A steady-state solution for warped accretion discs

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ABSTRACT
We consider a thin accretion disc warped due to the Bardeen–Petterson effect, presenting both analytical and numerical solutions for the situation in which the two viscosity coefficients vary with radius as a power law, with the two power-law indices not necessarily equal. The analytical solutions are compared with numerical ones, showing that our new analytical solution is more accurate than the previous one, which overestimated the inclination change in the outer disc. Our new analytical solution is appropriate for moderately warped discs, while for extremely misaligned discs only a numerical solution is appropriate.

Key words: accretion discs – black hole physics – galaxies: nuclei.

1 INTRODUCTION
Observational evidence is accumulating that accretion discs around black holes can be warped. Warped accretion discs have been directly observed by water maser observations in NGC 4258 (Miyoshi et al. 1995; Neufeld & Maloney 1995; Herrnstein, Greenhill & Moran 1996) and the Circinus galaxy (Greenhill et al. 2003). The lack of correlation of the radio jets in active galactic nuclei (AGNs) and the disc plane of the host galaxy (Kinney et al. 2000; Schmitt et al. 2002) can also be explained by disc warping. Wu, Wang & Dong (2008) discussed the possibility that double-peaked Balmer lines in AGNs might be emitted by warped discs. Possible evidence for disc warping is also found in X-ray binaries, including the misalignment between the jets and the orbital plane in GRO J1655−40 (Greene, Bailyn & Orosz 2001; Hjellming & Rupen 1995) and the precessing of jets in SS433 (Blundell & Bowler 2004).

Theoretically, warping can be caused by various mechanisms, including tidally induced warping by a companion in a binary system (Terquem & Bertout 1993; Larwood et al. 1996; Terquem & Bertout 1996), radiation-driven or self-induced warping (Maloney, Begelman & Pringle 1996; Maloney & Begelman 1997; Maloney, Begelman & Nowak 1998; Pringle 1996, 1997), magnetically driven disc warping (Lai 1999, 2003; Pfeiffer & Lai 2004) and frame-dragging-driven warping (Bardeen & Petterson 1975). Herein we consider the shape of a disc warped by the last mechanism.

Bardeen & Petterson (1975) pointed out that the combined results of the Lense–Thirring effect and the viscosity within the disc cause the inner part of the disc to be aligned with the central black hole, while the outer part of the disc remains tilted, thus resulting in a warped disc. Pringle (1992) derived the dynamical equations of such a warped disc. Scheuer & Feiler (1996, hereafter SF96) analytically solved the equation with a first-order approximation, assuming constant viscosity coefficients. Lodato & Pringle (2006) numerically solved the equations, also assuming constant viscosity coefficients. Martin, Pringle & Tout (2007, hereafter MPT07) generalized SF96’s analytical solution to the situation in which the viscosity coefficients vary as a power law, and then Martin (2008, hereafter M08) used this solution to fit the maser observation of NGC 4258’s disc.

We carried out a numerical calculation for a warped disc with power-law-varying $v$, and compared the results with MPT07’s analytical solution. The importance of this work lies in the fact that MPT07’s analytical solution (and SF96’s, as well) is based on a first-order approximation, under the assumption of a small inclination angle, $\theta_{\text{out}} \ll 1$, while real accretion discs can be strongly misaligned, $\theta_{\text{out}} \sim 1$: for example the fitting of NGC 4258 shows a strong misalignment. A numerical calculation is needed to show exactly how the error grows. Our calculation shows a prominent deviation between the analytical solution and the exact solution when the inclination angle is large, suggesting that the analytical solution is not appropriate for the study of NGC 4258 or other strongly misaligned discs.

We then proposed another way to extrapolate the small $\theta_{\text{out}}$ solution to a large $\theta_{\text{out}}$ situation, and thus find a new analytical solution. The new solution is also compared with numerical calculations and proves to be more accurate for the large $\theta_{\text{out}}$ situation. We also generalized the analytical solutions to the situation in which $v_1$ and $v_2$ have different power indices.

