Near-ground turbulence profiles from lunar scintillometer

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ABSTRACT

A simple and inexpensive six-channel array of photodiodes, LuSci, can measure weak moonlight scintillation produced by optical turbulence within few hundred metres above the ground. We describe the instrument, its operation and data reduction. Measured covariances of lunar scintillation are fitted to a smooth turbulence profile model with few parameters. Complete recipe for calculating the instrument response (including the effects of Moon’s phases) is given. The robustness of the results relative to various experimental factors and model assumptions is investigated. We give examples of the data and compare LuSci with other turbulence profilers. LuSci finds numerous applications in nighttime site testing and monitoring.

Key words: atmospheric effects – site testing.

1 INTRODUCTION

Optical turbulence in the terrestrial atmosphere critically influences the capabilities of ground-based telescopes and interferometers. Atmospheric distortions can be partially corrected with Adaptive Optics (AO). Design and operation of AO systems and interferometers need detailed information on the optical turbulence profiles (OTPs) and other parameters such as atmospheric time constant. New instruments are being developed and tested to answer this need.

A large fraction of optical turbulence is typically concentrated in the surface layer (SL) within a few hundred metres above the ground. At some sites, such as Dome C in Antarctica, the SL completely dominates the overall seeing. This is also true for daytime (solar) astronomy. It has been known since a long time (Codona 1986; Seykora 1993) that weak scintillation of extended sources such as the Sun or the Moon, shadow bands, is mostly produced in the SL. Beckers (2001) was the first to use this phenomenon for measuring the SL turbulence with an array of six detectors which record fast fluctuations of the solar flux. This instrument, called SHABAR, was used in the site survey for a modern solar telescope and played a decisive role in the final site selection (Socas-Navarro et al. 2005).

Moon can be used in a similar way to measure the nighttime SL turbulence, as demonstrated by Hickson & Lanzetta (2004). A detailed analysis of this method was made by Kaiser (2004) in an unpublished report. A lunar scintillometer consists of an array of small detectors. Compared to SHABAR, the task of measuring and interpreting scintillation has some additional challenges (Moon’s phases, smaller flux, etc.). Nevertheless, this method delivers robust estimates of OTP near the ground. In this paper, we study various instrumental and theoretical aspects of this technique.

A nighttime SL turbulence monitor finds the following applications.

(i) Measuring the strength of the SL turbulence and its vertical distribution to predict the performance of ground-layer AO, as has been done, for example, for the Gemini-North telescope (Chun et al. 2009).

(ii) Measuring the SL at new or existing sites to predict the seeing above a certain level or to determine the height of a telescope building.

(iii) Translating the measurements of seeing obtained by a site monitor located in a small tower to the level of the telescope.

Compared to the standard technique of measuring the SL turbulence with microthermal probes, a lunar scintillometer, LuSci, has the advantage of being a direct optical method that is self-calibrated. It does not require a tower. Other optical methods for SL turbulence measurements are the SLODAR (Wilson, Butterley & Sarazin 2009) or the low-layer Scidar (LOLAS; Avila et al. 2008), but LuSci is much simpler. Obviously, LuSci works only when the Moon is above the horizon, which makes it unsuitable for continuous SL monitoring. It is appropriate for working in campaign mode or for calibrating other methods, e.g. microthermals or acoustic sounders.

We begin by describing the LuSci instrument in Section 2. The method of OTP restoration from the measured signals is developed and tested in Section 3. Examples of LuSci applications are given in Section 4 and conclusions in Section 5.

2 THE INSTRUMENT

2.1 Operational principle

The scintillometer consists of a linear array of photodetectors pointed at the Moon (Fig. 1). Small fluctuations of the photocurrent are recorded with a time resolution of 2 ms during an accumulation time of the order of 1 min. Covariances between each pair of signals

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(i, j) are computed:

\[ B_{i,j} = \frac{1}{K} \sum_{k=1}^{K} (\xi_i \xi_j) k, \]

(1)

where \( \xi_i = I_i / \langle I_i \rangle - 1 \) is the normalized fluctuation of the photocurrent \( I_i \) at the detector \( i \), \( K \) is the number of signal samples collected during the accumulation time.

Theory (Appendix A) relates the measured covariances to the distribution of the refractive index structure constant \( C_n^2(z) \) along the line of sight:

\[ B(r) = \int_0^{\infty} dz \, W(r, z) C_n^2(z). \]

(2)

The coordinate axis \( z \) is directed from the instrument to the Moon, and the transverse coordinates are \( r = (x, y) \). The weighting functions (WFs) \( W(r, z) \) are calculated from the known instrument parameters, Moon’s image and statistical turbulence model. The WFs are measured in \( m^{-1/3} \).

Given the set of measured covariances at a number of baselines \( r_i \) (including the zero baseline, i.e. the variance), equation (2) is inverted to infer the OTP \( C_n^2(z) \). Practically, the OTP is represented by a smooth function of \( z \) with a small number of parameters which are fitted to the data.

### 2.2 Hardware

The scintillometer hardware should record small-amplitude fast fluctuations of Moon’s flux at several locations. Different engineering solutions are possible to do this. We assembled LuSci from cheap and readily available commercial components (Table 1). Copies of the instrument can be made with a minimal amount of in-house work.

