization of microphysics in this model can be valid at most in a limited time interval. Each regime should be separately compared to analytical and numerical results for extracting the parameters. The results will show how parameters that are considered as constant in this model evolve during the lifetime of the burst. This is the best we can do until a better understanding of relativistic shock models and Fermi processes become available. If the model and the estimation of its parameters for each regime are sufficiently correct, adding them together should give us an overall consistent picture of characteristics of the burst, its afterglow and its surrounding material. A priori, this knowledge should help us to better understand the engine activities and eventually its nature and classification.

The model presented here depends on a large number of parameters, and we need an extraction procedure permitting us to extract as much information as possible about the physical properties of the shock from the available data. In Section 3, we explain how in the frame work of this model one can extract various quantities from the data. Evidently, the success of the modelling strongly depends on the availability and quality of simultaneous multi-wavelength observations.

The Swift observations show that during the first few hundred seconds after the trigger, there is usually a very close relation between the prompt gamma-ray emission and the emission in lower energy bands (Kumar et al. 2006; Mészáros 2007), therefore most probably they have a common origin, presumably internal shocks. However, historically and even in the present literature (and sometimes in the beliefs), any emission after the prompt gamma-ray is called the afterglow – meaning due to a shock with the interstellar medium (ISM) or surrounding material, presumably external shocks. Therefore, for clarity of context, here we define the afterglow as the emission in any energy band and at any time after the main prompt peaks regardless of its origin. If by afterglow we mean the external shocks, this is mentioned explicitly in the text.

We finish this paper with some outlines and two Appendices containing the details of calculation of the dynamics and flux for a power-law distribution of the electron Lorentz factor.

2 Shock Model

In this section, we first give a sketch description of a relativistic collision between two shells of material and processes which produce gamma-ray and other radiations. Then, we discuss a simplified mathematical formulation of the evolution of the shock and synchrotron emission. By restricting the model to a thin layer and to the early times after beginning of the collision, we can analytically calculate various observables.

2.1 Qualitative description of a relativistic shock and its simplified model

We begin this section by a pictorial description of present beliefs about the origin of GRBs and their afterglow. A central source – supernova, collapsar, collision of two compact objects, etc. – ejects highly relativistic cold baryon dominated material as spherical, jet or torus-like shells called the fireball. Collision between faster later ejected shells and slower earlier ejected ones produces what is called the prompt emission (Rees & Mészáros 1994).

A priori, there is no reason why faster shells should be ejected later. One way of explaining this paradigm is the deceleration of the front shells (Fenimore & Ramirez-Ruiz 1999) by surrounding material which are observed around massive objects such as Wolf–Rayet (WR) stars (Eldridge et al. 2006). It is possible that this initial deceleration is the source of a weak emission which has been seen before the main spike in many bursts. A relatively weak and soft precursor spike has been also observed in some bursts and can be related to this deceleration (Umeda et al. 2005). Although other origins such as jet–star interaction (Lazzati, Morsony & Begelman 2006, 2007) and fallback to the collapsar (Wang & Mészáros 2007) are also suggested to explain precursors, deceleration of the initial shell seems to be a more natural explanation and does not need any fine-tuning of the progenitor models and their parameters. In contrast, the jet–star interaction scenario cannot explain large time lag – a few hundreds of seconds between the precursor and the main spike in some bursts, e.g. GRB 050820A (Page et al. 2005; Cenko et al. 2006), GRB 060124 (Holland et al. 2006; Romano et al. 2008) and GRB 070721B (Ziaeepour et al. 2007b). In the fallback scenario, a weak jet is produced during the formation of a transient neutron star which later collapses to a black hole. The main jet in this scenario is produced by the accretion of the material from a disc to the black hole. In this case, the lag depends on the lifetime of the proto-neutron star and the rate of the accretion from the surrounding disc. Although these parameters can be tuned to explain the lag, a priori much longer lags should also be possible but never observed. In the deceleration scenario, the maximum possible lag is the duration of the central engine main activity – a few hundreds of seconds according to the observations of the main flares in X-rays, and is consistent with all the observations. The UV emission from the precursor should ionize the unshocked material in front of the first shell, and therefore there would be little additional absorption of soft X-rays later (Watson et al. 2007). In fact, in GRB 060124 (Holland et al. 2006; Romano et al. 2008), which had a long lag between the precursor and the main peak, a slight increase in $N_H$ column density at late time with respect to the initial density has been detected. In Paper II, we argue that another possible origin of the precursor is the temporary dynamical reduction of the emission in the early stage of the shock that makes the burst unobservable for a short time. Then, with the progress of coalescence of the shells the emission resumes and is observed as the main peak.

When two high-density shells collide, in the most general case they partially coalesce. Then, the most energetic particles get ahead of the rest in the downstream, and at the end of the collision the configuration includes again two shells. The back shell consists of slower particles.
Figure 1. A sketch of a relativistic shock. (a) Beginning of the shock: a high Lorentz factor cold shell (violet/dark grey) moving from left to right collides with a slower shell or ISM (cyan/light grey). (b) During the passage of the shells through each other, at the shock front an active region is formed where large electric and orthogonal magnetic fields are induced by plasma instabilities and charged particles are accelerated. They lose part of their kinetic energy as synchrotron emission. The concept of an active (emitting) region is not new and is commonly used in the context of the formation of cosmic rays in relativistic and non-relativistic shocks by Fermi processes (Virtanen & Vainio 2005a,b). A fraction of these highly accelerated particles escape downstream. In contrast, particles that have lost their energy move towards upstream. This process extends the active region but reduces the gradient of quantities such as density, thus gradually weakens the shock. (c) and (d) show two possible outcomes of the collision: total coalescence of the shells (radiative collision) (c), or after the passage of the fast shell a leftover slow shell is formed behind the fast shell (d). In these figures, the colour gradient presents the velocity and density distributions: darker colours correspond to higher velocity/density.

that have lost their kinetic energy (shocked particles). The front shell is the remnant of faster unscattered particles in the shells. In practice, we expect that the kinetic energy difference is continuous and slower particles become an expanding tail behind a faster and most probably denser head. In a few bursts such as GRB 060607A (Ziaeepour et al. 2006b, 2008a) and GRB 070107 (Stamatikos et al. 2007), it seems that we are seeing the separation between these components in the X-ray afterglow. Fig. 1 shows a sketch of the shock processes.

In the simplest case, the shock between two shells is radiative. This means that for an observer in the rest frame of the fast shell the kinetic energy of the falling particles from the other shell is immediately radiated and particles come to rest and join the shell. For a far observer at rest with respect to the engine, the difference between the kinetic energies of the two shells is partially radiated and partially transferred to the particles of the slower shell. The fast shell is decelerated until the totality of the slow shell is swept.

Not all the particles in the fast shell are decelerated with the same rate. Therefore, after the coalescence of the shells there is a gradient of the Lorentz factor from the shell front head with highest $\Gamma$ to the back tail with lowest $\Gamma$. In the macroscopic treatment of the shock processes, it is usually assumed that during the collision two distinguishable shocked layers and corresponding discontinuities are formed in each side of the boundary between the shells. They are called forward and reverse shocks according to their evolution direction with respect to the initial discontinuity (Sari et al. 1996; Nakar & Piran 2004; Zhang & Kobayashi 2005). According to these models and depending on the density difference of the shells and their relative Lorentz factor, the induced electric and magnetic fields and thereby the distribution of accelerated electrons and their synchrotron emission in these shocked regions can be very different. Notably, the reverse shock is expected to emit mostly at optical wavelengths with a relatively large lag with respect to the prompt gamma-ray as the emission must traverse the width of both shocked layers. Therefore, it should appear as an optical flash asynchronous from gamma-ray peaks (Kobayashi 2000).

On the other hand, if shells have similar densities and a small relative Lorentz factor, the scattering of particles in the two sides of the boundary quickly homogenizes the shocked regions, and therefore one can consider a single shocked zone with a relatively slow gradient in density and fields. If one of the shells has a density much larger than the other, the width of its shocked layer would be very small with respect to the shocked region in the other shell, and again the assumption of a single shocked layer is a good approximation for internal shocks. This schematic view corresponds well to what is observed in the simulations of Fermi acceleration in the ultra-relativistic shocks (see fig. 1 of Spitkovsky 2008 and Keshet et al. 2009).

For a distant observer who only detects photons from the synchrotron emission of the shocked material, it is very difficult to distinguish between photons coming from distinct regions unless they are well separated in time and in energy. The lack of clear evidence for a reverse shock emission in the Swift bursts, especially during the prompt emission, means that the reverse shock in the prompt GRB emission is weak and the assumption of just one shocked region is a good approximation. Most of the synchrotron emission is expected to be emitted by charged particles in the shocked region. But as electric and magnetic fields, and accelerated particles, are not really confined in this region.
and penetrate to a larger area, the region that emits radiation can be much more extended than the shocked region. We call this emitting zone the active region.

Evidently, in a real situation multiple shells are ejected in a short time interval. In this case, both pair collisions between late shells and collision-coalescence of the late shells with the main ejecta – prompt shell – are possible. These events happen at different points of space–time, but their synchrotron emissions can arrive at the observer separately, partially overlapping, or simultaneously. Therefore, it is not always possible to distinguish separate collisions and their characteristics. To this complexity one must also add the variation of quantities such as density and the Lorentz factor in a single shell that increases the variability of observed emissions. On the other hand, overlapping emissions make the comparison of data with the models based on a simple parametrization of the physical properties of a shell ambiguous. Despite these complexities, one should be able to consider peaks as separate collisions and find an effective set of parameters to model each one separately.

2.2 Shock evolution

A shock is defined mathematically as a discontinuity in the density distribution of a flow. It should satisfy at each point of the space–time the total energy-momentum and current/particle flux conservation equation (Anile 1989):

$$T_{\mu \nu} = 0$$  \hspace{1cm} (1)

$$\sum_i (\rho_i u_i^\mu)_{\mu \nu} = 0,$$  \hspace{1cm} (2)

where \( T_{\mu \nu} \) is the total energy-momentum tensor; \( \rho \) is density and \( u^\mu \) is velocity vector; the index \( i \) indicates species of particles/fluids with a conserved number. They also include the interactions between these particles/fluids. When there is no interaction, these equations must be satisfied separately for each species. In particular, at the shock front where these quantities are discontinuous the conservation equations take the form of jump conditions across the discontinuity surface:

$$[T_{\mu \nu}] \Sigma_{\mu} = 0$$  \hspace{1cm} (3)

$$\sum_i [\rho_i u_i^\mu] \Sigma_{\mu} = 0.$$  \hspace{1cm} (4)

The symbol \([ \ ] \) means the difference of the quantities between square brackets on the two sides of the shock front. Solutions of these equations determine the evolution of the shock front. As for the state of the shocked material behind the discontinuity, when there is no energy dissipation jump conditions can be used to obtain a self-similar solution (Blandford & McKee 1977). In the presence of energy injection or dissipation, however, the self-similarity solutions are only approximations and in general an exact self-similar solution does not exist. Moreover, energy dissipation by synchrotron emission changes the form of the conservation equations, i.e. it cannot be treated as an inhomogeneous term in the differential equations describing the dynamics. Thus, it is not possible to extend the solutions of the non-dissipative case to this case even as an approximation. We will discuss this issue in detail later in this section. Therefore, in a dissipative shock the jump conditions are applied only at the initial time when a shell meets another shell or the ISM, and they should be considered as boundary conditions when conservation equations (1) and (2) are solved.

Intuitively, we expect that with time the discontinuous distributions of quantities in the shock front, i.e. the initial jump conditions, change to continuous distributions which at far downstream distances should asymptotically approach the slow shell values, and at far distances in the upstream direction the characteristics of the fast shell (see Fig. 1 for a schematic illustration). In this transient region, instabilities form electric and transversal magnetic fields. They accelerate electrons which subsequently lose their energy by synchrotron and inverse Compton emission. A distant observer receives the signature of the shock mainly through the detection of these radiations as a \( \gamma \)-ray burst or X-ray flare. Therefore, finding the evolution of physics of this region is the main purpose for solving conservation equations.

A full solution of equations (1) and (2) with the initial conditions (3) and (4) a priori includes all the necessary information about the physical processes and their evolution. However, the complexity of the problem cannot permit solution of them analytically. On the other hand, numerical simulations are both complex and time consuming, and it would be very difficult to cover the full parameter space and obtain results that can be compared with the observations. At present, simulations are only able to demonstrate the validity of ideas about processes involved and the role of the instabilities in the formation of coherent fields (Spitkovsky 2008; Keshet et al. 2009).

Here, we consider an intermediate strategy. We assume a spherical\(^1\) thin and optically transparent active region. Its average distance from the central source is \( r(t) \) and its thickness \( \Delta r(t) \). We neglect the variation of quantities inside the active region and consider the average value through the region. This means that the evolution depends only on \( r(t) \), the mean distance of the active region from central engine. As for the energy dissipation, we assume a radiative shock, i.e. for an observer at rest with respect to the active region the incoming particles lose their energy through synchrotron radiation and join this region. Despite the possibility of a large contribution from Compton cooling of electrons in the GRB prompt emission (Stern & Poutanen 2004; Piran, Sari & Zou 2009), X-ray flares (Kobayashi et al. 2007) and afterglow (Wang, Dai & Lu 2001), there is no strong evidence of this process in the Swift data. Cases such as the early flare of GRB 050406 (Romano

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\(^1\) Most of the formulations presented here are also applicable to a beamed ejecta/jet if the transverse dispersion of matter in the jet is negligible with respect to the boost in the radial direction. We consider the effect of beaming in Section 2.6.
et al. 2006; Corsi & Piro 2006) that was once attributed to Compton cooling of a reverse shock are now understood to be related to the late activity of the central engine like all other observed flares and have the same origin as the gamma-rays. It is also suggested that the anomalous behaviour of GRB 060218/SN 2006aj (Soderberg et al. 2006) is due to inverse Compton cooling (Dai, Zhang & Liang 2006), but this is an exception. Therefore, here we only consider synchrotron radiation in a relativistic shock as the source of emission.

From now on we consider the active region as an isolated shell of baryonic material. The effect of the ISM/slow shell appears as an incoming flux, and the effect of diffusion of shocked material into the upstream is included in the evolution of the thickness of the active region and other physical parameters such as density variation with time. The initial value of the density and Lorentz factor corresponds to the values in the fast shell, i.e. the shock front at the time of encounter between the two shells (internal shock) or the fast shell and the ISM (external shock). We also assume that thermal energy and pressure are negligible with respect to the relativistic boost. This assumption of cold matter is especially justified for the prompt emission because the temperature of ejected material from the progenitor is expected to be at the most a few hundred MeV, where the kinetic energy of relativistic baryons must be of the order of a few GeV or larger. This approximation is probably not suitable for late interaction of the shells when they have lost most of their kinetic energy and probably turbulence and scattering have transferred part of this energy to thermal.