2 THE BASIC SCENARIO AND EQUATIONS
We use the same assumptions as adopted by Pringle (1992). The disc is assumed to be a thin one, consisting of concentric (but misaligned) circular gas rings. Each ring can be totally described by its surface density $\Sigma$, its angular velocity $\Omega$ and its radial velocity $V_r$. Note that $\Omega$ is a vector, so that it describes both the speed of
the rotation $\Omega = |\Omega|$ and the orientation of the ring $I = \Omega / \Omega$. Thus the state of the disc can be completely described by the distribution of the three quantities with radius $R$: $\Sigma = \Sigma(R)$, $\Omega = \Omega(R)$ and $V_r = V_r(R)$. Each ring will receive viscous torque from neighbouring rings whenever the angular velocity $\Omega$ changes with radius, $\partial \Omega / \partial R \neq 0$. Each ring also receives a Lense–Thirring torque from the central black hole whenever it is misaligned with the black hole. The dynamical equations under such assumptions are

$$\Sigma = -\frac{1}{R} \left( R \Sigma V_r \right)$$

$$\dot{L}_{\text{surf}} = -\frac{1}{R} \left( RV_r \dot{L}_{\text{surf}} \right) + \frac{1}{R} \dot{T}_{\text{vis}} + \Omega_{\text{pre}} \times L_{\text{surf}},$$

$$T_{\text{vis}} = R^3 \Sigma \left( \nu_1 \Omega I + \nu_2 \Omega I' \right),$$

$$\Omega = \Omega_{\text{K}},$$

(1)

where $\Omega_{\text{pre}}$ is the Lense–Thirring precession frequency

$$\Omega_{\text{pre}} = \omega_p / R^3 = \frac{2GJ_H}{c^2R^3}.$$ (2)

$J_H$ is the angular momentum of the black hole, $L_{\text{surf}} = \Sigma L_{\text{a}}$ is the surface density of angular momentum and $L_a = R^2 \Omega$ is the specific angular momentum, i.e. the angular momentum carried by unit mass. Here we use a dot above a quantity to stand for $\partial / \partial t$, and the prime symbol ($'$) to stand for $\partial / \partial \theta$.

In this work, we use logarithmic coordinate $x = \ln(R/R_0)$, where $R_0$ is an arbitrarily defined length-scale, so that all the physical quantities can be written as functions of $x$. The mass of a ring $x$ to $x + dx$ is $dm = \Sigma 2\pi R dR = 2\pi \Sigma_\alpha dx$, where annulus density $\Sigma_\alpha = R^2 \Sigma$ is the mass per unit $x$ interval and unit arc angle. The angular momentum of the ring is $dL = 2\pi L_{\text{a}} dx = 2\pi \Sigma_\alpha L_{\text{a}} dx$, where annulus angular momentum density $L_{\text{a}} = \Sigma_\alpha L_a$ is the angular momentum per unit $x$ interval and unit arc angle. We also describe the radial motion of rings with $V_r = (1/R) V_x$, which is the $x$ interval moved by the ring in unit time. The disc can therefore be described by $(\Sigma_\alpha, L_{\text{a}}, V_x)$ as functions of $x$, and the evolution of the disc is described by the evolution of the functions with time $t$. In the following we use a dot above a quantity to stand for $\partial / \partial t$, and the prime symbol to stand for $\partial / \partial x$.

With the notation defined above, the equations can be written in a simpler form (nevertheless equivalent to the previous form):

$$\Sigma_a = -\left( \Sigma \nu_1 \right)' = (L_a)_{\text{adv}} + (L_a)_{\text{vis}} + (L_a)_{\text{pre}}$$

$$= -\left( V_r L_a \right)' + T_{\text{vis}} + \Omega_{\text{pre}} \times L_a,$$

$$T_{\text{vis}} = \Sigma_\alpha \left( \nu_1 \Omega I + \nu_2 \Omega I' \right),$$

$$\Omega = \Omega_{\text{K}}.$$ (3)

Note that the prime here means $\partial / \partial x$ instead of $\partial / \partial R$, and $\partial / \partial x = R \partial / \partial R$.