Our instrument has evolved with time (e.g. Rajagopal et al. 2008). Here, we describe its current version (Fig. 2). Six individual modules with photodiodes and amplifiers are located inside the \( \Pi \)-shaped aluminium profile of \( 440 \times 102 \times 40 \mathrm{~mm}^3 \) size with a thin cover. The distances between the detectors (counting from the top) are \( 0, 12, 15, 17, 21, 40 \) cm, forming a set of 15 non-redundant baselines from 2 to 40 cm. The method also works with smaller (four) or larger number of detectors. The choice of the configuration is motivated by the need to sample a range of baselines, form smallest to largest. Each detector is behind a circular aperture of 1 cm diameter. Baffling prevents grazed reflections from the walls and restricts the unvignetted field of each detector to 10° diameter. Walls separate the light paths of individual detectors, except for the closest detector pair where the wall is shorter and the outer holes of 25.4 mm diameter partially overlap.

Signals from the detector modules are wired to the eight-channel, 16-bit analogue-to-digital converter (ADC) housed in the same box. The ADC is connected to the personal computer (PC) via universal serial bus (USB), which also delivers the power. The power to the detector modules can be provided either by the 5 V USB line with an additional DC/DC converter or by an external stabilized supply of \( \pm 15 \) V. The box also contains a web camera (webcam) with a 25-mm lens, enabling remote pointing and tracking. The field of this camera/lens combination is about 10°.

The silicon photodiodes FDS1010 have a square active area of \( 10 \times 10 \mathrm{~mm}^2 \). The responsivity is around \( 0.65 \mathrm{~A/W} \) at 900 nm, with a noise-equivalent power of \( 5.5 \times 10^{-14} \ \mathrm{W Hz}^{-1/2} \) at this wavelength. We use the photodiodes with zero bias voltage and transform the photocurrent into voltage with a 9.1-M\( \Omega \) resistor in the feedback loop (Fig. 3). The full Moon gives a photocurrent of about 90 nA, or a signal voltage of about 1 V. The zero bias helps to

<table>
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maintain a low dark current (otherwise, at the nominal bias of −5 V the dark current of 600 nA is larger than Moon’s signal).

The relative fluctuations of the photocurrent caused by scintillation can be as small as $10^{-4}$ (see Figs 4 and 5). Direct digitization of the signal with 16-bit resolution would be barely sufficient, provided that the full dynamic range of the ADC is used. This is why we separate the variable (AC) part of the signal above 0.1 Hz by a high-pass filter and amplify it by 90 times with the second stage before digitization. However, the average (DC) level of the signal must be monitored as well. In the previous four-channel versions of the instrument, we recorded the DC signal at the output of the first stage separately. This is not possible with six detectors and eight ADC channels, therefore a small fraction of the DC signal is transmitted by the second stage. The ratio of the amplification coefficients of AC and DC parts $K_{\text{ampl}} \approx 45$ is determined by the ratio of the resistors, then measured accurately to confirm, and used to renormalize the fluctuations. Overall, the electronics behave as though all fluctuations above 0.1 Hz were amplified by $K_{\text{ampl}}$ times relative to the average signal level. Of course, all signal transients are amplified as well, so a stabilization time of ~30 s after pointing the Moon is needed before the measurements can start.

The ADC reads all channels sequentially at a rate of 5 kHz. The signals are averaged by the acquisition software to emulate the synchronous sampling of all channels at 500 Hz (10 reads in 2-ms time) and to average any rapid noise. Low-pass filtering in the amplifiers also helps to suppress noise outside the acquisition bandwidth. There was some concern that the large capacity of the un-biased photodiode could smooth the signal. A direct test with faint light flickering at 110 Hz confirmed that the detector and its electronics behave like an RC filter with a time constant $\tau = 0.36$ ms (3-dB bandwidth 440 Hz).

2.3 Software and operation

The software to acquire signals and control the instrument is written in C++ and works under Windows. It was developed under Visual Studio 6.0. The configuration of the system is schematically shown in Fig. 1. Parameters relevant to the operation and data acquisition are stored in a configuration file. They include geographical site coordinates, number of channels (six), sampling frequency, accumulation and averaging time, as well as technical data needed to acquire the signals and control the mount.

We do not rely on the pointing and tracking capability built into the mount and use it simply as a two-axis pointing device under computer control. The program calculates Moon’s elevation and azimuth and points the instrument. The azimuth axis is aligned vertically during mount installation using the bubble level. The zero-points in elevation and azimuth are set by powering the mount with the instrument pointed to the north. Small offsets are introduced to correct the pointing if necessary. As the required accuracy is only

![Figure 3. Electronic scheme of the detector module.](https://example.com/fig3)

![Figure 4. Temporal covariance between two detectors separated by 38 cm (full line) and autocovariance of each detector (dotted and dashed lines). Data from Cerro Tololo, 2007 February 5. The distance between the detectors was 0.38 m.](https://example.com/fig4)

![Figure 5. Covariances measured at Paranal averaged over one night versus baseline are plotted as full lines for the nights of 2009 February 6 (top panel) and 2009 February 12 (bottom panel). The asterisks show averaged covariances calculated from the fitted OTP models.](https://example.com/fig5)
of few degrees, this procedure works well and permits a ‘cold start’
of the mount remotely (without human presence) using inclination
sensor and digital compass (or home switch) to initialize the point-
ing. The signals of those sensors are read through the free channels
of the same ADC device.

The operation is controlled via a graphic user interface. Moon’s
image from the webcam is displayed to check or correct the point-
ing, if necessary. We are also developing an automatic tracking on
webcam images.

At the start, the software offsets the pointing in altitude and/or
azimuth to measure the sky background. The sky measurement is
repeated after a certain number of Moon measures are collected.
This way, we monitor the background and electronic offset, to be
subtracted from Moon’s flux for proper normalization of the covari-
ces. Parameters defining the offsets and the numbers of Moon
and sky measurements are read from the configuration file.