With these approximations, we write the energy-momentum conservation equations for the active region in the rest frame of the slow shell. The reason for choosing this frame is that the same formulation can be applied to external shocks where the slow shell is the ISM or surrounding material around the engine. The latter is considered to be at rest with respect to the observer (after correction for the expansion of the Universe).\(^2\) Note that for consistency and conservation of energy and momentum, we have to integrate equations (1) and (2) along the active region. As we only consider the average value of quantities, the integration is trivial.\(^3\)

\[
\frac{d(n'\Delta r')}{dr'} = \gamma' \left[ \frac{\gamma^2}{\gamma' - 2} \frac{d(n'\Delta r')}{dr'} + 2r'(n'\Delta r') \frac{d\gamma'}{dr'} \right] = n_0(r') c^2 - \frac{dE_{\gamma}}{4\pi mc^2},
\]

\[
\frac{d(n'\Delta r'\beta' \gamma')}{dr'} = \beta' \gamma' \left[ \frac{\gamma^2}{\gamma' - 2} \frac{d(n'\Delta r')}{dr'} + 2r'(n'\Delta r') \frac{d\beta' \gamma'}{dr'} \right] + r^2(n'\Delta r') \frac{d\beta' \gamma'}{dr'} = -\frac{dE_{\gamma}}{4\pi mc^2},
\]

where \(r'\) is the distance from the central engine, \(n'\) is the baryon number density of the fast shell measured in the slow shell frame, \(n_0\) is the baryon number density of the slow shell in its rest frame and, in general, it can depend on \(r'\). Here, we assume that \(n_0(r') = N_0 \left( r'/r_0 \right)^{-\kappa}\).

For ISM \(\kappa = 0\), i.e. no radial dependence. For a wind surrounding the central engine, \(\kappa = 2\) is usually assumed (Chevalier & Li 2000). For a thin shell or jet expanding adiabatically also \(\kappa = 2\) if we neglect the transverse expansion in the case of a jet (collimated ejection). But for the collision between two thin shells if the duration of the collision is much smaller than \(r_0/c\) we can neglect the density change due to expansion during the collision, and assume \(\kappa = 0\). \(\Delta r'\) is the thickness of the shocked synchrotron emitting region, \(\gamma'\) is the Lorentz factor of the fast shell with respect to the slow shell, \(\beta' = \sqrt{\gamma'^2 - 1}/\gamma'\), \(m = m_p + m_e \approx m_p\), \(E_{\gamma}'\) is the total emitted energy and \(c\) is the speed of light. The evolution of the radius with time is

\[
r'(t') = c \int_{t_0'}^{t'} \beta'(t') dt',
\]

where the initial time \(t_0'\) is considered to be the beginning of the collision.

A priori, we should also consider a conservation equation for the baryon and lepton numbers. However, with approximations explained above the thickness of the active region is a parameter which is added by hand. The simplest assumption is that the active region is formed only from particles that fall from the slow shell to the shock front and they stay there. In this case, the baryon number conservation is simply

\[
\frac{d(n'\Delta r')}{dr'} = n_0.
\]

On the other hand, the assumption of particles staying for ever in the active region seems quite unphysical because scattering, acceleration and dissipation move shocked particles to both upstream and downstream, and gradually many of the particles are in a region where instabilities are too weak to make the electric and magnetic fields necessary for the acceleration and synchrotron emission. Therefore, in this approximation the active region cannot be considered as a completely isolated system with full conservation laws applied to it. The consequence of this is that we cannot determine \(\Delta r\) from first principles. This point can be also interpreted as the manifestation of the fact that there is not an abrupt termination of the active zone, and therefore there is no conservation for the artificial boundary we have added by hand. In fact, it is equally valid if we consider a conserved number of particles and the corresponding conservation equation but give up energy or momentum conservation. In this definition, the active region follows active particles (in a Hamiltonian formulation sense). However, the only measurable quantity for a distant observer is the energy dissipated as radiation. Therefore, it is more useful to define the active region based on the energy-momentum conservation and leave the number of these particles as a free parameter. Note also that in the left-hand side of (5) and (6), as well as in the synchrotron term under some conditions (see below), \(\Delta r'\) always appears as \(n'\Delta r'\), i.e. the column density. It is more relevant for the observations and does not depend on the way we define the active region.

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\(^2\) Throughout this work, frame-dependent quantities with a prime are measured with respect to the rest frame of the slow shell and without a prime with respect to a distant observer at the redshift of the central engine. Parameters used for parametrization do not have a prime even when parametrization is in the slow shell frame.

\(^3\) Note also that a priori we should consider an angular term presenting the collimation of the radial element. But for an infinitesimal element the angular dependence is negligible and we do not add it to this formulation.
We can formally integrate equation (5) by replacing the synchrotron term in the left-hand side of (6), and determine the column density of the active region.\footnote{We recall that column density and total power are scalars and therefore their value is frame independent.}

\[
n' \Delta r' = \frac{N_0 \gamma_0^3 \left( \frac{\tau_t}{\tau_i} \right)^{3-k} - 1 + (3 - \kappa) r' n'(r_0) \Delta r'(r_0) \gamma_0^2 (1 - \beta_0^2)}{(3 - \kappa) \gamma_0^2 \delta (1 - \beta')}
\]

where \(\gamma_0 \equiv \gamma'(r_0)\) and \(\beta_0\) is the corresponding \(\beta'\). This solution depends on the evolution of \(\gamma'\) which can be obtained by solving (6), but we first recall the dependence of the synchrotron term on the microphysics of the shock. The power of synchrotron radiation emitted by the active shell is

\[
P' = \frac{dE}{dr'} = c \beta' \frac{dE}{dr'} = \frac{16\pi}{3 \gamma^2} \sigma_T \gamma^2 B^2 \frac{\gamma}{8\pi} \int n'_e(\gamma_e) \gamma_e^2 d\gamma_e,
\]

where \(n'_e\) is the number density of accelerated charged leptons – electrons and possibly positrons – with a Lorentz factor \(\gamma' > \gamma_0\) and \(\gamma'_e\) is the number density of accelerated charged leptons in the active region frame. \(B'\) is the magnetic field – \(B^2/8\pi\) is the magnetic energy density in the active region frame and \(\sigma_T\) is the Thomson cross-section. We define the normalization of the electron distribution as the following:

\[
\int_{\gamma_m}^{\infty} n'_e(\gamma_e) d\gamma_e = n'_e
\]

\[
\int_{\gamma_m}^{\gamma_c} \gamma_e n'_e(\gamma_e) d\gamma_e = \frac{\gamma^2 \gamma'_e m_p n'_0 e}{m_e},
\]

where \(n'_e\) is the number density of accelerated charged leptons and \(\epsilon_e\) is the fraction of the kinetic energy of the falling baryons transferred to the accelerated leptons in the active region frame.

In GRB-relativistic shock literature, it is usually assumed that \(n'_e \approx \gamma'^2 n'_0\), i.e. only falling leptons are accelerated. However, the validity of this assumption is not certain because once the electric and magnetic fields are produced by the instabilities, all the charged leptons are accelerated. If the initial number density of charged leptons in the rest frame of the shells is similar to the slow shell, then as the flux of falling leptons is enhanced by a factor of \(\gamma'\), the density of local leptons can be neglected. However, in the prompt collision one expects that the relative Lorentz factor of the shells would be \(O(1)\). In this case, the local density of leptons is not negligible and \(n'_e \approx (n'_{\text{le}} + n'_{\text{pe}})\), with \(n'_{\text{le}}\) and \(n'_{\text{pe}}\), respectively, the total number density of electrons and positrons. For a neutral matter with negligible positrons content \(n'_e \approx n'_{\text{le}}\).

Motivated by the power-law distribution of accelerated charged particles in other astronomical shocks, e.g. supernovae and cosmic rays, it is usually assumed that the distribution of accelerated electrons responsible for the GRB prompt and afterglow emission is a power law:

\[
n'_e(\gamma_e) = N_e \left( \frac{\gamma_e}{\gamma_m} \right)^{-(p+1)} \quad \text{for} \quad \gamma_e \geq \gamma_m
\]

\[
N_e = \frac{n'_e p}{\gamma_m} = \frac{p^2 m_e n'_0^2}{(p - 1) \epsilon_e \gamma^2 m_p n'_0}
\]

\[
\gamma_m = \frac{(p - 1) \epsilon_e \gamma^2 m_p n'_0}{p m_e n'_e}
\]

Recent simulations of particle acceleration by the Fermi process in relativistic shocks show (Spitkovsky 2008) that \(n'_e(\gamma_e)\) is best fitted by a 2D Maxwellian distribution plus a power law with an exponential cut-off:

\[
n'_e(\gamma_e) = C_1 \gamma_e \exp\left(\gamma_e / \gamma_1\right) + C_2 \gamma_e^{-4 \min[1, \exp(\gamma_e / \gamma_{\text{cut}})]},
\]

where \(C_2 = 0\) for \(\gamma_e \) less than a minimum value \(\gamma_{\text{min}}\). The typical values of parameters obtained from the fit to simulations for an initial Lorentz factor of \(\gamma_0 = 15\) are \(\gamma_{\text{min}} = 40, \gamma_1 = 6, \gamma_{\text{cut}} = 300\), and \(\delta = 2.5\) (Spitkovsky 2008). Implementation of this distribution makes the model presented here significantly more complex. None the less, in the range of energies relevant to the prompt and early afterglow emission of GRBs, the first term in (16) is negligible, and for \(\gamma_e > \gamma_1\), the distribution has the form of a power law with exponential cut-off. Conservation conditions similar to (14) and (15) for this distribution lead to the following relations:

\[
n'_e(\gamma_e) = N_e \left( \frac{\gamma_e}{\gamma_m} \right)^{-(p+1)} \min[1, \exp(\gamma_e / \gamma_{\text{cut}})]
\]

\[
n'_e = \frac{N_e \gamma_e}{p} \left[ 1 - \left( \frac{\gamma_e}{\gamma_m} \right)^{-p} \Gamma \left( -p, \frac{\gamma_e}{\gamma_{\text{cut}}} \right) \right]^{-1} \left( \frac{\gamma_e}{\gamma_m} \right)^{-(p-1)} \exp \left( \frac{\gamma_e}{\gamma_{\text{cut}}} \right) \Gamma \left( -(p-1), \frac{\gamma_e}{\gamma_{\text{cut}}} \right)
\]

\[
\frac{\gamma^2 m_p n'_0 \epsilon_e}{m_e} = \frac{N_e \gamma_e^2}{(p - 1)} \left[ 1 - \left( \frac{\gamma_e}{\gamma_m} \right)^{-(p-1)} \Gamma \left( -(p-1), \frac{\gamma_e}{\gamma_{\text{cut}}} \right) \right],
\]
where \( \Gamma(\alpha, x) \) is the incomplete \( \Gamma \) function. As the number of parameters in this distribution is larger than the number of conservation conditions, in contrast to the power-law distribution, it is not possible to find an expression for \( N_e \) and \( \gamma_m \) with respect to the total density and the fraction of electric and magnetic energies transferred to leptons. A more simplified version of this model is a power law with an exponential cut-off:

\[
n'_e(\gamma_e) = N_e \left( \frac{\gamma_e}{\gamma_m} \right)^{-p-1} \exp \left( - \frac{\gamma_e}{\gamma_m} \right) \quad \text{for } \gamma_e \geqslant \gamma_m, \quad \gamma_{\text{cut}} \gg \gamma_m. \tag{20}
\]

Note that due to the exponential cut-off, the restriction to \( p \geq 2 \) does not apply and the index of the power-law term can be negative. Using conservation conditions (11) and (12), we can find relations between parameters of this distribution:

\[
\gamma_{\text{cut}} = \frac{\epsilon_e \gamma^2 m_p n'_0}{|p| m_e n'_0} \tag{21}
\]

\[
\gamma_m N_e = n'_e \left( \frac{\gamma_{\text{cut}}}{\gamma_m} \right)^p \left[ -p - \frac{\gamma_{\text{cut}}}{\gamma_m} \right]^{-1}. \tag{22}
\]

As we have only two constraints, it is not possible to find expressions for three constants \( N_e, \gamma_{\text{cut}} \) and \( \gamma_m \), and one of them will stay as a free parameter. In Section 2.4, we show that this type of electron distribution is necessary to explain the hard spectrum of the short hard and some of the long bursts.

In the Introduction, we mentioned that the induced transverse magnetic field is produced by the Weibel instability in the active region. The magnetic energy density is parametrized by assuming that it is proportional to the energy density of in-falling particles to the shock front/active region:

\[
B^2 = \frac{\epsilon_B^2}{8\pi} \tag{23}
\]

It is expected that both \( \epsilon_e \) and \( \epsilon_B \) evolve with time/radius. If the central engine is magnetized, the external magnetic energy can be very important and an external field should be added to the right-hand side (RHS) of (23). Here, for simplicity, we neglect such cases.

Considering the simplest case of a power-law distribution for electrons and also assuming that only in-falling electrons are accelerated, i.e. \( n'_e = n'_e \gamma/\gamma'_0 \) (the minimum of \( n'_e \) for a radiative shock), the synchrotron term in (5) and (6) becomes

\[
\frac{\text{d}E_{\gamma}}{\text{d}\tau} = \frac{4\alpha_\gamma \gamma^2 m_e n'_0 \epsilon'_e \epsilon_B \gamma' r^2 \Delta \tau'}{3m_e^2 \beta'}, \quad \alpha = \frac{(p - 1)^2}{p(p - 2)}. \tag{24}
\]

For the reasons explained in the Introduction and in Section 2.1, we believe that in a realistic model of relativistic shocks one should consider the time evolution of electric and magnetic fields. Here, we assume a simple power-law evolution with a constant index:

\[
\epsilon_e = \epsilon_e(\gamma'_0) \left( \frac{r}{\gamma'_0} \right)^{a_\epsilon}, \quad \epsilon_B = \epsilon_B(\gamma'_0) \left( \frac{r}{\gamma'_0} \right)^{a_B}. \tag{25}
\]

Using (24), the expression for the column density (9) and the momentum conservation equation (6), we obtain the following equation for the evolution of the relative Lorentz factor:

\[
\frac{\text{d}}{\text{d}\tau'} \left[ N_e \gamma'_0^3 \left( \frac{r}{\gamma'_0} \right)^{3-k} - 1 + (3-k)r'_0 n_0(1 - \beta'_0) \Delta \tau'(r_0) \gamma'_0(1 - \beta'_0) \beta' \right] = - \frac{4\alpha_\gamma \gamma^2 m_e n'_0 \epsilon'_e \epsilon_B \gamma' r^2 \Delta \tau'}{3m_e^2 \beta'}. \tag{26}
\]

The parameter \( \eta = 2\alpha_\epsilon + \alpha_B + 2k \) is the evolution index of the density and fields. Although this differential equation is a first-order equation, it is highly non-linear. To solve it, we proceed with a perturbative method based on iteration. Moreover, it depends explicitly on \( \Delta \tau' \), and as we discussed in Section 2.2 to be able to find an explicit solution for \( \gamma'(r') \), we have to model its evolution. We consider two models.