### 3 STEADY-STATE SOLUTION FOR A SLIGHTLY MISALIGNED DISC

Under the Keplerian assumption, the disc can be entirely depicted by a distribution of $L_a$. Eliminating redundant variables, equations (3) can be rewritten as

$$\dot{L}_a = -\left( \frac{3}{2} \nu_1 \frac{1}{R^2} L_a \right)' + \left( \nu_2 \frac{1}{R^2} \left( I' \right)^2 \right)'$$

$$+ \left[ \left( \frac{1}{R^2} \left( L_{\text{a}} \right)' \right) I \right] + \left( \nu_2 \frac{1}{R^2} \left( I' \right)^2 \right)'$$

$$+ \Omega_{\text{pre}} \times L_a,$$ (4)

By taking $I' = \dot{L}_a$, we obtain the parallel part of the equation:

$$L_a = -\left( \frac{3}{2} \nu_1 \frac{1}{R^2} L_a \right)' - \left( \nu_2 \frac{1}{R^2} \left( I' \right)^2 \right)'$$

$$+ 3 \left( \frac{1}{R^2} L_a \right)' - \left( \nu_2 \frac{1}{R^2} \left( I' \right)^2 \right)'$$.$$ (5)

By taking equation (4)$- I' \times$ equation (5), we obtain the perpendicular part of the equation:

$$L_a I' = -\left( \frac{3}{2} \nu_1 \frac{1}{R^2} L_a \right)' + \left( \nu_2 \frac{1}{R^2} \left( I' \right)^2 \right)'$$

$$+ \left( \nu_2 \frac{1}{R^2} \left( I' \right)^2 \right)'$$.$$ (6)

When the disc is only slightly misaligned, equation (4) can be linearized. Taking the $z$-axis along the direction of $\Omega_{\text{pre}}$, we have $l_x = l_x e_x + l_y e_y + l_z e_z \approx e_x + l_z$, where $l_x, l_y, l_z$ are small enough for their second-order term to be neglected. Then $\dot{I}' = I_{\text{pre}}' = l_x' e_x + l_y' e_y$, $I_{\text{pre}} = l_x' e_x + l_y' e_y$ and $I' \cdot I = (l_x')^2 = 0$. Using these approximations, the two parts of the angular momentum equations become

$$L_a = -\left( \frac{3}{2} \nu_1 \frac{1}{R^2} L_a \right)' + \left( 3 \nu_1 \frac{1}{R^2} L_a \right)'$$

$$L_a I_{\text{pre}}' = -\frac{3}{2} \nu_1 \frac{1}{R^2} L_a I_{\text{pre}}' + \left( \nu_2 \frac{1}{R^2} L_a I_{\text{pre}}' \right)'$$

$$+ 3 \left( \nu_1 \frac{1}{R^2} L_a \right)' I_{\text{pre}} + \Omega_{\text{pre}} \times \left( L_a I_{\text{pre}} \right).$$ (7)

Further, using SF96's and MPT07's symbol $W = l_x + i l_y$, where $i = \sqrt{-1}$, the equations become

$$L_a = -\left( \frac{3}{2} \nu_1 \frac{1}{R^2} L_a \right)' + \left( 3 \nu_1 \frac{1}{R^2} l_x \right)'$$

$$L_a W = -\frac{3}{2} \nu_1 \frac{1}{R^2} L_a W + \left( \nu_2 \frac{1}{R^2} L_a W \right)'$$

$$+ 3 \left( \nu_1 \frac{1}{R^2} L_a \right)' W + i \Omega_{\text{pre}} L_a W.$$ (8)

It is not surprising that the first part is the same as that for a planar disc. This means that for a slightly warped disc we can find the solution in two steps. In the first step the evolution and distribution of $L_a$ are solved, with the misalignment omitted and the disc looked upon as planar. In the second step the inclination at each radius is found, with $L_a$ already known. This two-step method is much easier than finding the exact solution.
To find a steady-state solution, we set the left side of equations (8) to zero:

\[
0 = -\left(\frac{3}{2}v_1 \frac{1}{R^2} L_w \right) + \left( 3v_1 \frac{1}{R^2} L_w \right)' ,
\]

\[
0 = -\frac{3}{2}v_1 \frac{1}{R^2} L_w W' + \left( \frac{v_2}{2} \frac{1}{R^2} L_w W' \right)' + 3 \left( \frac{1}{R^2} L_w \right) W + i\Omega_{\nu} L_w W .
\]