Segments of data of 5-s duration are acquired into the computer
memory. Average values of signals, their variances and covariances
are calculated and stored in a text file, together with the time stamp.
The signal values are stored on the disc as unsigned 16-bit integers
in another, binary file. Binary data can be accessed by means of a
pointer which accompanies each text record. The text file is suf-
cient for calculating normalized covariances, but the binary data
are used for off-line control: checking random and periodic noise,
temporal power spectra and covariances. New text and binary files
are opened each night.

The data saved on the disc are pre-processed by an idl program
which calculates the variances and covariances of the signals nor-
malized by their average values, as required for the OTP restoration
(equation 1). The sky level is subtracted from the measured fluxes
and the AC/DC amplification coefficients are accounted for. The
covariances are averaged over time (usually 1 min) and written
to another file. The same program filters the data, removing erro-
neous measurements. The filtering algorithm approximates the flux
in each channel by a polynomial as a function of time and removes
measurements with flux deviations relative to the fit or flux fluctua-
tions within 1 min larger than 2 per cent. Such data can be affected
by clouds, pointing failures, etc. Other criteria to select valid data
are low sky background (less than 5 per cent of Moon’s flux) and
sufficient number of valid 5-s data segments within each minute.
For a normally operating instrument, the fraction of valid data in
clear conditions is usually larger than 90 per cent.

2.4 Tests

Various tests can be made to assure the good quality of the data.
Temporal spectra calculated from the saved binary data usually
show a smooth decline with frequency, spanning as much as four
orders of magnitude (40 dB). In some instances, there are narrow
peaks caused either by electronic pickup noise (e.g. at 50 Hz) or
by variable light sources which contribute to the flux (e.g. 100 Hz
and harmonics from the street lights in the cities). No such peaks
are seen in the data acquired in the LuSci campaigns at various
observatories. The temporal power spectra of the dark noise are flat.

Fig. 4 shows the temporal autocovariance functions of two chan-
nels and their mutual covariance. The signal variance in all six
channels is equal to better than 5 per cent, showing that the ampli-
fication coefficients and flux normalization are correct. The mutual
covariance is wider than the autocovariance, and its maximum is
displaced from the coordinate origin by the transit time of shadow
bands moving with the projected wind speed. Slow signal fluctua-
tions also cause ‘wiggles’ in the covariance and are the major source
of statistical measurement errors (Appendix C). Wind velocity near
the ground can be estimated by fitting a model to the temporal spec-
trum of the signal together with the measured OTP (Rajagopal et al.
2008).

The covariances decrease with the baseline, as plotted in Fig. 5.
On 2009 February 6, this dependence was smooth, indicating that
the SL turbulence was distributed over altitude. In contrast, the
covariances decline very rapidly on February 12, showing that most
of the SL turbulence was below 3 m.

3 PROFILE RESTORATION

Several approaches can be used to derive the OTP from measured
covariances (inversion of equation 2). First, a simple linear method
is outlined. It is replaced now by fitting data to a smooth OTP model
with few parameters.

3.1 Weighting functions

The calculation of covariances and WFs is described in detail in
Appendices A and B. The WFs do not depend on the wavelength,
so there is no need to specify the spectral response of the instrument.
At distances larger than 100 m, the WFs depend substantially on the
turbulence outer scale $L_0$, which is usually not measured (we assume
$L_0 = 25$ m). Fig. 6 plots the WFs for LuSci (1-cm detectors, full
Moon). Signals of a pair of detectors separated by baseline $r$ become
 correlated at distances $z > r / \theta \sim 100 \theta$, where the cones with
Moon’s angular diameter $\theta$ start to overlap (Fig. 1). At somewhat
shorter distances, the covariance is slightly negative. The variance
$B(0, z)$ falls down at $z < 3 \text{ m}$ because of the finite detector size.

Averaging of scintillation by a detector of diameter $d$ can be
neglected for $z \gg d / \theta$, in which case the transverse scale of the
coherence is determined only by the projected Moon’s diameter,
r \sim \theta. As shown by Kaiser (2004), the change of variables from $(r, z)$
to $(\log r, \log z)$ reduces the integral (2) to a simple convolution. It
is natural to use the logarithmic grid in $z$ for calculations of the WFs
and for restoration of the OTP. The resolution $\Delta z$ should be
approximately constant. In this respect, LuSci differs from SLODAR,
where the vertical resolution $\Delta z$ is constant.

3.2 Linear restoration

It can be guessed from Fig. 6 that a difference between two co-
variances at bases $r_1$ and $r_2$ will provide information on turbulence

![Figure 6. WFs $W(z)$ for the six-element array and full Moon. Full line –
variance, dashed lines – covariances for baselines from 2 to 40 cm.](https://academic.oup.com/mnras/article-abstract/404/3/1186/1049024/1049024)
located between \( r_1/\theta \) and \( r_2/\theta \) because the difference of the corresponding WFs will peak in this range. The idea can be developed further by finding linear combinations of covariances having near-constant response over some range and near-zero response outside it. Such combinations can be interpreted as integrals of \( C^2 \).

We define a logarithmic distance grid of \( N = 100 \) points from \( z = 0.3 \) m to 10 km. The fractional step \( \epsilon_n = z_{n+1}/z_n = 1.11 \) is fine enough to capture the details of the WFs. The OTP is represented by the \( n \)-element vector \( C \) of \( C^2 \) values on this grid, the WFs – as a matrix \( W \) of size \( L \times N \), where \( L \) is the number of measured covariances \( B \). In this discrete formulation, the integral (2) is replaced by a matrix product

\[
B = W^T C. \quad W_{n,l} = W(r_l, z_n)\epsilon_n z_n. \quad (3)
\]

Any linear combination of the WFs with coefficients \( A = \{a_l\} \) corresponds to the OTP integral \( J_k \) with some response function \( R(z) \),

\[
J_k = \sum_{l=1}^{L} a_l B_l = \int_0^\infty dz \, R(z) \, C^2(z), \quad (4)
\]

where

\[
R(z) = \sum_{l=1}^{L} a_l W(r_l, z), \quad \text{or} \quad R = W^T A. \quad (5)
\]

We can find linear combinations of WFs which approximate some desirable responses. The mathematical details are omitted for brevity because this method is only of historic and didactic interest. It was used in the early LuSci campaigns (Thomas-Osip et al. 2008). Fig. 7 gives an example of response functions for the six-element array.