### 2.2.1 Dynamically driven active region

Assuming that the shock strength and consequently \( \Delta \tau \) depend mainly on the density difference, and that the densities of the shells in their rest frame are roughly the same, we expect smaller \( \Delta \tau' \) for larger \( \gamma' \). On the other hand, when the relative Lorentz factor is small and the shock is soft, \( \Delta \tau' \) should be proportional to \( \beta' \) and \( \Delta \tau' \to 0 \) when \( \beta' \to 0 \). The simplest parametrization of \( \Delta \tau' \) with these properties is

\[
\Delta \tau' = \Delta \tau'_0 \left( \frac{\gamma'_0 \beta'}{\beta'_0 \gamma'} \right)^\iota \Theta(r' - r'_0), \tag{27}
\]

where \( \Delta \tau'_0 \) is a thickness scale. A \( \Theta \) function is added to (27) to explicitly indicate that the expression is valid only for \( r' \geq r'_0 \). Note that in this model the initial thickness is not null, and therefore it is assumed that it was formed in a negligible time or the value of \( \Delta \tau'_0 \) is the final value from a previous regime that makes the initial active region before (27) can be applied. This model should be suitable for the prompt/internal shocks in which two high-density narrow shells pass through each other and one expects that roughly instantly a narrow and dense active

region forms around the shock discontinuity (see also next section for other cases). As $\beta' / \gamma' < 1$ is expected to be a decreasing function of $r'$, for $\tau > 0$ the width of the active region decays and for $\tau < 0$ it grows. But it is not always the case, see simulations in Paper II.

2.2.2 Quasi-steady active region

At the beginning of a strong shock, presumably an internal shock or when the slow shell is extended, roughly homogeneous, and has a low density, we expect that after a transient time in which the active region grows, its thickness arrives at a stable state determined by the relative Lorentz factor, density, synchrotron emission and expansion of the shells. This stability should persist until the loss of kinetic energy due to radiation and mass accumulation become important, or the fast shell passes through the slower one (this does not happen for a radiative shock). In this case, we parametrize the time evolution of $\Delta r'$ as

$$\Delta r' = \Delta r_{\infty} \left[ 1 - \left( \frac{r'}{r_0} \right)^{-\delta} \right] \Theta(r - r_0'),$$

where $\Delta r_{\infty}$ is the final width when the equilibrium is achieved. For the decay of the active region at the end of this regime, one can use the dynamical model. Another possibility is to consider

$$\Delta r' = \Delta r_{\infty} \left( \frac{r'}{r_0} \right)^{-\delta} \Theta(r - r_0').$$

In Appendix A, we argue that with small modifications the calculation of dynamical evolution can be used for this model too.

This model is especially suitable for studying the external shocks of the ejecta from the central source with diffuse material or wind surrounding it, and/or the ISM. One expects that in these cases the density of the relativistic ejecta – the fast shell – will be much higher than the wind or ISM and its extension much smaller. In Paper II, we show that such a model along with a late emission from internal shocks can explain the shallow regime observed in the X-ray light curve of the majority of GRBs detected by Swift.

For studying the evolution of the active region, a priori we should also take into account the total size of the shells and the passage or coalescence time. However, observations show that synchrotron emission continues for a significant time after the end of the shock – when shells passed through each other or coalesced. For a distant observer what matters is the emission rather than physical encounters between shells. All these stages can be modelled by one of the models explained here or similar models for the evolution of $\Delta r'$. In this case, the difference between various stages of the collision is reflected in the different value of parameters.

2.3 Evolution of relative Lorentz factor

To solve equation (26), we use a perturbative/iterative method based on the assumption that the dimensionless coupling in the RHS of this equation is smaller than one. By dividing both sides of (26) with $n_0 r_0^2$, one can extract the coupling $A$:

$$\frac{d}{d \left( \frac{r'}{r_0} \right)} \left[ \left( \frac{r'}{r_0} \right)^{3-\kappa} - 1 + \frac{(3-\kappa)\Delta r'(r_0)}{n_0 r_0^2} \gamma_0 \left( 1 - \beta_0' \right) \beta' \right] = -\frac{A \beta} {\beta' \Delta r' (r_0)} \left( \frac{r'}{r_0} \right)^{2-\kappa} \gamma_0.$$

$$A = \frac{4 \sigma_\gamma^2 \epsilon_e^2 \Delta r'(r_0) \epsilon_e (r_0) \epsilon_B (r_0)}{3 m_e^2 N_0 \Delta r'(r_0) \epsilon_e (r_0) \epsilon_B (r_0)}.$$

It is straightforward to see that if the initial column density of the slow shell/ISM $n_0 \Delta r'(r_0) \lesssim 10^{22} \text{ cm}^{-2}$, for any value of $\epsilon_e (r_0) < 1$ and $\epsilon_B (r_0) < 1$, the coupling $A \ll 1$. This upper limit on the shell column density is in the upper range of the observed total $N_0 \text{ cm}^{-2}$ column density of GRBs. However, the real $N_0$ can be much higher than that measured from the absorption of the soft X-ray at least $10^2 \text{ s}$ after the trigger, because it is in conflict with $N_0$ estimated from Lyman $\alpha$ absorption (Watson et al. 2007). The difference can be due to the ionization of the neutral hydrogen by UV emission from the prompt emission. None the less, simulations of the formation of the electric and magnetic fields in the shocks show that the fraction of the kinetic energy transferred to the fields, especially to the magnetic field, is much less than 1 (Keshet et al. 2009). Therefore, even with larger column densities, the value of $A$ should be less than 1 and the validity of the perturbative method is justified.

The zeroth-order approximation corresponds to $A \to 0$. In this case, equation (30) is a pure differential and its solution is trivial:

$$\beta_0'(r') = \begin{cases} \frac{(r'/r_0)^{3-\kappa}}{((3-\kappa)r_0^2)} & \kappa \neq 3 \\ \frac{1 - (\kappa + 1)D}{\kappa r_0^2} & \kappa = 3 \end{cases}$$

$$D = \frac{n'(r_0') \Delta r'(r_0) \beta_0'(r_0)}{N_0 r_0^2}.$$
where $\beta_0^\prime$ indicates the zeroth-order approximation for $\beta(r')$. In the rest of this work, we only concentrate on $\kappa \neq 3$, but calculations can be easily extended to this exceptional case.

The physical interpretation of (32) is quite evident. $\beta(r')$ changes inversely proportionally to the total mass of the shell including the accumulated mass of the swept material. This zeroth-order solution does not take into account the energy necessary to accelerate particles of the slow shell. Thus, its use without radiation corrections will lead to a violation of energy conservation. Parameter $D$ presents the strength of the shock; the smaller the $D$, the faster the constant term in the denominator becomes negligible with respect to the growing radial term and $\beta_0^\prime(r')$ approaches a cubic decline (for $\kappa = 0$) due to the adiabatic expansion. The origin of term $(3 - \kappa)$ is partially geometrical and partially related to the density variation with $r$ in the slow shell. It is the effective mass accumulation index of the shock.

As (32) is the dominant component of the dynamics and is used throughout this work, it is useful to have its asymptotic behaviour for $(r_0') \gtrsim 1$ and $(r_0') \gg 1$:

$$
\beta_0^\prime(r') \approx \begin{cases} 
\left. \frac{D}{r_0'} \approx \beta_0^\prime \left(1 - \frac{r_0^\prime}{D} \right), \quad (r_0') - 1 \equiv \varepsilon \gtrsim 0 \\
(3 - \kappa) D \left(\frac{r_0^\prime}{D} \right)^{3-k}, \quad \beta_0^\prime = \frac{\beta_0}{r_0} \quad \left(\frac{r_0^\prime}{D} \right)^{3-k} \gg \left(\frac{r_0^\prime}{D} \right)^{3-k} \gg 1. 
\end{cases}
$$ (34)

We use the zeroth-order solution in the RHS of (30) to obtain the first-order correction of the solution. The $n$-order approximation corresponds to using the $(n-1)$-order approximation in the RHS of (30) and solving the equation:

$$
(\frac{3-k}{3-k})^n \left(3-k\right) D \left(\frac{r_0^\prime}{D} \right)^{3-k} = \mathcal{D}_{(n)} - \frac{A}{\Delta r' r_0' \int_1^{r_0'} \frac{\beta_0^\prime(r')}{\beta_0^\prime(1 - \beta_0^\prime)} dx.
$$ (35)

where $\mathcal{D}_{(n)}$ is an integration constant. For any order of correction (35) must satisfy the initial condition, i.e. $\beta_{(0)}^\prime(r_0') = \beta_0^\prime$ and from this constraint one can determine $\mathcal{D}_{(n)}$. It is easy to see that $\mathcal{D}_{(n)} = \mathcal{D}$ for all orders of perturbation. Calling the integral term $\mathcal{M}_{(n-1)}(r')$ for the $(n-1)$-order solution, we find the following recursive expression for $\beta_{(n)}^\prime$:

$$
\beta_{(n)}^\prime = -\frac{D}{r_0'} \left(\frac{3-k}{3-k}\right)^n \left(3-k\right) D \left(\frac{r_0^\prime}{D} \right)^{3-k} \int_1^{r_0'} \frac{\beta_0^\prime(r')}{\beta_0^\prime(1 - \beta_0^\prime)} dx
$$ (36)

At this point, we have to consider a model for $\Delta r'$. For the dynamical model, $\mathcal{M}_{(n)}$ becomes

$$
\mathcal{M}_{(n)}(r') = \Delta r_0^\prime \left(\frac{\beta_0^\prime}{\beta_0^\prime} - \frac{1}{\beta_0^\prime(1 - \beta_0^\prime)} \right) x^{2-k} dx
$$ (37)

The second form of the integral is obtained using the $\beta_{(0)}^\prime$ solution. Unfortunately, this integral cannot be determined analytically. None the less, by expanding one of the two terms in the integrand, it is possible to find an approximation which is useful for getting an insight into the behaviour of $\beta_{(1)}^\prime$, its dependence on various parameters and the importance of radiation correction. They are summarized in Appendix A.

For a quasi-steady model $\mathcal{M}_{(0)}(r')$ is

$$
\mathcal{M}_{(0)}(r') = \Delta r_0^\prime \left(\frac{1}{1 - \beta_0^\prime} \frac{1}{\beta_0^\prime(1 - \beta_0^\prime)} \right) x^{2-k} dx = -\frac{\Delta r_0^\prime}{3-k} \left((3-k) D \right)^{1-k} \int_1^{r_0^\prime} dy \frac{y^{3-k}}{(1 - y^{3-k})^2} \left(1 + \left(\frac{1}{(3-k) D} - \frac{1}{\beta_0^\prime} \right) y \right)^{-\frac{1}{2-k}}
$$ (38)

The similarity of the second term in (38) to the integrand of (36) shows that at first order of radiative correction the two terms in (28) act independently. The constant term leads to the first term in (38) which is equivalent to (36) with $\tau \to 0$, i.e. an active region with constant thickness. The term proportional to $(r_0^\prime/r')^k$ in (28) is responsible for the second term in (38). Up to a constant, it is also the expression for a constant thickness model with $\eta \to \eta + \delta$. This means that the effect of the power-law term in (28) is very similar to power-law dependence of fields and shell density on $r'$. This behaviour justifies the name quasi-steady we have given to this model. Analytical approximations of these integrals can be found in Appendix A.

In the last paragraph, we considered a power-law dependence on $r'$ for the electric and magnetic fields. The case of an exponential rise or fall of the fields is important at the beginning and at the end of gamma-ray spikes or early X-ray flares. For this case, the only modification in (37) and (38) is the replacement of $x^{-\eta}$ with $\exp(-\eta x)$ where here $\eta$ is a dimensionless coefficient determining the speed of exponential variation. It is negative for rising fields and positive when fields are declining, similar to the power-law case. In the same way, if the electron distribution $n_e(y_e)$ includes an exponential cut-off (Dempsey & Duffy 2007), an exponential term similar to the term for fields appears in the expression for $\mathcal{M}_{(0)}(r')$. Therefore, in a general case, the integrand in (37) and (38) includes an exponential term which makes it even more complex. None the less, the expansion of the exponential permits us to obtain the analytical approximation given in Appendix A.
Finally, by using \( M_{\text{g}}(r') \) in (36) and (9), we can determine \( \beta'_{1}(r) \), the first-order radiation corrected evolution of \( \beta' \) and similar concentration \( n'(r') \Delta r' \) with radius/time. The complexity of expressions for \( M_{\text{g}}(r') \) and consequently for \( \beta'_{1}(r) \) does not permit us to investigate the effect of various quantities from the exact calculations, and we leave this for Paper II where we numerically evaluate quantities characterizing the kinematics of the shock and its synchrotron emission. Here, we just consider the simplest cases when in (34), \( r'/r_0 > 1 \) or \( r'/r_0 \gg (3-\kappa)D/\beta_0 \).

Using (A6) and (A7), respectively, for large and small \( \eta \), we find the following expressions for \( M_{\text{g}}(r') \) when \( \varepsilon \ll 1 \):

\[
M_{\text{g}}(r') \approx \Delta r_{\text{g}} \eta B \left( 1 - \frac{1}{\tau} \right) \left( 1 + \left( 3 - \frac{\eta}{\tau} \right) \frac{\beta_{0}^{2}}{\tau} \right) \varepsilon \quad \text{for} \ |\eta| \gg 0 \tag{39}
\]

\[
M_{\text{g}}(r') \approx \Delta r_{\text{g}} \eta B \left( \frac{\eta}{3 - \kappa} \right) \left( \frac{\eta}{2 - \kappa} \right) \left( \frac{\beta_{0}^{2}}{\tau} + \frac{\kappa}{\beta_{0}^{2}} \right) \left( \frac{\eta}{\tau + \frac{\eta}{\tau}} \right) \varepsilon \quad \text{for} \ |\eta| > 0 \tag{40}
\]

\[
B = \frac{\gamma_{0}}{(3 - \kappa)\beta_{0}^{2}} \frac{\tau}{\beta_{0}}, \quad C = \frac{(3 - \kappa)D}{\beta_{0}}. \tag{41}
\]

Therefore, \( M_{\text{g}}(r') \propto \Delta r_{\text{g}} B \varepsilon \) for \( \varepsilon \ll 1 \). The constant coefficient is expected to be of the order of 1. By applying these results to (36), we find

\[
\beta'_{1}(r) \approx \frac{D}{\beta_{0}^{2}} + (1 - AB^2) \varepsilon . \tag{42}
\]

where \( B \) is the variable by the corresponding constant coefficients in (39) or (40) depending on the value of \( \eta \). Comparing this result with the corresponding \( \beta'_{1}(r) \), we conclude that the strength of the radiation correction of \( \beta' \) and its effect on the kinematics of the ejecta/jet depend on \( S \equiv AB \). As \( B \) is proportional to \( D \), for a positive \( \eta \), larger \( D \) (stronger shock), smaller \( S \). In this case, the kinetic energy of the shock is much larger than radiation, and therefore the synchrotron emission does not significantly modify the evolution of the shock. Note also that although \( S \) is linearly proportional to the synchrotron total coupling \( A \), it depends non-linearly on \( D \) through a power which depends on the time/radius variation of the electric and magnetic fields as well as the density of the shells. The quantity \( S \) in this model depends also on \( \gamma_{g} \), larger \( \gamma_{g} \), larger influence of radiation. This simply means that the effective thickness of the active region decreases faster when the effect of radiation is stronger. This behaviour is consistent with the phenomenology of the model described here.