The solutions of \( L_w \) are simple:

\[
v_1 L_w = C_0 R^{5/2} + C_1 R^2 ,
\]

where \( C_0 \) and \( C_1 \) are constants. \( C_1 \) is connected with the condition at the inner boundary, and always becomes unimportant when the region concerned is much larger than the inner radius. We therefore discard \( C_1 \) and obtain

\[
v_1 L_w = C_0 R^{5/2} .
\]

Substituting the \( L_w \) value back, we obtain

\[
0 = \left( \frac{v_2}{2v_1} R^{1/2} W' \right)' + \frac{i\omega}{v_1} R^{-1/2} W .
\]

If \( v_1 \) and \( v_2 \) vary with radius as the power laws \( v_1 = v_{10}(R/R_0)^{\beta_1} \) and \( v_2 = v_{20} \exp(\beta_2 x) \), the equation for \( W \) becomes

\[
\left( \frac{v_{20} R_0}{2} \right) \exp[(1/2 + \beta_2 - \beta_1) x] W' + i\omega \exp[-(1/2 - \beta_1) x] W = 0 .
\]

Physically we have the boundary conditions

\[
W \to 0 , \quad R \to 0 ,
\]

\[
W \to W_{\infty} , \quad R \to \infty .
\]

Solving equation (13) under such boundary conditions, we obtain

\[
W = f W_{\infty}
\]

where

\[
\begin{align*}
 f &= \frac{2^{1-s}}{\Gamma(s)} K_n(s) , \\
n &= \frac{1/2 + \beta_2 - \beta_1}{1 + \beta_2} , \\
s &= \frac{2}{1 + \beta_2} (1 - i) \sqrt{\frac{\omega_0}{v_{20} R_0}} \exp \left( -\frac{1 + \beta_2}{2} x \right) ,
\end{align*}
\]

and \( K_n \) is the nth-order modified Bessel function of the second kind.

The solution reduces to the MPT07 one (see equation 24 therein) when the two viscosity coefficients vary with the same index \( \beta_1 = \beta_2 = \beta \), and further to the SF96 solution when \( \beta_1 = \beta_2 = 0 \).

By defining the warping radius as

\[
R_w = \left( \frac{\omega_0}{v_{20} R_0} \right)^{1/(1 + \beta_2)} ,
\]

the parameter \( s \) can be written as

\[
s = \frac{2}{1 + \beta_2} (1 - i) \left( \frac{R}{R_w} \right)^{-(1 + \beta_2)/2} .
\]

Hereafter we always set \( R_0 = R_w \), i.e. use the warping radius as our length unit, thus making the problem scale-free, and turning the equation into

\[
s = \frac{2}{1 + \beta_2} (1 - i) \exp \left( -\frac{1 + \beta_2}{2} x \right) .
\]

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{figure1}
\caption{Analytical solutions, \( i_1/W_\infty \) against \( i_1/W_\infty \). Solid lines, \( \beta_2 = \beta_1 \); long-dashed lines, \( \beta_2 = \beta_1 + 0.1 \); short-dashed lines, \( \beta_2 = \beta_1 - 0.1 \). For each line style the three lines are for \( \beta_1 = 0, 1, 2 \), respectively, from top to bottom.}
\end{figure}

It is easy to see that the warping radius thus defined is the one where the Lense–Thirring precessing time-scale and viscosity time-scale are equal:

\[
\frac{1}{\Omega_{\nu}(R_w)} = \frac{R_w^3}{\omega_p} = \frac{R_w^2}{v_0(R_w)} .
\]