The simplicity of this linear method is appealing. The integrals \( J_k \) are calculated directly as weighted sums of the measured covariances. However, these integrals are defined along the line of sight, making it difficult to account for the zenith angle \( \gamma \). If turbulence in the SL is concentrated in thin layers, the measured \( J \) should be multiplied by \( \cos \gamma \) in order to reduce them to the zenith. On the other hand, if \( C^2(z) = \text{constant} \), the integrals do not depend on \( \gamma \).

To complicate things further, the response functions \( R \) depend on Moon’s phase and baseline orientation. Therefore, the linear method is now replaced by model fitting.

### 3.3 OTP restoration by model fitting

Representing a continuous unknown OTP by a coarse model with few parameters is a kind of regularization necessary to solve the inverse problem. One does not expect a miracle, i.e. that the model would render accurately any profile. Instead, we hope that the model will reproduce correctly the total intensity of turbulence and its location. The experience of site testing with SHABAR (Socas-Navarro et al. 2005) shows that individual OTPs are not very useful, as they contain excessive information. What is really needed usually is the measurement of turbulence integrals over specific ranges within the SL.

The OTP model is specified as a set of \( C^2(Z_k) \) values at \( K = 5 \) fixed pivot points \( Z_k = (3, 12, 48, 192, 768) \) m (here we refer to the six-channel array). To enforce the non-negativity of the OTP, the model parameters are \( Y_l = \log_{10} C^2(Z_k) \). Values between the pivot points are calculated by linear interpolation of \( Y_k \) on the logarithmic distance grid. This is equivalent to representing the OTP by power-law segments. Below \( Z_1 \) and above \( Z_5 \), the model OTP is extrapolated by constants equal to its values at the first and last pivot points.

Linear interpolation is expressed in the compact form by introducing the \( N \times K \) matrix \( T \) of triangular functions:

\[
T_i(z_n) = 1 - |\log_{10}(z_n/Z_k)|/\log_{10}(4)
\]

for \( |\log_{10}(z_n/Z_k)| < \log_{10}(4) \) and zero otherwise. This formula takes advantage of the fact that \( Z_n+1/Z_n = 4 \). Knowing the vector of model parameters \( Y \), the OTP is calculated simply as

\[
\log_{10} C = T Y. \quad (7)
\]

This corresponds to the model covariances \( B = W^T C \). We fit model to the measurements by minimizing the \( \chi^2 \) metric

\[
\chi^2 = \frac{1}{L} \sum_{l=1}^{L} [(B_l - \hat{B}_l)/\sigma_B]^2 + \beta S, \quad (8)
\]

where the OTP smoothness \( S \) is defined as

\[
S = \sum_{k=2}^{K-1} [(Y_k - 0.5(Y_{k-1} + Y_{k+1}))]. \quad (9)
\]

The rationale for this metric is as follows. First, we normalize the residuals simply by the measured variance \( \sigma_B \) rather than by the estimated measurement errors of \( B \) because these errors are not known; they are mutually correlated and of comparable magnitudes (Appendix C). Secondly, we add a smoothness penalty with the regularization parameter \( \beta = 10^{-4} \). If the restored OTP has a spike of 1 dex (i.e. 10 times), the typical \( \chi^2 \) will increase by 25 per cent. Regularization helps to select the smoothest solution among many solutions compatible with the data.

If an OTP is represented by linear (rather than power-law) segments between the pivot points, the values at these points can be found immediately because the unknowns and data are related to each other linearly. However, the non-negativity and smoothness of the OTP are not guaranteed. The linear model serves to find preliminary values of \( Y \) which are then used as a starting point in the minimization of (8). An even better choice is to use the previously measured OTP (when available) as a starting point. The minimization is done with the AMOEBA method (Press, Flannery & Teukolsky 1986). Typically, \( \chi^2 \sim 0.02^2 \) is reached (see asterisks in Fig. 5).

The OTP model consisting of power-law segments can reproduce very well smooth profiles such as exponential. When the OTP changes abruptly, e.g. from high to low level, the fitted model is necessarily inaccurate. Even in this worst-case situation (Fig. 8) the turbulence integrals are recovered with errors less than 10 per cent. When we add to all covariances a large constant to emulate...
the effect of transparency fluctuations, the dashed curve in Fig. 8 remains practically unchanged (the constant offset is absorbed by increasing the last $Y_1$ without affecting values at other pivot points). However, the fitted model underestimates turbulence integrals for very steep OTPs because it does not allow for strong turbulence in the immediate vicinity of the instrument. For example, for an OTP $C_2^i(h)\propto h^{-2.5}$ the estimated SL integral is only 0.6 of its true value. Nevertheless, $C_2^i$ above 3 m is measured correctly. This bias can be easily fixed by adding another pivot point at 0.75 m.