Similar expressions can be found for the other extreme case, i.e. when \( r'/Cr_{0} \gg 1 \):

\[
M_{\text{g}}(r') \approx \frac{\Delta r_{\text{g}} \eta C^{1 - \frac{\eta}{\tau}}}{(3 - \kappa)^{2}} \left( \frac{1}{\tau - 1 + \frac{\eta}{\tau}} + \frac{3 - \frac{\eta}{\tau}}{\tau + 1 + \frac{\eta}{\tau}} \right) \varepsilon \quad \text{for} \ |\eta| > 0 \tag{43}
\]

\[
M_{\text{g}}(r') \approx \frac{\Delta r_{\text{g}} \eta C^{1 - \frac{\eta}{\tau}}}{(3 - \kappa)^{2}} \left( \frac{1}{\tau - 1 + \frac{\eta}{\tau}} + \frac{3 - \frac{\eta}{\tau}}{\tau + 1 + \frac{\eta}{\tau}} \right) \varepsilon \quad \text{for} \ |\eta| > 0 \tag{44}
\]

When \( r'/Cr_{0} \to 0 \), \( M_{\text{g}}(r') \to c, \text{ if } \tau - 2 + \frac{\eta}{\tau} > 0 \text{ for } |\eta| > 0 \text{ or } \tau - 1 + \frac{\eta}{\tau} > 0 \text{ for } |\eta| > 1 \). This can be interpreted as a saturation state for the synchrotron radiation which has been observed especially in bright bursts where peaks of the prompt gamma-ray emission are roughly square-like. Examples are GRB 060105 (Ziaeepour et al. 2006), GRB 061007 (Schady et al. 2006b, 2007), GRB 060813A (Moretti et al. 2006) and GRB 070427 (Sato et al. 2007a). Even the super burst GRB 080319B (Racusin et al. 2008a,b) seems to be consisting of three overlapping square-shaped peaks (see also simulations in Paper II).

The conditions mentioned above can be considered as consistency conditions because if they are not satisfied \( M_{\text{g}}(r') \to \infty \) which is not physically acceptable. Therefore, these conditions constrain parameters of the model – \( \eta, \tau, \kappa \). For instance, for a slow shell with a roughly constant density \( \kappa \approx 0 \). Therefore, \( \eta \) depends only on the behaviour of electric and magnetic fields. If the variation index of these fields is small and positive, the value of \( \tau \) can be small, i.e. the radiation will not vary very quickly. By contrast, a negative index – increasing fields – cannot last for a long time and imposes a large value for \( \tau \). This simple argument shows that there is an intrinsic relation between these parameters. However, only a detailed modelling of the microphysics of the shock will be able to determine possible relations and their physical origin. None the less, the above discussion shows that the simple model studied here is consistent and includes some of the important properties of the phenomena involved in the production of GRBs.

Using (36), one can see that a constant \( M_{\text{g}}(r') \) is equivalent to redefinition of \( D \). Therefore, when the radiation term arrives at its maximum, \( \beta' \) evolution becomes like a non-radiating ejecta.

In the case of a quasi-steady active region, the behaviour of \( M_{\text{g}}(r') \) is essentially similar because of the similarity between two models explained above. However, the time-scales and indices are different. For instance, when \( \varepsilon \ll 1 \) the value of \( M_{\text{g}}(r') \) is proportional to \( \varepsilon \) with a coefficient equal to (39) or (40) and \( \tau = 0 \), minus the same term with \( \eta \to \eta + \delta \). In this case, the initial \( \gamma_{g} \) does not have an explicit
contribution, and the slope of $\mathcal{M}_{0\theta}(r')$ is smaller than the dynamical model. For $r'/C_{\theta}^0 \gg 1$ if other parameters are the same as the dynamical model, the absence of $\gamma_0$ and smaller power of $C_{\theta}^0/r'$ mean that $\mathcal{M}_{0\theta}(r')$ approaches its maximum value more slowly. There is also a slower change when one of the two contributor terms becomes too small and negligible. As expected, all these properties make this model more suitable for modelling the afterglow.

Up to now, we have only discussed the solutions of dynamical equation (26) corresponding to the rise of the synchrotron emission. The falling edge of the emission, i.e. when $\mathcal{M}_{0\theta}(r') \rightarrow 0$, can be obtained simply by time/radius reversal of rising solutions (see equation 29). For instance, in (39) and (40) if $r' < r_{\theta}^0$, $\epsilon < 0$. By moving the initial condition from $r_{\theta}^0$ to $r'$, we obtain a positive but decreasing value for $\mathcal{M}_{0\theta}(r')$ which becomes zero at $r' = r_{\theta}^0$. These approximations, however, do not permit us to determine when the radiation begins to decrease. For this, we need a detailed study of the evolution of fields and other shock properties.

In summary, the evolution of $\beta'$ determines the kinematics of the burst and is important for the estimation of all observables such as those we will discuss in the next sections – synchrotron flux, hardness ratios, etc. $\beta'$ is also important for determining the evolution of other parameters that are not directly observable and a model must be used for their extraction from data. A good example is the time variation of $\omega_{\theta}^0$, the minimum characteristic frequency of the synchrotron emission. It determines the behaviour of the spectrum and light curves (see expression (78) below for its definition). Assuming the simplest case of $n' = n_0$ in (15), from the definition of $\omega_{\theta}^0$ one can see that $\omega_{\theta}^0 \propto \gamma^{1/2} \epsilon_\theta$. The proportionality coefficient is time/radius independent.

### 2.4 Synchrotron flux and spectrum

In this section, we first recall the synchrotron emission for the purpose of completeness and then use the results for determination of lags between the light curves of different energy bands.

The ejecta from a central engine that produces the GRB is most probably collimated and jet-like, otherwise the observed energy is not explainable. On the other hand for the distant observers, even a spherical relativistic emission looks collimated to an angle $\theta < 1/\Gamma$ along the line of sight where $\Gamma$ is the Lorentz factor of the emitting matter in the observer rest frame (Rees 1967). Therefore, in any case we need to consider the angular dependence of the synchrotron emission. Moreover, we need to consider the delay as well as angular dependence of the Doppler shift of the emission. Simulations show that even with an angle-independent emission, these effects can a priori explain the lag between different bands observed in both Burst And Transient Source Experiment (BATSE) and Swift bursts (Qin 2002; Qin et al. 2004; Lu et al. 2006). However, in these simulations the spectrum and the time profile of emission have been put by hand.

There are a number of pieces of evidence against a high-latitude/Doppler effect origin of the observed lags. First of all, the total effect of high-latitude emission decreases when the Lorentz factor is very high. It is expected that in GRBs $\Gamma \gtrsim 100$, and therefore this effect should be very small. In addition, even in the early X-ray emission where it was expected that this effect dominates, it has not been observed. On the other hand, for a given category of bursts, short or long, it does not seem that there is any relation between lags and spectrum as expected on the other hand for the distant observers, even a spherical relativistic emission looks collimated to an angle $\theta < 1/\Gamma$, and the intensity exponentially decreases for $\gamma_{\theta} \beta' \gtrsim 1$. Therefore, it is a good approximation for any angle (Jackson 2001).

For instance, in (39) and (40) if $r' < r_{\theta}^0$, $\epsilon < 0$. By moving the initial condition from $r_{\theta}^0$ to $r'$, we obtain a positive but decreasing value for $\mathcal{M}_{0\theta}(r')$ which becomes zero at $r' = r_{\theta}^0$. These approximations, however, do not permit us to determine when the radiation begins to decrease. For this, we need a detailed study of the evolution of fields and other shock properties.

In summary, the evolution of $\beta'$ determines the kinematics of the burst and is important for the estimation of all observables such as those we will discuss in the next sections – synchrotron flux, hardness ratios, etc. $\beta'$ is also important for determining the evolution of other parameters that are not directly observable and a model must be used for their extraction from data. A good example is the time variation of $\omega_{\theta}^0$, the minimum characteristic frequency of the synchrotron emission. It determines the behaviour of the spectrum and light curves (see expression (78) below for its definition). Assuming the simplest case of $n' = n_0$ in (15), from the definition of $\omega_{\theta}^0$ one can see that $\omega_{\theta}^0 \propto \gamma^{1/2} \epsilon_\theta$. The proportionality coefficient is time/radius independent.

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In the model presented here, the synchrotron emitting matter is confined to the active region. Therefore, we identify the bulk Lorentz factor $\Gamma$ with respect to the observer with the average Lorentz factor of the active region with respect to the observer. It can be related to the relative Lorentz factor $\hat{\gamma}'(r')$ obtained in Section 2.2 and the final Lorentz factor when two shells are coalesced:

$$\Gamma(r) = \Gamma_f \hat{\gamma}'(r)(1 + \beta_f \beta')(r'),$$

where $\Gamma_f$ is the Lorentz factor of coalesced shells with respect to the observer. The initial value of $\hat{\gamma}'(r)$ is the relative Lorentz factor of the colliding shells. In the case of external shocks on low-velocity surrounding material or the ISM $\Gamma_f \approx 1$ and $\Gamma(r) \approx \hat{\gamma}'(r)$. Note that here we have written $\gamma$ with respect to the observer coordinate $r$, because concerning the synchrotron emission, only the observations of distant observers matter.

The energy (intensity) angular spectrum of synchrotron emission (Jackson 2001) by one electron or positron in a frame where it is accelerated to a Lorentz factor of $\gamma_e$ is

$$\frac{d^2 I}{d\omega d\Omega} = \frac{3e^2 \omega^2 \gamma_e^2}{4\pi^2 \omega_e^2 c} \left( 1 + \gamma_e^2 \theta'^2 \right)^2 \left[ K_{3/3}(\zeta) + \frac{\gamma_e^2 \theta'^2}{1 + \gamma_e^2 \theta'^2} K_{1/3}(\zeta) \right],$$

$$\omega_e = \frac{3}{2} \frac{\beta'(r')}{c} \frac{c}{\rho'_0} = \frac{3\gamma_e^2 B'}{2cm_e}$$

$$\zeta = \frac{\omega_e' \left( 1 + \gamma_e^2 \theta'^2 \right)^2}{2\omega_e}.$$  \hspace{1cm} (48)

The expression (46) is valid for small $\theta'$ angles. However, the main part of the emission is in $\gamma_e \theta' < 1$, and the intensity exponentially decreases for $\gamma_e \theta' \gtrsim 1$. Therefore, it is a good approximation for any angle (Jackson 2001).
Quantities with a prime in (46) to (48) are with respect to the frame where the Lorentz factor of electrons is \( \gamma_e \). Here, we identify this frame as the rest frame of the active region. \( \rho' \) is the Larmor radius of electrons. The angle \( \theta' \) is the angle between the acceleration direction and emission. Without loss of generality, it can be assumed to be in the \( x-z \) plane. Therefore, \( d\Omega = \cos \theta \, d\theta \, d\phi \) (Jackson 2001). When \( |\theta| \gtrsim 0 \), \( d\Omega \approx d\theta d\phi \).

To obtain the power spectrum that is the measured quantity, we must divide the intensity by the precession period of electrons \( 2\pi \rho' / c \).

We expect that accelerated electrons have a range of Lorentz factors, therefore we should also integrate over their distribution:

\[
\frac{d^2 P'}{\omega' d\omega' d\Omega} = \frac{\pi^2}{4c^2} \int_{\theta_{\min}}^{\theta_{\max}} d\gamma_e \, n_e'(\gamma_e) \gamma_e^{-1} \left( \frac{\omega'}{\omega} \right) \left( 1 + \frac{\gamma_e^2 \omega'^2}{\Gamma^2 + \gamma_e^2 \omega'^2} K_{2,\beta}(\xi) \right)^2 K_{2,\beta}(\xi). \tag{49}
\]

Note that we have divided the angular power spectrum by \( \omega' \) to make it dimensionless. In the observer’s rest frame, the spectrum is transferred as (Sadun & Sadun 1991; Huang et al. 2000; van Eerten & Wijers 2009)

\[
\frac{d^2 P}{\omega d\omega d\Omega} = \frac{1}{\Gamma^2 (1 - \beta \cos \theta)^2} \frac{d^2 P'}{\omega' d\omega' d\Omega} \tag{50}
\]

\[
\omega' = \frac{\omega}{\Gamma (1 - \beta \cos \theta)} \tag{51}
\]

\[
\cos \theta' = \frac{\cos \theta + \beta}{1 + \beta \cos \theta}, \quad \cos \theta' = \frac{\cos \theta - \beta}{1 - \beta \cos \theta}, \tag{52}
\]

where in equations (50) to (52), \( \beta \) is related to \( \Gamma \), the Lorentz factor of the active region with respect to the observer. To find the total power at a given frequency \( P(t, \omega) \) in the frame of a distant observer, we integrate over the distribution of accelerated electrons and the emitting volume but constrain it to the emission in the direction of the observer. As we assumed that the active region is thin, we neglect the absorption of synchrotron photons inside the active region itself. Without loss of generality, we put the observer at \( \Omega = \Omega' = 0 \). In this case, for an electron moving at angle \( \Omega' \) with respect to the observer, only photons emitted in the direction of \( \Omega' = \Omega' \) are detected by the observer. Therefore, we need to integrate either on \( \Omega' \) (or equivalently \( \Omega_1 \)) or on \( \Omega (\Omega) \). For simplicity of notation, we use the latter. As the synchrotron angular distribution does not depend on \( \phi \) (or \( \phi' \)), we only need to integrate over \( \theta \) (or equivalently \( \theta' \)):

\[
\frac{dP}{d\Omega d\omega} = 2\pi \int_{\omega}^{\omega + \Delta\omega} d\omega \int_{\theta_{\min}}^{\theta_{\max}} d\theta \cos \theta \frac{d^2 P(t - \Delta t, \theta, \omega)}{d\omega d\Omega}. \tag{53}
\]

\( \theta_{\min} \) and \( \theta_{\max} \) are minimum and maximum visible angles for the observer with the constraint \( |\theta_{\max}| < 1/\Gamma \) and \( |\theta_{\min}| < 1/\Gamma \). We define \( \Delta t = |\theta_{\max} + \theta_{\min}| / 2 \) as the viewing angle of the observer with respect to the ejecta/jet axis. These angles are not directly measurable and therefore the simplest assumption is a symmetric ejecta \( \theta_{\max} = -\theta_{\min} = 1/\Gamma \), i.e. \( \Delta t = 0 \). Note that photons coming from \( \theta \neq 0 \) arrive at the observer with a time delay \( \Delta t \) where \( \Delta t(\theta) \) is (Rybicki & Lightman 2004):

\[
\Delta t(\theta) = \frac{r (1 - \cos \theta)}{c \beta(r) (1 + \beta^2 \Gamma^2 \sin^2 \theta)^2}. \tag{54}
\]

In (54), we have assumed that the initial radius from which the shells are ejected from the central engine is much smaller than their distance from it when they collide. Therefore, the initial radius is neglected. Due to the direct relation between radius and time in this model, \( t - \Delta t(\theta) \) can be replaced by \( \Delta r(\theta) = r - c \beta(r) \Delta t(\theta) \). The quantity \( \Delta r(\theta) \) should not be confused with the thickness of the active region \( \Delta r \).