We present here the analytical solutions for several sets of \( \beta_1 \) and \( \beta_2 \). The \( \beta_1 \) values are 0, 1, 2, respectively, and for each \( \beta_1 \) we calculated solutions for \( \beta_2 = \beta_1, \beta_2 = \beta_1 + 0.1, \beta_2 = \beta_1 - 0.1 \). The plane of the \( z \)-axis and \( \Phi \) at infinite radius, \( I_{out} \), is set to be the \( xz \) plane, so that \( I_{out} = (\sin \theta_{out}, 0, \cos \theta_{out}) \) and \( W_{\infty} = \sin \theta_{out} \). For each solution we plot in Fig. 1 the \( f \) value in the complex plane, which is equivalent to an \( I_{out}/W_{\infty} \) against \( I_{out}/W_{\infty} \). Plot in Fig. 2 we plot the absolute value and angle of \( f \) (divided by \( 2\pi \)) against radius \( x \). The angle of \( f \) equals the azimuthal angle \( \psi \) of \( I \). The absolute value of \( f \) is \( |f| = |W|/W_{\infty} \) and equals \( \theta/\theta_{out} \) for small \( \theta_{out} \). Fig. 2 is therefore also a \( \theta/\theta_{out} \) against \( x \) and \( \phi/2\pi \) against \( x \). We divide \( \phi \) by \( 2\pi \) so that the value is now the number of turns \( I \) has precessed around the \( z \)-axis. The very fast growth of \( \psi \) in the innermost part of the disc is not important, because \( \theta \) is already very small there, meaning the disc is almost aligned with the black hole spin.

In the following we refer to equation (15) as ‘solution A’.

4 NUMERICAL SOLUTION

We developed a finite-differential code to solve the evolution of a disc. The state of the disc at each time point is represented by the \( L_w \) value upon a uniform grid of \( x \) (logarithmic grid of \( R \)). The time differentials of \( L_w \) are then evaluated, and then the \( L_w \) values at the next time point. We used upstream differencing for the advective part of the equation. The code is designed with the flexibility to solve various physical problems by adjusting the initial condition and boundary condition.

The code can also be used in finding a steady-state solution. If the boundary condition is fixed to the desired setting, and the evolution lasts long enough, in principle the disc will always arrive at the required steady-state solution. However, the computational cost can be enormous, due to the large time-scale range involved in the system. To ensure the solution reaches the steady-state value, the time \( T \) of disc evolution must be at least several times larger than the viscosity time-scale, \( T > R^2/\nu_0 \). On the other hand, the
maximum time-step $\Delta t$ to keep the algorithm numerically stable is determined by the time-scale for angular momentum to diffuse viscously over only one grid, $2\Delta t < (R \times \Delta x)^2 / v_1$, where $\Delta x$ is the grid size. These two conditions must hold for the whole calculating region $R_{in}$ to $R_{out}$. The number of time-steps needed is therefore determined by

$$\left( \frac{R^2}{v_1} \right)_{max} / \left( \frac{(R \times \Delta x)^2}{v_1} \right)_{min} = \Delta x^{-2} \left( \frac{R_{out}}{R_{in}} \right)^{2-\beta_1}.$$  

As an example, supposing $\beta_1 = 0$, $R_{out} = 10^4$, $R_{in} = 10^{-4}$, $\Delta x = 0.01$, we find $T > 10^8 R_0^1 / v_{10}$, $2\Delta t < 10^{-12} R_0^2 / v_{10}$, so that the time-steps needed are dozens of $10^20$, absolutely unaffordable. Our way out of this difficulty is to add a ‘speeding up’ factor $K(R)$ artificially to the evolutionary equation (4), changing it to

$$L_x = K(R) \mathcal{F}(L_x),$$  

(20)

where $\mathcal{F}(L_x)$ is the time differential of $L_x$ given in equation (4). This new equation leads to the right steady-state solution $\mathcal{F}(L_x) = 0$, though its intermediate results (the $L_x$ values found before the disc becomes steady) are physically meaningless. We find that $K(R) = R^{2-\beta_1}$ will make the equation converge stably and quickly.