The data product of LuSci is a text file. Each line contains the Julian date of the measurement, air mass sec $\gamma$, SL seeing and fitting error $\chi^2$. Then, the values of $Y_1$ are listed, followed by the turbulence integrals up to several user-defined heights. Turbulence integrals over any height interval $(h_1, h_2)$ can easily be calculated from $Y_1$. To do this, we take the $Z_k$ listed at the beginning of the file, compute the matrix $T$ on a fine logarithmic $z$-grid and use (7) to reconstruct the OTP. The integral is found by summing up $C_2^i(z_k)\Delta z_k$ in the interval between $h_1$ sec $\gamma$ and $h_2$ sec $\gamma$ and multiplying the result by $\cos \gamma$. The piece-wise power-law OTP can also be integrated analytically.

A variant of the pivot-point method has been used in processing the SHABAR data (Socas-Navarro et al. 2005). On the other hand, Hickson, Pfommer & Crotts (2009) fitted scintillation covariances to an OTP model consisting of two decaying exponents. This imposes an additional constraint on the modelled OTP which can only be fitted on the immediate vicinity of the instrument. For example, for an OTP that is very steep because it does not allow for strong turbulence in the immediate vicinity of the instrument, the fitted model underestimates turbulence integrals for very steep OTPs because it does not allow for strong turbulence in the immediate vicinity of the instrument. For example, for an OTP $C_2^i(h)\propto h^{-2.5}$ the estimated SL integral is only 0.6 of its true value. Nevertheless, $C_2^i$ above 3 m is measured correctly. This bias can be easily fixed by adding another pivot point at 0.75 m.

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A variant of the pivot-point method has been used in processing the SHABAR data (Socas-Navarro et al. 2005). On the other hand, Hickson, Pfommer & Crotts (2009) fitted scintillation covariances to an OTP model consisting of two decaying exponents. This imposes an additional constraint on the modelled OTP which can only be fitted on the immediate vicinity of the instrument. For example, for an OTP $C_2^i(h)\propto h^{-2.5}$ the estimated SL integral is only 0.6 of its true value. Nevertheless, $C_2^i$ above 3 m is measured correctly. This bias can be easily fixed by adding another pivot point at 0.75 m.

3.4 Robustness of LuSci results

Consistency. Fig. 9 shows an example of the OTPs measured during one night. A general tendency of turbulence decreasing with height is seen, but there are some exceptions. Note the slow variation of the model parameters with time. This shows that the restoration is not dramatically affected by the random noise and that the two OTPs measured within 1 min from each other are similar.

Sky background measured by offsetting the instrument is underestimated because the sky around the Moon is brighter. This leads to underestimating measured covariances and $C_2^i$ by a factor of $(1-\epsilon)^2$, where $\epsilon$ is the fraction of Moon’s flux scattered by the sky in the 10° instrument field and unaccounted for by the sky measurements. The scattered light cannot exceed the total extinction, so we can safely assume that $\epsilon < 0.1$. The bias can possibly be removed by modelling the sky brightness around the Moon, for now we estimate that it is <20 per cent.

Choice of the pivot points is not critical. Average OTP at a given site shows a smooth dependence on height, without any details near $Z_k$. We tried OTP restoration with different sets of $Z_k$ and obtained very similar results. A version of the algorithm using six $Z_k$ with a ratio of 3 (rather than 4) also works well.

Temporal filtering of the scintillation signal can cause underestimation of near-ground turbulence. This effect becomes important when $V\tau/\max(z, d)$, the ratio of the wavefront translation during exposure time $\tau = 2$ ms by the transverse wind speed $V$ to the spatial scale of scintillation, becomes comparable to 1 or larger. Maximum effect is observed on the variance $B(0)$.

The temporal filtering is evaluated by including additional factor in the calculation of the WFs (Appendix B). The ratio of filtered to unfiltered scintillation variance is plotted in Fig. 10 for the worst-case scenario: 8 d after new Moon, wind direction perpendicular to the terminator. In this case, $C_2^i$ can be underestimated by as much as two times at close distances, but the effect is small at $z > 10$ m. For the full Moon or other wind directions, the temporal bias is even

![Figure 8. Test of modelling errors. The input OTP abruptly changes from $5 \times 10^{-14}$ to $10^{-18}$ m$^{-2/3}$ at 24 m. Its integral is plotted as full line. The integral of the model OTP fitted to the covariances is plotted as a dashed line.](https://academic.oup.com/mnras/article-abstract/404/3/1186/1049024/1186-1196)

![Figure 9. Results of the OTP measurement on the night of 2009 January 9/10 at Paranal. The $C_2^i$ values at five pivot points are plotted as a function of time in the upper panel. The lower panel shows seeing in the SL (at 500 nm at zenith) integrated from the instrument to the heights of 4, 16, 64 and 256 m calculated from these OTPs.](https://academic.oup.com/mnras/article-abstract/404/3/1186/1049024/1186-1196)
The spatial scale of optical distortions which produce lunar scintillation ranges from 1 cm to 1 m at distances from 1 to 100 m. These scales encompass the range of Fried’s $r_0$ values relevant to optical propagation. The validity of the Kolmogorov model in this restricted range is of little doubt, while potential deviations of the power-spectrum index from its canonical value $−11/3$ will have only a mild effect. In contrast, extending the measurement range to smaller or larger scales increases the sensitivity to the turbulence model. For this reason, LuSci uses 1-cm detectors and makes no attempt to measure turbulence very close to or very far from the instrument.

Optical propagation is usually treated in the small-perturbation approximation (Tatarskii 1971; Roddier 1981). Situations where this approximation fails are not uncommon. For example, interpretation of stellar scintillation in the Multi-Aperture Scintillation Sensor (MASS) instrument fails for scintillation indices above 0.6 and requires semi-empirical corrections otherwise (Tokovinin & Kornilov 2007). Fortunately, lunar scintillation is described by the geometric optics and is so small (so far from the focusing regime) that the small-perturbation theory works perfectly (Kaiser 2004). Therefore, the relation of the LuSci signal to $C_n^2$ is very well defined. The signal itself is just a flux variation. LuSci does not need any calibration and measures the $C_n^2$ on absolute scale. It is a good method to calibrate other, less direct techniques of turbulence profiling.