In this model, we have considered \( \Delta r \ll r \), and the physical properties of the active region are close to uniform. Thus, the integral over the interval \( r < r + \Delta r \) becomes trivial if we consider \( r \) to be the average distance of the active region. This is similar to the way we calculated kinematic quantities in Section 2.2. With this simplification, the total spectrum becomes

\[
\frac{dP}{d\Omega d\omega} = 2\pi \int_{\theta_{\min}}^{\theta_{\max}} d\theta \cos \theta \frac{d^2 P(r - \Delta r(\theta), \theta, \omega)}{d\omega d\Omega}. \tag{55}
\]

If we use (7) to describe \( r \) as a function of time, equation (55) depends only on \( t \) and \( \omega \).

The width \( \Delta r \) can in general depend on the energy. In Section 2.1, we explained that when electrons lose their energy, they get distant from the shock front or, in other words, are pushed to the upstream. Although in this region the magnetic field is expected to be weaker, it can be enough for the low-energy emissions, UV, optical, IR. Moreover, in a collimated or structured jet, less these less accelerated particles emit mostly in lower energies. In this simple model of a shock, one way of taking these effects into account is to consider that \( \Delta r \) as well as \( \theta_{\min} \) and \( \theta_{\max} \) depends on the energy. For \( \Delta r \), we can simply assume that \( \Delta r_0 \) in the dynamical model or \( \Delta r_{\infty} \) in the quasi-steady model are energy dependent. Estimation/modelling of the energy dependence of \( \theta_{\min} \) and \( \theta_{\max} \) is more difficult because the only way to modify them is through \( \Gamma \). Assuming \( \theta_{\min} > 1/\Gamma \), we must consider that \( \Gamma \) is \( \theta \)-dependent. This needs a modification of the dynamics and makes the model too complicated. For this reason, here we ignore the energy dependence of the opening angle of the jet.

\[^6\]A structured jet is usually considered to have a transverse gradient in density (Xu et al. 2005; Takami et al. 2007). However, it is expected that gradually a transverse gradient in the Lorentz factor forms too.
The differential term in (55) can be replaced by (50) and (49). We must also take into account the proper time delay discussed above. Therefore,

\[
\frac{dP}{\omega \, d\omega} = \frac{e^2}{2\pi^2} r^2 \Delta r \left( \frac{\omega_e'}{\gamma_e'} \right) \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} d\theta \cos \theta \left[ \Gamma^2 \left( r - \frac{r(1 - \cos \theta)}{(1 + \beta^2 \Gamma^2 \sin^2 \theta)^{1/2}} \right) (1 - \beta(t \cos \theta))^2 \right]^{-1} \\
\times \int_{\gamma_{\text{min}}}^{\infty} d\gamma_e \frac{n_e'(\gamma_e)}{\gamma_e^2} \left( 1 + \gamma_e^2 \theta^2 \right)^2 \left[ K_{2/3}'(\xi) + \frac{\gamma_e^2 \theta^2}{1 + \gamma_e^2 \theta^2} K_{2/3}'(\xi) \right] \quad (56)
\]

\[
\omega_e' = \frac{\omega_e'}{\gamma_e'} = \frac{3eB'}{2cm} \quad (57)
\]

As the angle \( \theta \) and therefore the delay are small, we use the Taylor expansion around \( r \) to obtain the explicit expression for \( \Gamma \) and \( \beta \) in the integrand of (56):

\[
\Gamma \left( \frac{r(1 - \cos \theta)}{(1 + \beta^2 \Gamma^2 \sin^2 \theta)^{1/2}} \right) \approx \Gamma(r) \left( 1 - \frac{r(1 - \cos \theta)}{(1 + \beta^2 \Gamma^2 \sin^2 \theta)^{1/2}} \frac{d\Gamma}{dr} + \cdots \right) \quad (58)
\]

In a similar way, we can expand \( \beta (r, \theta) \) around \( r \):

\[
\beta \left( \frac{r(1 - \cos \theta)}{(1 + \beta^2 \Gamma^2 \sin^2 \theta)^{1/2}} \right) \approx \beta(r) \left( 1 - \frac{r(1 - \cos \theta)}{(1 + \beta^2 \Gamma^2 \sin^2 \theta)^{1/2}} \frac{d\beta}{dr} + \cdots \right) \quad (59)
\]

As \( |\sin \theta| \sim \theta \leq 1/\Gamma \), we can also expand the \( \theta \)-dependent terms in (58) and (59). It is more convenient to use the combination \( \beta' \) and \( \theta' \) in (52) and transfer \( \theta \) to \( \theta' \). We keep only terms up to \( \theta' \) order. With these simplifications, the total energy spectrum can be written as

\[
\frac{dP}{\omega \, d\omega} = \frac{e^2}{2\pi^2} r^2 \left( \frac{\omega_e'}{\gamma_e'} \right) \frac{\Delta r}{\Gamma(r)} \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} d\theta \left( \cos \theta + \beta' \right) \left[ 1 + \varrho(r) \theta'^2 + \cdots \right] \int_{\gamma_{\text{min}}}^{\infty} d\gamma_e \frac{n_e'(\gamma_e)}{\gamma_e^2} \left( 1 + \gamma_e^2 \theta'^2 \right)^2 \left[ K_{2/3}'(\xi) + \frac{\gamma_e^2 \theta'^2}{1 + \gamma_e^2 \theta'^2} K_{2/3}'(\xi) \right] \quad (60)
\]

\[
\varrho(r) = \frac{\beta'^2(r)(1 - \beta)}{2(1 + \beta)} \left( \frac{\beta_0' - \beta(r)}{\beta_0' - \beta(r)} + \frac{1}{D} \right) \left( \frac{\beta_0'}{1 + \beta_0' \beta(r)} - \gamma^2 \beta'(r) \right) \quad (61)
\]

The coefficient \( \varrho(r) \) in (60) presents the lowest order correction due to the Doppler effect and delay of high-latitude emissions. As expected, for \( \Gamma \to \infty \), \( \varrho(r) \propto 1/\Gamma^2 \to 0 \). Moreover, \( \varrho(r) \) does not depend on \( \omega' \) (or equivalently \( \omega \)), and therefore it does not affect the lag between different energy bands itself.

Apart from the energy dependence of \( \Delta r, \theta_{\text{min}}' \), and \( \theta_{\text{max}}' \) which has an intrinsic origin, i.e. it is related to the evolution of the active region and the structure of the jet/ejecta, there are two other effects that can create a lag: angular dependence of the emission and time/radius dependence of \( \gamma_{\infty} \) and \( \omega_{\infty}' \). Assuming a homogeneous density for the colliding shells, equation (15) shows that the time dependence of \( \gamma_{\infty} \) is mainly due to variation of the relative Lorentz factor \( \gamma' \) and the electric field energy fraction index \( \epsilon_e \) with time. The variation of \( \omega_{\infty}' \) is also possible if the magnetic field changes with time. If fields are considered to be constant during the emission – as is the case for many GRB models in the literature – only dynamical friction and the decrease in \( \epsilon_e \) remain. They may be insufficient enough to explain the observed lags because the Doppler and latitude corrections are usually small, especially when \( \Gamma \) is large. In fact, simulations show that with an angular independent emission flux, large lags can be obtained only if the emission happens at large radius with respect to the central engine, \( \gtrsim 10^{15} \) cm (Lu et al. 2006). This is orders of magnitude larger than that expected from internal shocks. As synchrotron emission is highly angle dependent, lags are expected not only from the Doppler and latitude corrections but also when these effects are ignored. In fact, this can be seen in (60). The integration over \( \gamma_e \) can be expressed as \( f(\xi_m) \) where \( \xi_m = \zeta \) [defined in (48)] for \( \gamma_e = \gamma_{\infty} \). It is then clear that the \( r \) and \( \omega' \) dependences of the integrand are not factorizable, and therefore even when \( \varrho(r) \) is ignored, the \( r \) dependence of \( dP/\omega \, d\omega \) cannot be factorized. Therefore, given the expectation of high collimation and large Lorentz factor, the main cause of the observed lags in the GRBs seems to be the time variation of physical properties of the active region such as \( \Delta r, \gamma_{\infty} \), and electric and magnetic fields.

We note that the zeroth-order term in (60) is the same as the expression (49) for the synchrotron emission by one particle. Therefore, for this part of the integration in (60) we can use the well-known expression for the spectrum (Schwinger 1949; Westfield 1959; Jackson 2001; Rybicki & Lightman 2004). For determining the Doppler and high-latitude correction terms, we can again use the integration method used for the zeroth-order term and express (60) as a sum of Bessel functions (Westfield 1959). Finally, we find the power spectrum with the first-order correction for the Doppler effect and high-latitude delays:

\[
\frac{dP}{\omega \, d\omega} = \frac{\sqrt{3 \epsilon_e}}{3\pi} r^2 \frac{\Delta r}{\Gamma(r)} \int_{\gamma_{\text{min}}}^{\infty} d\gamma_e \frac{n_e'(\gamma_e)}{\gamma_e^2} \left[ 2 \int_{\xi_m}^{\infty} K_{3/2}(\xi) d\xi + \frac{7}{12} \varrho(r) \left( \frac{\omega_e'}{\omega} \right) \right] \times \left[ \frac{32}{7} K_{1/3} \left( \frac{\omega_e'}{\omega} \right) - 9 \left( \frac{\omega_e'}{\omega} \right) K_{2/3} \left( \frac{\omega_e}{\omega} \right) + 3 \left( \frac{\omega_e'}{\omega} \right) \right] \right] \quad (62)
\]

\( ^7 \) For simplifying the integration over \( \theta' \) in (60), we consider \( \cos \theta' + \beta \approx 2 \). This is a valid approximation when \( \beta \to 1 \) and \( \theta' \approx 0 \) dominates the synchrotron emission.
We can further integrate (62) if we consider a specific distribution for accelerated electrons $n'_e(\gamma_e)$. In Appendix B, we calculate $dP/\omega \, d\omega$ for the power-law distribution (13) and comment on the case for a power law with an exponential cut-off distribution. Here, we use those results to discuss the lowest order properties of the spectrum and light curves.

2.5 Lag between light curves

The presence of a lag between various energy bands of the GRBs has been detected in BATSE (Kouveliotou et al. 1993) and INTEGRAL (Foley et al. 2008) light curves and confirmed by Swift-BAT (Sakamoto et al. 2008). Observations show that in most long bursts there is a significant lag – from tens to a few hundreds of milliseconds between soft and hard bands. By contrast, in short bursts the lags are very small and within the present sensitivity and time resolution of gamma-ray telescopes it is consistent with zero or at most a few milliseconds. These two classes are respectively associated with the explosion of massive stars (collapsars, hypernovae) and with the merging of two compact objects – a neutron star with a black hole, two neutron stars, or a neutron star and a white dwarf. There is, however, evidence for a difference between lags of separate peaks in the same burst. Moreover, some bursts that according to their $T_\infty$ can be classified as short, such as GRB 080426 (Ziaeepour et al. 2008b), have relatively long lags of a few tens of milliseconds. On the other hand, some apparently long bursts, such as GRB 060614 (Mangano et al. 2007) and GRB 080503 (Mao et al. 2008), have small lags similar to the short bursts. Various explanations have been put forward for these out of norm behaviours: sensitivity of detectors only to the peak of a long burst leading to its misclassification as short, existence of a separate class of GRBs with intermediate durations and lags, long tail emission in an otherwise short burst for bursts with long $T_\infty$ and small lags. Some authors even rule out the association of long GRBs – hypernovae, short GRBs – collision of compact objects, and suggest that they should be classified according to their lags: short lags old population, long lags young population (Zhang et al. 2007a). Apart from the classification of progenitors of the GRBs, lags along with luminosity have been also used as proxy for the GRBs’ redshift determination (Norris, Marani & Bonnel 2000).

In summary, lags seem to be important quantities related to the nature of the central engine of the GRBs, properties of the ejecta and the surrounding material. They can be relatively easily measured, and therefore it is important to be able to relate them to these phenomena. Some authors have tried to explain lags just as a geometrical effect related to the high-latitude emission and associated Doppler shift (Qin 2002; Qin et al. 2004; Lu et al. 2006). As we have discussed in the previous section and also regarding the $r$ and $\omega$ dependence of the spectrum in (62), it is evident that even without Doppler shift and high-latitude corrections the $r$ dependence – equivalent to time dependence in the model discussed here – and $\omega$ dependence are not factorizable, and therefore light curves in different energy bands cannot be the same even when they are normalized to smear the amplitude difference.

To define and determine lags, we need fast varying features such as peaks. In the GRB light curves, peaks are mostly observed in the prompt gamma-ray energy bands. None the less, fast slew of the Swift satellite and some of the ground-based robotic telescopes have permitted observation of the counterpart peaks or at least evidence of their presence in X-ray and optical bands. A realistic model for the lags should be able to predict the lag in all these energy bands if they have the same origin. On the other hand, a deviation of some bands from predictions can be used as evidence for a different origin of the corresponding feature.

The nature of peaks and their profile is not well understood. In the framework of the synchrotron emission from internal shock model as the origin of the prompt gamma-ray emission, the rising side of a peak indicates the beginning of the collision between shells and formation of the electric and magnetic fields that leads to the acceleration of charged particles and synchrotron emission in the induced magnetic field. The decreasing edge corresponds to the separation and/or the total coalescence of the shells. However, it is expected that even before separation/coalescence, microphysics in the active region arrive at a roughly steady state during which only slight changes due to, e.g., density fluctuation in the shells will occur. In particular, when the initial evolution of the microphysics is much faster than the time of the passage of the shells through each other, we expect that for a limited duration the active region has quasi-steady characteristics. Assuming such a case – in accordance with the discussion about the evolution of $\Delta r$ in Section 2.2 – a peak corresponds to

$$\frac{d}{dr} \left( \frac{dP}{\omega \, d\omega} \right) = c \beta(r) \frac{d}{dr} \left( \frac{dP}{\omega \, d\omega} \right) = 0.$$  \hspace{1cm} (63)

The lag for a given peak corresponds to the difference between peak time/radius in two frequencies or two energy bands. Usually, observations are performed in known energy bands. Therefore, the purpose of the lag measurement is to determine the difference between $r_{peak}$, the solution of equation (63), at two different energies. In the rest of this section, we use the results of Section 2.4 to determine the lags.