In this work we set a uniform grid of $x$ from $x_{in} = -9.2$ to $x_{out} = 9.2$ (corresponding to $R_{in} \approx 10^{-4}$ and $R_{out} \approx 10^4$, the latter large enough to represent nearly infinity), and space resolution $\Delta x = 0.01$. We used a $(v_1 L_x) / (2/5 v_1 L_x)$ inner boundary condition, by adding a ‘ghost grid’ at $x = x_{in} = \Delta x$, and keep $L_x(x) = e^{-(2.5-\beta_1)\lambda x} L_x(x_{in})$ in order to imitate a planar disc obeying $v_1 L_x \propto R^{2/3}$ inside the inner boundary. At the outer boundary we set a fixed $L_x(x_{out})$, with an inclination angle to the black hole spin axis (set as the $z$-axis). The plane of the $z$-axis and $L_x(x_{out})$ is set to be the $xz$-plane. Therefore, $I(x_{out}) = (\sin \theta_{out}, 0, \cos \theta_{out})$, or $W(x_{out}) = \sin \theta_{out}$. We use ‘solution B’ (explained later) as the initial condition to save in computational cost, although the calculation can converge to the steady-state solution from an arbitrary initial condition.

In Figs 3, 4 and 5, the numerical solution is shown and compared with solution A. As an example, we show the results for $\beta_1 = \beta_2 = 3/4$, and the inclination angle at the outer boundary $\theta_{out} = \arcsin(0.01), 30^\circ, 85^\circ$, or equivalently $W_{\infty} = 0.01, 0.5, 0.9962.$
The numerical solution and analytical solution A coincide well when the disc is only slightly misaligned, $|W_i| \ll 1$, but when the inclination angle is large the two solutions deviate strongly. We therefore conclude that solution A is not appropriate for large inclination angles. In the plot of mass distribution, we use $R^\beta/\Sigma$ because the analytical solution predicts $\Sigma \propto R^{-\beta}$. (similar to a planar disc). The numerically calculated mass distribution differs from the analytical solution mainly in the vicinity of the warping radius, showing a dip there. This is natural because the warping there brings forth additional angular momentum transfer, so that the gas there falls faster than in the planar disc, and thus causes a lower density there.

5 A NEW ANALYTICAL SOLUTION FOR A MODERATELY MISALIGNED DISC

To find a better analytical solution for a moderately misaligned disc, we define another measure of misalignment $V = \theta \cos \psi + i \sin \psi$, where $\theta$ and $\psi$ are the inclination angle and azimuthal angle of $I$, respectively. To a first-order approximation of $\theta$, $W$ and $V$ are equal: $W = \sin \theta \cos \psi + i \sin \psi \approx V$. Then all the equations for $W$ in Section 3 also hold for $V$, hence we write

$$V = f V_\infty = V_\infty \frac{2^{1-n}}{1} e^{-n \theta / \theta_{out}}. \tag{21}$$

Hereafter we call this ‘solution B’, and equation (15) ‘solution A. The two solutions are equivalent for slight misalignment, but behave differently when extrapolated to large inclination angle $\theta_{out}$. When $\theta_{out}$ varies, solution A keeps $\sin \theta / \sin \theta_{out}$ constant at each $R$, while solution B keeps $\theta / \theta_{out}$ constant. Thus solution A causes too quick a decrease of $\theta$ at the outer disc, while solution B gets rid of this disadvantage.

Solution B is plotted in Figs 3 and 4 for comparison with solution A and the numerical solutions. In the $W/W_\infty$ plot, solution A remains unchanged with different $\theta_{out}$, while solution B predicts increasing $|W|/W_\infty$ for increasing $\theta_{out}$, which is closer to the numerical results. In the $\theta/\theta_{out} \sim x$ plot, solution B remains unchanged, while solution A predicts a decrease in $\theta/\theta_{out}$ with increasing $\theta_{out}$, which contradicts the numerical results. However, when $\theta_{out}$ is as large as $85^\circ$, even solution B become very inaccurate. On the other hand, for very small $\theta_{out}$ the two analytical solutions are equivalent and both very accurate.

6 CONCLUSIONS

We generalized MPT07’s analytical solution for warped accretion discs to the situation in which the power-law indices of the two viscosity coefficients are not necessarily equal (solution A). We then proposed a new analytical solution (solution B), which is intended to be more accurate than solution A. We also presented numerical solutions of the dynamical equations for a warped disc. Our comparison between the two analytical solutions and the numerical results shows that solution B is indeed better and can be recommended for moderately or slightly misaligned discs. For extremely misaligned discs, only the numerical solution is appropriate. Regarding the situation in NGC 4258, M08’s fitting suggested a large inclination angle, so that a numerical solution is needed for more accurate fitting.

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