4 SOME RESULTS

Several LuSci instruments have been fabricated by the European Southern Observatory (ESO) for the site selection programme of the future European 42-m telescope, E-ELT. In 2008–2009, these lunar scintillometers were extensively tested at the Paranal observatory in Chile together with other instruments.

The SLODAR turbulence profiler (Wilson et al. 2009) was modified to measure the SL turbulence with increased resolution (Osborn et al., in preparation) and operated at Paranal simultaneously with LuSci in 2009 February and April by J. Osborn and H. Shepherd. Wide binaries were observed to measure the OTP up to ~60 m height, in eight equally spaced bins. In Fig. 11, we compare the turbulence integrals $J_{\text{SLODAR}}$ and $J_{\text{LuSci}}$ from 6 m height above ground to the upper limit of SLODAR range (which varied from 52 to 78 m).
mean 65 m) calculated from the SLODAR and LuSci data matched in time to within 1 min. This comparison avoids the first few metres affected by the strong local turbulence. The medians of 2096 integrals are $0.64 \times 10^{-13}$ and $0.47 \times 10^{-13}$ m$^{-1/3}$ for SLODAR and LuSci, respectively. These median values correspond to $\sim$0.2-arcsec seeing, so at Paranal the SL turbulence above 6 m is typically weak. Large scatter between the two instruments is mostly caused by statistical difference of turbulence which they sample on different paths, rather than by measurement errors. The systematic difference (LuSci integrals are smaller by 30 per cent) is likely explained by the fact that SL at Paranal is slightly tilted. The SLODAR observed mostly stars in the southern part of the sky; LuSci was pointed to the Moon in the north. A small, but significant trend of the ratio in air mass between the instruments supports this explanation.

Integrated turbulence strength in the ground layer is deduced by the difference between turbulence integrals measured with the DIMM and MASS instruments (Tokovinin & Kornilov 2007). These integrals $J_{MD}$ correspond to a response function which starts at 6 m above ground (the height of the site monitor) and smoothly falls to zero between 250 and 500 m. We model this falling part as linear in height, but this assumption is not critical because $J_{MD}$ is usually dominated by the first few metres. OTPs measured by LuSci were converted to turbulence integrals with the same response and compared to $J_{MD}$. Fig. 12 shows such comparison at Paranal for 2009 February (six nights, 110 integrals averaged in common 5-min intervals). The correlation and systematic difference are obvious, with median integrals $3.6 \times 10^{-13}$ and $0.41 \times 10^{-13}$ m$^{-1/3}$ for the MASS–DIMM and LuSci, respectively. The MASS–DIMM measured turbulence is almost an order of magnitude stronger than LuSci and SLODAR.

The difference between MASS–DIMM and LuSci at Paranal is not constant. For example, the median integrals are closer to each other in 2009 July (MASS–DIMM: $5.4 \times 10^{-13}$, LuSci: $2.6 \times 10^{-13}$ m$^{-1/3}$ ). The difference becomes smaller or even changes sign when LuSci integrals are calculated from the instrument level rather than from 6 m above ground. Most likely, the Paranal site monitor is strongly affected by local turbulence, making wrong the default assumption that both instruments measure the same horizontally stratified OTP. Systematic excess of the DIMM seeing compared to the seeing in the VLT is a known feature of the Paranal observatory (Sarazin et al. 2008). It is not our purpose here to investigate the SL at Paranal. The point is that the new instrument, LuSci, brings new insights.

5 CONCLUSIONS

Our experience with LuSci and the studies reported here show that this is a robust and cheap method to measure OTP in the first 100–200 m above nighttime astronomical sites. The OTPs are self-calibrated and derived from the optical propagation – an obvious advantage over in situ microthermal probes. Compared to masts and towers, remote turbulence sounding by moonlight is non-intrusive; it does not create additional man-made turbulence and could help to detect such effects in other instruments. This will be particularly relevant at sites with excellent natural seeing, where even a small internal turbulence in a DIMM matters.

After testing at Paranal, the LuSci instruments will be used in the E-ELT site programme. The Giant Magellan Telescope project also plans to use lunar scintillometers for characterizing the SL. The first LuSci campaign at Las Campanas observatory with a four-channel array gave encouraging results (Thomas-Osip et al. 2008). This early instrument also worked at Paranal in 2007 December. The 12-channel lunar scintillometer built at the University of Vancouver is deployed at Cerro Tololo since 2006 to study the optimum height of future telescope domes (Hickson et al. 2009). Our model fitting was successfully tested on the data from this instrument, and a good agreement with our six-channel array was found during the comparison campaign in 2009 March.

A new exciting application of lunar scintillometers will be the study of intense surface turbulence at Arctic and Antarctic sites. The simplicity and robustness of this method are the key features in this application. Such instruments are being developed now.

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APPENDIX A: WEIGHTING FUNCTIONS

Here, we give the recipe for calculating the WFs which relate OTP with measured covariances (equation 2). Similar derivations can be found in Codona (1986), Kaiser (2004) and Hickson & Lanzetta (2004).