Using (62), the peak equation (63) can be written as a partial differential equation:

$$\mathcal{K}(r) \frac{dP}{\omega \, d\omega} + D(r) \frac{\partial}{\partial \gamma_m} \left( \frac{dP}{\omega \, d\omega} \right) + \mathcal{G}(r) \frac{\partial}{\partial \mathcal{G}} \left( \frac{dP}{\omega \, d\omega} \right) = 0$$  \hspace{1cm} (64)

$$\mathcal{K}(r, \omega') \equiv \frac{1}{r^2 \Delta r} \frac{d}{dr} (r^2 \Delta r) + \Gamma(r) \frac{d}{dr} \left( \frac{1}{\Gamma(r)} \right)$$  \hspace{1cm} (65)

$$D(r) \equiv \frac{d\gamma_m}{dr}.$$  \hspace{1cm} (66)
Note that the functions $D$ and $G'$ depend only on $r$ and not on $\omega'$. If $\Delta r$ is energy independent, so is $K(r)$. The spectrum in (62) can be written as
\[
\frac{dP}{d\omega d\omega'} = F(r, \omega') \int_{\gamma_m}^{\infty} d\gamma n'(\gamma) \gamma^{-2}H(r, \omega').
\]
(67)

\[
F(r, \omega') = \frac{\frac{\Delta r}{\gamma'} \gamma^2 \Delta \tau}{\Gamma^4(r) (1 - \beta(r))^3}
\]
(68)

\[
H(r, \omega', \omega) = \left\{ 2 \int_{\omega}^{\infty} K_{3/3}(\xi) d\xi + \frac{7}{12} G(r) \left( \frac{\omega'}{\omega} \right)^5 - \frac{22}{7} K_{1/3} \left( \frac{\omega'}{\omega} \right) - 9 \left( \frac{\omega'}{\omega} \right) K_{2/3} \left( \frac{\omega'}{\omega} \right) + 3 \left( \frac{\omega'}{\omega} \right) \int_{\omega}^{\infty} K_{1/3}(\xi) d\xi \right\}.
\]
(69)

With this definition, the partial differentials of $dP/\omega d\omega'$ in the peak condition equation (64) can be calculated:
\[
\frac{\partial}{\partial \gamma_m} \left( \frac{dP}{\omega d\omega'} \right) = -\gamma_m^{-2}n'(\gamma_m)F(r, \omega')H(r, \omega')
\]
(70)

\[
\frac{\partial}{\partial G} \left( \frac{dP}{\omega d\omega'} \right) = F(r, \omega') \int_{\gamma_m}^{\infty} d\gamma n'(\gamma) \gamma^{-2} \frac{\partial^2 H}{\partial \omega \partial G}
\]
(71)

\[K(r, \omega') \int_{\gamma_m}^{\infty} n'(\gamma) \gamma^{-2}H - n'(\gamma_m) \gamma_m^{-2}D(r)H + G' \int_{\gamma_m}^{\infty} n'(\gamma) \gamma^{-2} \frac{\partial H}{\partial G} = 0.
\]
(72)

This equation is obviously very complex, and solving it is not a trivial task. However, we are only interested in the change in the roots of respect to energy $\omega'$. Moreover, if we restrict ourselves to $|\Delta \omega'/\omega'| \equiv |\omega'/\omega' - 1| < 1$ and assume that in this case the corresponding roots $r_0$ and $r_1$ are close, i.e. $|r_1/r_0 - 1| < 1$, then we can expand functions in (72) around $\omega'_0$ and determine the lag in the observer frame, i.e. $c \beta \Delta t = r_1 - r_0$. Assuming that $\beta \approx 1$, we find
\[
c \Delta t \approx -\frac{\Delta \omega'}{\partial r} \frac{\partial P}{\partial r} \left( r_0, \omega'_0 \right)
\]
(73)

\[
P(r_0, \omega'_0) = K(r_0) \int_{\gamma_m}^{\infty} d\gamma n'(\gamma) \gamma^{-2} \frac{\partial H}{\partial \omega} - D(r_0) \frac{\partial}{\partial \omega} \left( n'(\gamma_m) \gamma^{-2}H \right) + G'(r_0) \int_{\gamma_m}^{\infty} d\gamma n'(\gamma) \gamma^{-2} \frac{\partial H}{\partial \omega} \frac{\partial^2 G}{\partial \omega} - \frac{1}{\omega'F} \frac{d}{dr} \left( \frac{dP}{\omega d\omega} \right).
\]
(74)

\[
Q \left( r_0, \omega'_0 \right) = \frac{dK}{dr}(r_0) \int_{\gamma_m}^{\infty} n'(\gamma) \gamma^{-2}H + K(r_0) \left[ G'(r_0) \int_{\gamma_m}^{\infty} n'(\gamma) \gamma^{-2} \frac{\partial H}{\partial \omega} - Dn'(\gamma_m) \gamma^{-2}H \right]
\]
\[
- \frac{dG}{dr} \left( Dn'(\gamma_m) \gamma^{-2}H \right) - G'(r_0) Dn'(\gamma_m) \gamma^{-2} \frac{\partial H}{\partial G} + \frac{dG}{dr}(r_0) \int_{\gamma_m}^{\infty} n'(\gamma) \gamma^{-2} \frac{\partial H}{\partial G} + \frac{K}{F} \frac{d}{dr} \left( \frac{dP}{\omega d\omega} \right).
\]
(75)

All the terms in (74) and (75) can be expressed as a sum of Bessel functions and their integrals using the definition of $H$. For the special case of a power-law distribution of electrons, most of the integrals can be calculated analytically. We present the results in Appendix B. Note also that $\Delta \omega'/\omega' = \Delta \omega/\omega$ and $\partial \omega'/\partial \omega' = \partial \omega/\partial \omega$. Other $\omega'$-dependent terms also are $\Gamma$-independent, and therefore the calculation of the lag does not need a pre-knowledge of the bulk Lorentz factor $\Gamma$.

The left-hand side of (63) or equivalently (64) is the slope of the spectrum (light curve) and therefore its energy dependence explains, for instance, why peaks or more precisely their auto-correlations are wider at lower energies (Fenimore et al. 1995). However, the complexity of expressions (73) to (75) makes the analytical estimation of the time and energy dependence of light curves and lags difficult. In Paper II, we present some simulations in which the lags are consistent with the long bursts or are very small, similar to short bursts (Sakamoto et al. 2006).

### 2.6 Break at low energies

Before finishing this section, we want to make a few comments about the jet break which is one of the most important aspects of the afterglow light curves predicted since the beginning of the modelling of GRBs.

Since the early days of the discovery of GRBs and measurement of apparently huge amounts of energy released in these phenomena – $\mathcal{O}(0.1–1) \times 10^{50}$ erg for long bursts or even larger in a few exceptionally bright bursts such as GRB 990123, GRB 080319B (Racusin et al. 2008a,b) and GRB 080607 (Mangano et al. 2008) – it has been suggested that these measurements are biased by the collimation of the fireball and the actual total emitted energy should be much smaller. To produce the gamma-ray prompt emission in a shock, the ejecta should be highly relativistic with a bulk Lorentz factor of the order of a few hundred. This leads to a strong apparent collimation of the radiation from a spherically symmetric ejecta to angles $\Theta < \Theta_{\text{boost}} \equiv 1/\Gamma$ for a distant observer (Rees 1967). It is also possible that the ejecta is not intrinsically spherical but jet like (Rees, Mészáros & Wijers 1998). In this case, after deceleration of the fireball during its propagation through the surrounding material or the ISM, at some radius/time $\Theta_{\text{boost}} > \Theta_{\text{open}} = \Theta_{\text{max}} - \Theta_{\text{min}}$ and radiation is no longer collimated. This
leads to a drop of observed flux (Sari, Piran & Halpern 1999). As this effect is purely kinematic/geometric, it should not depend on energy. Observations with the Swift and robotic ground telescopes, however, contradict this expectation. The breaks seen in the afterglow light curves are usually chromatic. In many bursts, no optical break has been observed up to millions of seconds after the prompt emission (Roming et al. 2009). In some bursts, mostly bright ones but not always, no break has been observed in X-rays either (Sato et al. 2007b). In a significant fraction of bursts, multiple breaks in the X-ray light curve have been observed. A priori, only one of these breaks (if any) can be due to the jet break. Therefore, the mechanism of the break is more complex than just a kinematical effect. Here, we want to argue that even in the simple case of the opening due to deceleration, we should not expect to have an achromatic break if the afterglow is synchrotron radiation.

To obtain the analytical expression (62) for the radiation spectrum, we made the simplifying approximation that \(|\theta_{\text{min}}| = |\theta_{\text{max}}| \to \infty\). This permitted us to analytically integrate the angular integral in (60). Without this approximation (60) depends on the opening angle. The justification for this approximation was that the synchrotron emission is highly directional, and for \(\theta \gamma \gg 1\) the intensity reduces exponentially because the value of \(\xi\) in the integrand of (60) grows. \(\xi\) increases with energy and therefore high-energy photons are preferentially emitted at small angles with respect to the electrons' boost direction. When the collimation is reduced, the observer receives less high-energy photons from high-latitude electrons (Panaiteascu & Mészáros 1998), and therefore the effect of the jet break influences high-energy light curves earlier than low-energy ones. Conceptually, this effect is very similar to the lag between light curves. It is, however, more difficult to investigate it mathematically because the angular integral (60) cannot be determined analytically as explained above.

On the other hand, it is not certain that the reduction of the Lorentz factor can explain the absence of a break in optical frequencies when it is observed in X-rays. A number of suggestions have been put forward to explain observations (Panaiteascu 2005; Misra et al. 2007). In the framework of the model presented here, it can be at least partly explained by the energy dependence of the geometry of the active region. In fact, the material that has lost its energy and has been decelerated is pushed behind the front edge of the shock. Due to the scattering, it also has a relatively larger transverse momentum (see simulations in Vigilieus et al. 2007 for non-relativistic shocks). Thus, less boosted particles in the ejecta have intrinsically a larger opening angle. This leads to a later jet break for the softer photons originating in this extended region. Therefore, one does not need additional processes (Rees & Mészáros 1998; Sari & Mészáros 2000), energy (Dai & Lu 1998; Rees & Mészáros 2000; Zhang & Mészáros 2002) and/or components (De Pasquale et al. 2009) to explain the breaks. The complex geometry of matter and fields in the material and shock can explain apparently contradictory observations. As for multiple breaks, they can be due to the fact that multiple shells are ejected by the central engine. This is in the same spirit as the structured jet models (Xu et al. 2005; Takami et al. 2007). If they do not completely coalesce both the tail emission and the afterglow due to the external shocks will be a combination of emissions from separate remnant shells. In this case, the break of radiation from each shell is independent of others, and a distant observer who detects the total emission will observe multiple breaks in the emission.

3 EXTRACTION OF PARAMETERS FROM DATA

The model described in the previous sections has a large number of parameters. It would not help to better understand the physics of GRB production and their engine if we cannot estimate the parameters of this or any other model from data. On the other hand, the only conveyor of information for us is the emission in different energy bands. Therefore, we must find the best ways to extract the information as efficiently as possible. In this section, we suggest a procedure for extracting the parameters of the model. We also discuss their degeneracies.

Since the massive detection of GRBs by BATSE, many efforts have been concentrated to understand their spectra. However, very little information could be extracted from the spectra. The complexity of time and energy dependence of the spectrum (62) explain why the time-averaged spectrum does not carry extractable information. As GRBs evolve very quickly both in time and in energy, integration over these quantities smears the useful information. Unfortunately, the effective detection surface of present gamma-ray detectors is too small, and we do not have enough photons to make a broad spectrum in small time intervals. In this situation, hardness ratios and their time variation are more useful quantities. Therefore, we first discuss what we can extract from hardness ratios.

Consider a power-law distribution for electrons; equation (B1) shows that at a given time the spectrum can be expanded as a power of \(\alpha' / \alpha'_w \gamma^2\). The coefficient in front of the integral term does not depend on energy. Therefore, if we neglect \(G(r)\), i.e. assuming the bulk Lorentz factor is very large, the hardness ratios depend only on the integral determined in (B4). We define the hardness ratio between two bands with logarithmic mean energies \(\omega_1\) and \(\omega_2\) as

\[
\text{HR}_{12}(r) = \frac{\int_{\omega_1}^{\omega_2} \text{d}N_{12} / \text{d}N_{\omega_2} \omega_2}{\int_{\omega_1}^{\omega_2} \text{d}N_{12} / \text{d}N_{\omega_1} \omega_1} \approx \frac{dP_{12}}{dP_{\omega_2}} \bigg|_{\omega_1}^{\omega_2} \frac{\Delta \omega_1}{\Delta \omega_2}
\]

(76)

\[
\Delta \omega_1 \equiv \omega_1^{\text{max}} - \omega_1^{\text{min}}.
\]

(77)

The approximations for integrals are valid when \(\Delta \omega_1 / \omega_1 \ll 1\). Expression (B4) shows that at a constant time/radius hardness ratio depends only on \(p\), electron distribution index and on the Lorentz invariant factor \(\alpha' / \alpha'_w\) where

\[
\omega'_w \equiv \omega'_w \gamma^2
\]

(78)
is the minimum synchrotron characteristic frequency. Therefore, by fitting observational data with (76) one can obtain the index of the electron distribution and the time variation of \(\omega_m\) which is a very important quantity. Note that as HR is time/radius dependent, we need multiple energy bands and their hardness ratios to make a statistically meaningful fit. If we release the assumption of a power-law distribution for electrons, we can use (62) to get insight into the distribution of the electrons' Lorentz factor and \(\omega_m\), but they will be degenerate.

Once the energy and temporal behaviour of the integral term in (62) are found, the extraction of the leading time-dependent coefficient \(F(r)\) [defined in (68)] from light curves is easy. Then, if we use one of the models considered in this work or another model for the evolution of \(\Delta r\), we can determine the time/radius variation of \(\Gamma'(r)\), the total Lorentz factor of the active region. Using (45) and (36), one can a priori estimate \(M(r')\). However, the latter depends on the relative Lorentz factor \(\gamma'(r')\). If we can roughly estimate the end of the collision from the form of the light curve – for instance if we assume that the end of the collision is when the continuous component begins an exponential decay – then

\[
\Gamma_f = \Gamma(r) \bigg|_{r=r_f}
\]  

where \(r_f\) is the radius at which the coalescence of the shells finishes.

\(\omega'/\omega_m\), \(M(r')\) and \(\gamma'(r')\) depend on the fundamental quantities such as magnetic field and thereby on the fraction of kinetic energy of the shell transferred to the electric and magnetic fields, the density of shells and the initial distance of the collision from the central engine. But, with only three quantities mentioned above, we cannot determine all these quantities and their time/radius variation. An important observable that can help extract more information from data is the lag. If we neglect the term proportional to \(G_{\alpha,\beta}(r)\) and its derivatives, we can determine functions \(P(r_0, \omega')\) and \(Q(r_0, \omega')\) in (74) and (75) from the knowledge about \(F(r)\) and the integral of \(H(r)\) [defined in (69)] explained above. Thus, in this way we can determine the lag between two energy bands. Comparison of the observed lags with the expectations from the model helps to estimate some of other quantities such as \(r_0\). However, as the analytical expression for the lags is very complex, only simulations and fitting can allow us to solve this inverse problem. Using a simpler version of the model presented here, this procedure has been applied to analyse the Swift data for GRB 060607a (Ziaeepour et al. 2006b) and to explain some of its peculiar properties (Ziaeepour et al. 2008a).

At the beginning of this section, we mentioned that the most complete information about GRBs is in the spectrum at short time intervals when the flux does not change significantly. In fact, in practice hardness ratios are calculated by adding together photons in a given energy band and time-rebinned event data to reduce the noise. When multiple simultaneous hardness ratios are available, they can be considered as a low-resolution normalized spectrum in a short time interval. Because these quantities give the most direct insight into the physics of the collision, in the rest of this section we investigate in more detail the spectral behaviour of the flux at constant time.

First, we consider a power-law distribution for electrons. When \(\omega'/\omega_m > 1\) hypergeometric functions \(I F_2\) in (B4) and (B5) can be expanded as a polynomial with positive or zero power of \(\omega'/\omega_m\). This shows that at the lowest order in \(\omega'/\omega_m\) the spectrum \(dP/\omega d\omega \propto (\omega'/\omega_m)^{-2/3}\) and therefore \(HR_{12} \propto (\omega/\omega_m)^{-2/3}\). The power of the coefficient in the second \(I F_2\) term is larger, and therefore for determining the spectrum at zero order it can be neglected. But for the first-order expansion of (B1) this term is significant and when it is added the spectrum becomes

\[
\frac{dP}{\omega d\omega} \propto \left(\frac{\omega'}{\omega_m}\right)^{-2/3} + \frac{\pi}{2} \frac{\omega}{\omega_m} \left(\frac{\omega'}{\omega_m}\right)^{1/3} \left[\Gamma\left(-\frac{p}{3}\right), \left(\frac{\omega'}{\omega_m}\right)^{1/3}ight] + \left(\frac{\omega}{\omega_m}\right)^{-2/3} \left\{2^{1/3} \Gamma\left(-\frac{2}{3}\right) \left(\frac{\omega}{\omega_m}\right)^{-1/3} I F_2\left(\frac{p}{4} + \frac{1}{6}, \frac{1}{3} \frac{p}{4} + \frac{1}{6}, \frac{1}{4}\right) + 2^{1/3} \Gamma\left(-\frac{2}{3}\right) \left(\frac{\omega}{\omega_m}\right)^{-1/3} I F_2\left(\frac{p}{4} + \frac{5}{6}, \frac{5}{3} \frac{p}{4} + \frac{11}{6}, \frac{1}{4}\right) \right\}
\]

for \(\omega'/\omega_m < 1\). (80)

The amount of deviation from the dominant power law only marginally depends on the index of the electron distribution function \(p\). When \(\omega'/\omega_m = 1\) the higher power of \(\omega'/\omega_m\) becomes important and should be considered. However, as they are all positive, they make the spectrum harder. This result shows that although a small hardening of the spectrum can be obtained in this regime, it is not possible to have a softer spectrum with an index smaller than \(-2/3\) and closer to what has been observed in many bursts (Sakamoto et al. 2006, 2008). Note also that the addition of terms proportional to \(G_{\alpha,\beta}(r)\) cannot make the spectrum softer because they have the same form of energy dependence as the dominant terms.

Next, we consider the regime where \(\omega'/\omega_m > 1\). In this regime, there is no analytical expression for \(I F_2\) functions. Therefore, we use the asymptotic behaviour of Bessel functions, \(K_\nu(\xi) \approx \sqrt{\pi/2} \xi^{-\nu} e^{-\xi}\), to estimate the integrals in (B1). Integration of the left-hand side of (B4) leads to

\[
\int_{\nu_m} d\nu \left(\frac{\nu}{\nu_m}\right)^{-(p+1)} \frac{\nu^{-2} K_{5/3}(\xi) d\xi}{\sqrt{\nu}} \approx \frac{1}{\nu_m} \left[\Gamma\left(-\frac{p}{3}\right) \left(\frac{\omega'}{\omega_m}\right)^{1/3} - \Gamma\left(-\frac{p}{3}\right), 1\right]
\]

\[+ \left(\frac{\omega}{\omega_m}\right)^{-2/3} \left\{2^{1/3} \Gamma\left(-\frac{2}{3}\right) \left(\frac{\omega}{\omega_m}\right)^{-1/3} I F_2\left(\frac{p}{4} + \frac{1}{6}, \frac{1}{3} \frac{p}{4} + \frac{1}{6}, \frac{1}{4}\right) + 2^{1/3} \Gamma\left(-\frac{2}{3}\right) \left(\frac{\omega}{\omega_m}\right)^{-1/3} I F_2\left(\frac{p}{4} + \frac{5}{6}, \frac{5}{3} \frac{p}{4} + \frac{11}{6}, \frac{1}{4}\right) \right\}\]

for \(\omega'/\omega_m > 1\). (81)

The constant coefficient in the last term is \(\sim 1.354[2^{5/3}/(p + 2/3) + 2^{1/3}/(p + 10/3)]\). Using an expansion in a series of the incomplete \(\Gamma\) functions:

\[
\Gamma(a, x) = e^{-x} x^a \left[1 + \frac{1}{x} + \frac{1}{x^2} \ldots \right],
\]

we see that in this regime at the lowest order the spectrum is a power law with an index \(\sim -(1 + p/2)\) which is much steeper (softer) than when \(\omega'/\omega_m < 1\). For energies between these two extreme regimes, one expects an index \(-(1 + p/2) \lesssim \alpha \lesssim -2/3\). They can be fitted by a
power law, but at high $\omega'/\omega_m$, a power law with an exponential cut-off gives a slightly better fit due to the presence of the exponential term in the expansion of the incomplete $\Gamma$ functions. None the less, if the distribution of electrons is truncated at high Lorentz factors or is a broken power law in which the index becomes smaller at higher energies, this constraint will not be applicable. As $\omega'/\omega_m > 1$, the second term in the sum and the last term in (81) are smaller than other terms considered above.

The two regimes explained above cover the maximum and soft wing of the index range observed by Swift (Sakamoto et al. 2008) and BATSE, but it cannot explain very hard short bursts with indices larger than the zeroth-order index $-2/3$. Moreover, when the observed spectrum is extended to high energies, $\geq 1$ MeV, a power law with an exponential cut-off at high energies is a better fit to GRB spectra. This type of spectra can be obtained if the distribution of the Lorentz factor of the accelerated electrons has an exponential cut-off at high energies. As we mentioned in Section 2.2, the spectrum of the accelerated electrons in the simulations of ultra-relativistic shocks is more sophisticated than a simple power law and is closer to a power law with an exponential cut-off [Spitkovsky 2008; see equation (16)]. Assuming for simplicity an exact power law with exponential cut-off for the Lorentz factor distribution of accelerated electrons like equation (20), the dominant term of the spectrum (62) becomes proportional to

$$\int_{\gamma_{\text{In}}}^{\infty} d\gamma \left( \frac{\gamma}{\gamma_{\text{In}}} \right)^{-(p+1)} e^{-\frac{\gamma}{\gamma_{\text{In}}}} \gamma^{-2} \int_{\gamma_{\text{In}}}^{\infty} K_{5/3}(\xi) d\xi$$

with the conditions mentioned in (20). When $\gamma_{\text{In}} \ll \gamma_{\text{cut}}$, the exponential term is close to 1 and the behaviour of the spectrum is indistinguishable from a simple power law. But at high energies photons are emitted preferentially by faster electrons for which the exponential term is important and leads to an exponential cut-off in the spectrum of the synchrotron emission. To prove such a behaviour, we concentrate on the part of the spectrum for which $\omega'/\omega_m \gtrsim 1$. In this case, the integration of (83) leads to

$$\int_{\gamma_{\text{In}}}^{\infty} d\gamma \left( \frac{\gamma}{\gamma_{\text{In}}} \right)^{-(p+1)} e^{-\frac{\gamma}{\gamma_{\text{In}}}} \gamma^{-2} \int_{\gamma_{\text{In}}}^{\infty} K_{5/3}(\xi) d\xi$$

$$\approx \sqrt{\frac{\pi}{\gamma_{\text{In}}}} \left[ \int_1^{\infty} \left( \frac{\gamma}{\gamma_{\text{In}}} \right)^{\frac{1}{2}} d\gamma \gamma^{-2} \left( \frac{\omega'}{\omega_m} \right)^{\frac{1}{2}} \Gamma \left( \frac{1}{2}, \frac{\omega'}{\omega_m} \gamma^2 \right) + 2 \int_{\gamma_{\text{In}}}^{\infty} \left( \frac{\gamma}{\gamma_{\text{In}}} \right)^{\frac{1}{2}} d\gamma \gamma^{-2} \left( \frac{\omega'}{\omega_m} \gamma^2 \right) K_{2/3} \left( \frac{\omega'}{\omega_m} \gamma^2 \right) \right].$$

For $\omega'/\omega_m \gg 1$, the second integral on the right-hand side of (84) is small and negligible. The first integral does not have an analytical solution, but using the asymptotic behaviour of the incomplete $\Gamma$ function (82) one can conclude that its behaviour at high energies must be close to an exponential with negative exponent proportional to $\omega'$.

In the beginning of this section, we mentioned that most probably $\Delta r$ is also somehow energy dependent, especially at very low and very high energies. In the latter case, it must be very small, probably exponentially decreasing with energy. This can also contribute to the existence of an exponential cut-off at high energies. As for the low-energy tail of the spectrum, the fact that even the residual energy of cooled electrons can be enough to emit soft photons means that the spectrum in this regime should be much flatter. This is consistent with the really flat spectrum and shallow temporal decline in optical and longer wavelength emissions as observed in most GRBs (Oates et al. 2008; Roming et al. 2009).

In conclusion of this section, we have shown that when a relativistic shock is modelled in detail and realistic distributions for electrons are considered, the synchrotron emission alone can explain different aspects of the time-averaged spectrum of GRBs as well as their spectrum in a short interval in which the evolution can be ignored. Evidently, other processes such as inverse Compton (Kobayashi et al. 2007; Piran et al. 2009) and pair production (Asano, Inoue & Meszaros 2008) contribute to the total emission but most probably are not the dominant component at low/intermediate-energy bands. None the less, their contribution should be more important at GeV and higher energies (Asano et al. 2008).

4 SUMMARY

We have presented a formulation of the relativistic shocks and synchrotron emission that includes more details than the dominant term considered in the previous calculations. Although we consider spherical shells, most of our results are valid also for non-spherical collimated jets as long as the collimation angle due to the relativistic boost is smaller than the intrinsic collimation angle.

We showed that the lags between light curves at different energies exist in the dominant order and are not only due to the high-latitude emissions which are negligible for ultra-relativistic ejecta. The main reason for such behaviour is the evolution of electric and magnetic fields as well as the evolution of the emitting region which can be in addition energy dependent. This fact is more evident in the simulations presented in Paper II. Despite the absence of high-latitude terms in our simulations, the presence of lags between the light curves of different energy bands is evident. For the same reasons, the change in the slope of the light curves – called breaks – is also energy dependent. This explains chromatic breaks of the GRBs detected by Swift.

The two phenomenological models we considered for the evolution of the active region are physically motivated, but do not have rigorous support from microphysics of the shock. None the less, they can be easily replaced if future simulations of Fermi processes lead to a better estimation of the size of the region in which electric and magnetic fields are formed and particles are accelerated and dissipated. The presence of an external magnetic field in the environment of the candidates for the central engine of GRBs is very plausible. The formulation presented...
here does not include such a possibility, but an external magnetic field can be added to (23). The modification of the evolution equation of $\beta'$ and the flux is straightforward.

Many other details, such as the effect of metallicity of the ejecta and surrounding material on both the low-energy emission and absorption, are not considered in this work. We have also neglected synchrotron self-absorption. It only affects the low-energy bands, none the less in hard bursts even optical emission can be affected by self-absorption. We leave the study of this issue, the effect of ionization on the emissions and the thermalization of shocked material to future works.

ACKNOWLEDGMENTS

I would like to thank Keith Mason for encouraging me to work on GRB science. The ideas presented in this work could not have been developed without long discussions with the past and present members of the Swift science team at MSSL: A. Blustin, A. Breeveld, M. De Pasquale, P. Kuin, S. Oates, S. Rosen, M. Page, P. Schady, M. Still and S. Zane, as well as the other members of the Swift team in particular: S. Barthelmy, Ph. Evans, E.E. Fenimore, N. Gehrels, P. Mészáros, J. Osborne and K. Page. I thank all of them.

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APPENDIX A: ANALYTICAL APPROXIMATION OF $\beta'_0$ (1)

The integral in (37) cannot be determined analytically. An analytical approximation is, however, useful for investigating the effect and importance of the radiation in the dynamics of the shell collision. Here, we consider a few approximations for various ranges of $\mathcal{D}$ and $\eta$ that determine, respectively, the strength of the shock and the radiation.