Standard theory of optical propagation through turbulence (Tatarskii 1971; Roddier 1981) provides expression for the spatial power spectrum of light amplitude at the ground created by a thin turbulent layer at a distance $z$:

$$
\Phi_x(f, z) = \alpha(2\pi)^3 \lambda^{-2} f^{-11/3} C_n^2(z) \sin^3(\pi \lambda f)^2.
$$

(A1)

The origin of the coordinate system is at the observer, the axis $z$ is directed towards the light source, other coordinates $x, y$ are in the wavefront plane, corresponding to the two-dimensional spatial frequency vector $f = (f_x, f_y)$ and $f = \sqrt{f_x^2 + f_y^2}$. The light source is monochromatic with wavelength $\lambda$, local turbulence strength is $C_n^2(z)dz$, the numerical coefficient $\alpha = 0.00969$. This equation is valid only for small amplitude fluctuations $\chi = \log[E(x, y)/\langle E \rangle] \ll 1$. To account for the finite outer scale $L_0$, additional multiplier $[1 + (f/L_0)^2]^{-11/6}$ must be included in (A1).

For an extended incoherent source such as Moon, scintillation patterns produced by different source elements superimpose, leading to a convolution with the projected source image $A(x, y, z)$. For example, a uniform disc of angular diameter $\theta$ projects to a circle of diameter $\theta z$ and creates the spatial filter

$$
\tilde{A}(f, z) = 2J_1(\pi f \theta z)/(\pi f \theta z),
$$

(A2)

where $J_1$ is the Bessel function. By definition, the filter is normalized so that $\tilde{A}(0) = 1$. Filtering by the source and detector of diameter $d$ limits the effective spatial frequencies to $f < 1/\max[\theta z, d]$. For Moon’s diameter $\theta \sim 10^{-2}$ rad, the argument under the sine in (A1) is always $\ll 1$. In other words, the Fresnel radius $\sqrt{\lambda z}$ is always much smaller than the projected source diameter $\theta z$ or the detector size $d$. Replacing sine with its argument and going from amplitude fluctuations to intensity fluctuations, $W_{\text{int}} = 4W_f$ [here, $\chi(x, y)$ is the normalized intensity fluctuation, $\chi = I/\langle I \rangle - 1$], we get

$$
\Phi_x(f, z) = \alpha(2\pi)^3 f^{11/3} \left[ 1 + (f/L_0)^2 \right]^{-11/6} C_n^2(z) \int_0^\infty dz \int_0^\infty dz C_n^2(z) \Phi_A(f, z).
$$

(A3)

$$
P_A(f, z) = |\tilde{A}(f, z)|^2.
$$

(A4)

This is the geometric–optics approximation where intensity fluctuations are produced by the local curvature of the wavefront,

$$
\chi(x, y, z) = z \nabla^2_{sz} \eta(x, y, z) \otimes A(x, y, z).
$$

(A5)

$\eta(x, y) = \lambda/(2\pi) \varphi(x, y)$ being the wavefront distortion, $\varphi(x, y)$ the phase and $\nabla^2_{sz}$ the Laplacian operator over $x, y$. The intensity fluctuations are achronic. The approximation of small perturbations which is essential in (A1) can now be dropped because, as demonstrated by Kaiser (2004), the intensity fluctuations from an extended source remain very small and equation (A3) is still valid even when the point-source scintillation is strong, $\chi \sim 1$. The validity of (A3) under strong scintillation is readily proved by numerical simulation. When $\chi \sim 1$ for a point source, the light is focused at spatial scales of the order of Fresnel radius $\sqrt{\lambda z}$ or smaller. For an extended source, these fluctuations are averaged out on larger scales, where the small-perturbation theory remains valid.

The last step in the derivation of the WFs is to combine the scintillation spectra produced by all turbulent layers, assumed to be statistically independent. This leads to

$$
\Phi_x(f) = \int_0^\infty dz \int_0^\infty dz C_n^2(z) \Phi_A(f, z).
$$

(A6)

The outer-scale factor is included here. The intensity covariance at baseline $r = (x, y)$ is calculated by the Fourier transform (FT) of (A6),

$$
B_\zeta(r) = \langle \chi(r + r') \chi(r') \rangle = \int d^2 f \Phi_x(f) \exp(2\pi i r f).
$$

(A7)

By changing the order of integration, we finally obtain the formula for calculating the WF,

$$
W(r, z) = \int d^2 f \int d^2 f \left[ 1 + (f/L_0)^2 \right]^{-11/6} \times P_A(f, z) \exp(2\pi i r f).
$$

(A8)

The spatial filter $P_A(f, z)$ combines the convolutions with the source image and detector averaging. It may also account for the finite exposure time, as detailed in the next section. If there were no $f^{11/3}$ factor under the integral, the scintillation covariance would be simply proportional to the autocorrelation function (ACF) of the source. In fact, it resembles the source autocorrelation and falls to zero at $r > \theta z$, with some negative ‘ringing’.

When going from (A3) to (A6), we made the usual assumption that turbulence can be represented by a combination of independent phase screens. This is not a very good approximation in the case of LuSci where the transverse distance $r$ can be of the same order as the propagation distance $z$. We repeated the derivation of the spectrum of intensity fluctuations starting from the geometric–optics formula (A5) integrated along the line of sight,

$$
\zeta(x, y) = \int_0^\infty dz \nabla^2_{sz} \eta(x, y, z).
$$

(A9)