If $\eta = \tau = 0$, the integral in (37) can be calculated analytically:

$$\int_{\beta'_0}^{\beta'_0(r')} \frac{dy}{y^3(1-y)^2} = \left[ A_1 \ln y + \frac{A_2}{y} + \frac{A_3}{2y^2} + B_1 \ln(y-1) + \frac{B_2}{y-1} + \frac{B_3}{2(y-1)^2} + C_1 \ln(y+1) + \frac{C_2}{y+1} + \frac{C_3}{2(y+1)^2} \right]_{\beta'_0}^{\beta'_0(r')} \quad (A1)$$

$$A_1 = 3, \quad A_2 = 0, \quad A_3 = 1, \quad B_1 = \frac{11}{8}, \quad B_2 = \frac{1}{2},$$
$$B_3 = \frac{1}{4}, \quad C_1 = \frac{a}{16}, \quad C_2 = -\frac{a}{16}, \quad C_3 = -\frac{a}{8}. \quad (A2)$$

When $\eta/(3 - \kappa)$ is large, as the value of $y$ in the integrand of (37) is always less than 1, we can formally expand the term $(1 - y^2)^{3-1/2}$ and integrate term by term:

$$M_{\phi}(r') = \frac{\Delta r'_0}{(3 - \kappa)} \left( \frac{y}{y'_0} \right)^{r'_0} ((3 - \kappa) \mathcal{D})^{1-\frac{r'_0}{r}} \left\{ \frac{1}{\tau - 2 + \frac{a}{2 \kappa}} \right.$$
$$\times \left[ \beta'_0 \right]_{\beta'_0}^{\beta'_0(r')} 2 F_1 \left[ \frac{\eta}{3 - \kappa}, \tau - 2 + \frac{a}{2 \kappa}; \tau - 1 + \frac{a}{2 \kappa} \right] \right.$$  
$$\left. \frac{(1 - \frac{a}{3 - \kappa})}{\tau + \frac{a}{2 \kappa}} \right.$$
$$\times \left[ \beta'_0 \right]_{\beta'_0}^{\beta'_0(r')} 2 F_1 \left[ \frac{\eta}{3 - \kappa}, \tau + \frac{a}{2 \kappa}; \tau + 1 + \frac{a}{2 \kappa} \right] \right.$$  
$$\left. \frac{(1 - \frac{a}{3 - \kappa})}{\tau + \frac{a}{2 \kappa}} \right.$$

Here, $2 F_1$ is the hypergeometric function and can be expanded as a polynomial of its last argument. When the latter is less than 1 and the power of the terms in the polynomial is positive, they converge rapidly to zero. When the last argument in $2 F_1$ is larger than 1 an analytical extension of this function with negative power in the polynomial expansion exists (Gradshteyn & Ryzhik 1980). Therefore, the dominant term is always $\beta'_0$ in front of each $2 F_1$ term. For small $\eta$, this approximation is not valid. In this case we expand $(1 + (\frac{1}{16-\frac{1}{3\kappa}} - \frac{1}{y'_0}) y)^{-\frac{a}{2\kappa}}$
and obtain
\[ M_{(0)}(r') = \frac{\Delta r_{(0)}}{(3 - \kappa)} \left( \frac{\gamma_0}{\beta_0} \right)^{\tau} \left[ 1 - \frac{\eta}{2(3 - \kappa)} \right] \left\{ \frac{1}{\tau - 2 + \frac{\eta}{3 - \kappa}} \right\} \]
\[ \times \left[ \beta_0^{r-2 + \frac{\eta}{3 - \kappa}} \right] \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 + \frac{\eta}{2(3 - \kappa)}; \frac{\tau}{2} + \frac{\eta}{2(3 - \kappa)} \right) \]
\[ - \beta_0^{r-2} \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 + \frac{\eta}{2(3 - \kappa)}; \frac{\tau}{2} + \frac{1}{2} \right) \]
\[ + \frac{\eta}{2(3 - \kappa)} - \beta_0^{r-1} \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 \right) \]
\[ + \frac{\eta}{2(3 - \kappa)} \times \frac{\eta}{2(3 - \kappa)} \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 \right) \]
\[ + \ldots \}. \] (A4)

In the same way, we find the following expression for \( M_{(0)}(r') \) for the quasi-steady model and large \( \eta/(3 - \kappa) \):
\[ M_{(0)}(r') = \frac{\Delta r_{(0)}}{(3 - \kappa)} ((3 - \kappa)D)^{\tau - \frac{\eta}{\gamma}} \left\{ \frac{1}{\tau - 2 + \frac{\eta}{3 - \kappa}} \right\} \]
\[ \times \left[ \beta_0^{r-2 + \frac{\eta}{3 - \kappa}} \right] \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 + \frac{\eta}{2(3 - \kappa)}; \frac{\tau}{2} + \frac{\eta}{2(3 - \kappa)} \right) \]
\[ - \beta_0^{r-2} \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 + \frac{\eta}{2(3 - \kappa)}; \frac{\tau}{2} + \frac{1}{2} \right) \]
\[ + \frac{\eta}{2(3 - \kappa)} - \beta_0^{r-1} \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 \right) \]
\[ + \frac{\eta}{2(3 - \kappa)} \times \frac{\eta}{2(3 - \kappa)} \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 \right) \]
\[ + \ldots \} - (3 - \kappa)D^{\tau - \frac{\eta}{\gamma}}. \] (A5)

For the case of a small \( \eta/(3 - \kappa) \), one can use (A4) and make an expression analogous to (A5), thus we do not repeat the details here. We note that when \( \tau = 0 \) and \( \delta \to \infty \), i.e., for a constant \( \Delta r' \), (A3) and (A5) are equal as expected. When \( \delta \to 0 \), the formation of the active region is very slow. In this case, the energy loss by radiation becomes negligible and \( \beta \) decreases only due to the increasing accumulated mass. Note also that \( M_{(0)}(r_0) = 0 \) as expected. If the active region varies according to (29), the expression for \( M_{(0)}(r') \) includes only the term \( \eta \to \eta + \delta \) in (A5) with a positive sign in front.

Although equations (A3), (A5) and (A4) seem quite sophisticated, due to the polynomial representation of \( 2 F_1 \), only a few dominant terms are of real interest to us. In most cases, we are only interested in the dominant power-law component. However, having expressions beyond the dominant power permits us to go much further and calculate quantities such as lags that in a simple power-law approximation cannot be determined.

When the kinetic energy of the fast shell does not change significantly, \( \beta_0^{(r')_0} \sim \beta_0^{(r')_0} \) and \( M_{(0)}(r) \to 0 \). This case happens when the radiation has a negligible effect on the kinematics of the shock. Assuming a strong shock, i.e., \( \beta_0 < 3D \), we can use the definition of \( 2 F_1 \) to investigate the behaviour of \( M_{(0)}(r) \) at the lowest order. For the dynamically driven active region with large \( \eta \) and a relatively soft shock, when we expand \( 2 F_1 \) terms in (A3) up to the first order, \( M_{(0)}(r) \) becomes
\[ M_{(0)}(r') \approx \frac{\Delta r_{(0)}}{(3 - \kappa)} \left( \frac{\gamma_0}{\beta_0} \right)^{\tau} \left[ 1 - \frac{\eta}{2(3 - \kappa)} \right] \left\{ \frac{1}{\tau - 2 + \frac{\eta}{3 - \kappa}} \right\} \]
\[ \times \left[ \beta_0^{r-2 + \frac{\eta}{3 - \kappa}} \right] \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 + \frac{\eta}{2(3 - \kappa)}; \frac{\tau}{2} + \frac{\eta}{2(3 - \kappa)} \right) \]
\[ - \beta_0^{r-2} \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 + \frac{\eta}{2(3 - \kappa)}; \frac{\tau}{2} + \frac{1}{2} \right) \]
\[ + \frac{\eta}{2(3 - \kappa)} - \beta_0^{r-1} \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 \right) \]
\[ + \frac{\eta}{2(3 - \kappa)} \times \frac{\eta}{2(3 - \kappa)} \times 2 F_1 \left( 3 - \frac{\tau}{2}, \frac{\tau}{2} - 1 \right) \]
\[ + \ldots \}. \] (A6)
Similarly, for small $\eta$ we use (A4) and obtain
\begin{equation}
\mathcal{M}_{00}(r) \approx \frac{\Delta r_0}{(3-\kappa)} (3-\kappa) \mathcal{D}^{(3-\kappa)/\kappa} \beta_0^{r-2+\frac{4}{\kappa}} \left\{ 1 + \frac{(3-\frac{1}{\kappa}) \beta_0^2}{\tau + \frac{\eta}{\kappa}} + \eta \frac{1}{\kappa} \left( 1 - \frac{\beta_0}{(3-\kappa) \mathcal{D}} \right) \left( 1 + \frac{(3-\frac{1}{\kappa}) \beta_0^2}{\tau + 1 + \frac{\eta}{\kappa}} \right) \right\} \beta_0^{r-2+\frac{4}{\kappa}} \left\{ 1 + \frac{(3-\frac{1}{\kappa}) \beta_0^2}{\tau + \frac{\eta}{\kappa}} + \eta \frac{1}{\kappa} \left( 1 - \frac{\beta_0}{(3-\kappa) \mathcal{D}} \right) \left( 1 + \frac{(3-\frac{1}{\kappa}) \beta_0^2}{\tau + 1 + \frac{\eta}{\kappa}} \right) \right\}. \tag{A7}
\end{equation}

For a quasi-static active region and the same shock conditions as in (A5), the first order $\mathcal{M}_{00}(r)$ is
\begin{equation}
\mathcal{M}_{00}(r) \approx \frac{\Delta r_0}{(3-\kappa)} (3-\kappa) \mathcal{D}^{(3-\kappa)/\kappa} \left( 1 - \frac{\beta_0}{(3-\kappa) \mathcal{D}} \right) \left\{ \frac{\eta}{3-\kappa} \left[ \frac{1}{\kappa} - 1 \right] \left( \frac{\beta_0^{n+2}}{\beta_0^{n+1}} - \frac{\beta_0^{n+1}(r')}{\beta_0} \right) \right. \\
+ \frac{3}{\kappa} \left. \left( \frac{\beta_0^{n+1}(r')}{\beta_0} - \frac{\beta_0^{n+1}(r)}{\beta_0} \right) \right\} + \ldots \tag{A8}
\end{equation}

**APPENDIX B: SPECTRUM AND LAGS FOR POWER LAW AND EXPONENTIAL ELECTRON DISTRIBUTIONS**

Using (13) to (15), after integration over $\gamma_e$ we obtain the following expression for the spectrum:
\begin{equation}
\frac{dP}{\omega d\omega} = \frac{\sqrt{3} e^2}{3 \pi} r^2 \Delta r \frac{N_e}{\Gamma^2(r) (1-\beta(r))^3} \left\{ 2 \int_{\gamma_m}^{\infty} d\gamma_e \left( \frac{\gamma_e}{\gamma_m} \right)^{-(p+1)} \right. \\
\times \gamma_e^{-2} \int_{\frac{\omega_m}{\omega}}^{\infty} K_{\gamma/3}(\xi) \xi \left( \frac{11}{12 \gamma_m} \right) \left[ 2^{-\frac{4}{3}} \left( \frac{\omega}{\omega_m} \right)^{-\frac{1}{3}} \Gamma\left( \frac{1}{3} \right) \right] \left( \frac{\omega}{2 \omega_m} \right)^{2} \\
\times F_2 \left( \frac{p}{4} + \frac{1}{3}, \frac{2}{3} ; \frac{p}{4} + \frac{4}{3}; \left( \frac{\omega}{2 \omega_m} \right)^{2} \right) \\
+ \frac{2^{-\frac{4}{3}} \left( \frac{\omega}{\omega_m} \right)^{-\frac{1}{3}} \Gamma\left( \frac{1}{3} \right)}{\frac{2}{3} + \frac{1}{3}} F_2 \left( \frac{p}{4} + \frac{2}{3}, \frac{1}{3} ; \frac{p}{4} + \frac{5}{3}; \left( \frac{\omega}{2 \omega_m} \right)^{2} \right) \\
\left. - \frac{21}{8 \gamma_m^2} \left[ 2^{-\frac{1}{3}} \left( \frac{\omega}{\omega_m} \right)^{-\frac{2}{3}} \Gamma\left( \frac{1}{3} \right) \right] F_2 \left( \frac{p}{4} + \frac{1}{3}, \frac{1}{3} ; \frac{p}{4} + \frac{5}{3}; \left( \frac{\omega}{2 \omega_m} \right)^{2} \right) \right] \\
+ \left. \frac{2^{-\frac{1}{3}} \left( \frac{\omega}{\omega_m} \right)^{-\frac{1}{3}} \Gamma\left( \frac{1}{3} \right)}{\frac{2}{3} + \frac{1}{3}} F_2 \left( \frac{p}{4} + \frac{4}{3}, \frac{5}{3} ; \frac{p}{4} + \frac{7}{3}; \left( \frac{\omega}{2 \omega_m} \right)^{2} \right) \right] \\
+ \left. \frac{7 \gamma_e^{-4} \left( \frac{\omega}{\omega_m} \right)^{2} \int_{\gamma_m}^{\infty} d\gamma_e \left( \frac{\gamma_e}{\gamma_m} \right)^{-(p+1)} \right. \gamma_e^{-4} \int_{\frac{\omega}{\omega_m}}^{\infty} K_{\gamma/3}(\xi) d\xi \right\}. \tag{B1}
\end{equation}
where $\omega_m' = \omega_c' \gamma_m^2$ is the minimum characteristic frequency of electrons. The double integrals in the first and last terms of (B1) do not have a simple analytical expression. An approximation can be obtained using an integral form of $K_\nu$ Bessel function:

$$K_\nu(\zeta) = \int_0^\infty dx e^{-\nu x} x^{-\nu-1} \frac{\omega}{\omega}.$$

The last integral is small and can be neglected. Therefore,

$$\int_{\gamma_m}^{\infty} d\gamma \frac{\gamma e}{\gamma_m} \frac{-(p+1)}{\gamma_m^2} \int_0^\infty K_{\gamma/\beta}(\zeta) d\zeta$$

\approx \frac{1}{2\gamma_m} \left[ 2^{-\frac{1}{2}} \left( \frac{\nu}{\omega m} \right)^{\frac{1}{2}} \Gamma \left( \frac{1}{2} \right) \right] \int F_2 \left( \begin{array}{c} p \frac{1}{4} + \frac{1}{2} \frac{1}{2} \\ 3 \frac{1}{4} \end{array} \right) \left( \frac{\omega}{2\omega_m} \right)^{\frac{1}{2}}$$

(B4)

The calculation is straightforward but long and laborious, and we do not present details here.

If the electron distribution is exponential, i.e. $n_e(\gamma) = N_e \exp(-\gamma \gamma_c)$, terms with $F_2$ must be replaced by terms proportional to Mejjer’s $G$-functions. These functions do not have simple analytical presentations. Therefore, only asymptotic behaviour of the power spectrum is described in Section 2.4.

In the same way, we can determine an analytical expression for lags when accelerated electrons have a power-law distribution using (65) to (75) and (B1) to (B5). In fact, it is easy to see that the functions $P(\gamma, \omega_m)$ and $Q(\gamma, \omega_m)$ are both functionals of $H$ which contains $\gamma_c$-dependent terms similar to the spectrum $dP/\omega d\omega$. The function $H$ includes Bessel and hypergeometric functions and their derivatives. Therefore, we can use (B1) to (B5) to determine them. As the derivatives of hypergeometric functions $F_2(\alpha, \beta, \gamma; z)$ are also hypergeometric, lags can be expressed as a sum of $F_2$ functions. The calculation is straightforward but long and laborious, and we do not present details here because their complexity does not permit us to investigate their properties, and numerical calculation is needed.

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