The FT over coordinates $x, y$ is taken and the spatial variations of the air refractive index $n(x, y, z)$ are related to the three-dimensional spatial spectrum of refractive index fluctuations

$$
\Phi_n(k) = \langle |\tilde{n}(k)|^2 \rangle = \alpha C_n^2(z)|k|^{-11/3},
$$

(A10)

where $k$ is the three-dimensional spatial frequency. The Kolmogorov turbulence model assumes isotropic and spatially stationary random process. By allowing the dependence $C_n^2(z)$, we formally commit an error. An attempt to constrain or measure $C_n^2(z)$ locally violates the statistical model which defines this parameter! Therefore, the theory can be approximately valid only in situations where the dependence of $C_n^2$ on coordinates is smooth, on length scales much larger than the spatial scales of the problem.
We do not reproduce here the full derivation, which leads to the same formula (A6) where only $P_A(f, z)$ is replaced by a slightly modified spatial filter

$$P_A(f, z) = f z \int_{-1}^{1} d \epsilon \, Y(\epsilon f z) \left(1 - \epsilon^2\right) \times \tilde{A}[f, z(1 + \epsilon)]\tilde{A}^*[f, z(1 - \epsilon)].$$  \hspace{1cm} (A11)

The function $Y(x)$ is defined as

$$Y(x) = \int_{-\infty}^{\infty} d \alpha \left(1 + \alpha^2\right)^{-1/2} \exp\left[-5.55\alpha^2/(x + 0.24)\right].$$  \hspace{1cm} (A12)

It is symmetric, falls exponentially to zero for arguments larger than 1, and its integral equals 1. The numerical approximation in (A12) is accurate to better than 0.5 per cent.

Analytical arguments and numeric calculation show that the difference between the exact filter (A11) and its approximation (A4) is small. The Moon filtering means that the spatial frequencies $f \sim (\theta z)^{-1}$ mostly contribute to the scintillation. The function $Y$ falls off rapidly, and the integrand is substantially non-zero for $\epsilon f z < 1$, which leads to $\epsilon < \theta \sim 0.01$. Therefore, averaging of the spatial filter in (A11) occurs over a 1 per cent fraction of the propagation distance and can be neglected. Some difference is found only at very low spatial frequencies where $zf \sim 1$, i.e. at spatial scales comparable to the propagation distance, but these scales make no effect on scintillation.

**APPENDIX B: MOON MODELS**

The aperture filter function in (A8) is a product of factors corresponding to Moon’s image and detector. Let $O(\xi, \eta)$ be the angular intensity distribution in Moon’s image, then

$$\tilde{A}_{\text{Moon}}(f_x, f_y, z) = \left[A_{\text{Moon}}(\xi, \eta)\right]^{-1} \int d\xi d\eta \, O(\xi, \eta) \times \exp[2\pi iz(f_x \xi + f_y \eta)].$$ \hspace{1cm} (B1)

For a circular detector of diameter $d$, the filter does not depend on $z$ and equals

$$\tilde{A}_{\text{det}}(f) = \pi f d / (\pi f d).$$ \hspace{1cm} (B2)

The full filter is

$$P_A(f, z) = \left|\tilde{A}_{\text{Moon}}(f, z)\right|^2 \left|\tilde{A}_{\text{det}}(f)\right|^2 \tilde{P}_{\text{wind}}(f, z).$$ \hspace{1cm} (B3)

The multiplier $\tilde{P}_{\text{wind}}(f, z) = \sin^2[V(z) f \tau]$ accounts for the averaging of scintillation signal during finite sampling time $\tau$. Here, $V(z)$ is the vector of the ground wind speed and $\sin(\pi x) = \sin(\pi x)/\pi x$.

In principle, it is possible to use a collection of Moon’s images in different phases and, for each observation, select the best match in phase for calculating the WFs or interpolate. This approach appeared too heavy, so we sought to approximate Moon’s filter. It turns out that a uniformly illuminated ellipse can serve to calculate the scintillation covariance to better than 10 per cent. A more sophisticated matrix model gives an even smaller error.

**B1 Ellipse model**

We used the collection of Moon’s images with daily sampling of phases posted by T. Talbott. The original 800-pixel images were rescaled and rebinned on a 128$^2$ grid in such way that the image diameter is always 128 pixels and the terminator is oriented vertically. The images were placed in a 1024$^2$ grid and Fourier transformed (zero padding increases the frequency sampling). We normalize the square modulus of each FT to one at coordinate origin, multiply it by $f^{1/3}$ and transform back to obtain the scintillation covariance $B_z$ modulo a constant coefficient.

Elliptical disc has similarities to the actual Moon’s shape, such as finite extend and sharp edge. Ellipse is characterized by its relative diameters $\delta_x$ and $\delta_y$, for a circle $\delta_x = \delta_y = 1$. The corresponding filter is

$$P_{\text{ell}}(f, z) = \left[\frac{2f \pi x}{\Delta x}\right]\left[\frac{\pi x}{f \Delta x}\right], \quad x = z(\pi f \delta_x)^2 + (f \delta_y)^2.$$ \hspace{1cm} (B4)

For each Moon’s image, we adjust the parameters $\delta_x$ and $\delta_y$ by matching the equivalent width of the energy spectra of ellipse and image. Then, the dependence of these parameters on Moon’s phase (measured by the time $t_M$ from the new Moon in days) is fitted by smooth curves:

$$\delta_y = 1.02 - 0.0004 (t_M - 14.75)^2,$$

$$\delta_x = 0.96/[1 + 0.0172 (t_M - 14.75)^2].$$ \hspace{1cm} (B5)

The scintillation covariance for an ellipse is calculated in the same way as for the true image. The difference normalized by variance, $E = |B_{\text{ell}}(r) - B_{\text{Moon}}(r)| / B_{\text{Moon}}(0)$, is a measure of the modelling error. As shown in Fig. B1, the largest errors occur near the first and last quarters. On the other hand, the model is good within ±5d from the full Moon. The largest errors are found at the shortest baselines.

The full Moon itself does not have a symmetric ACF and the model partially accounts for this: $\delta_x < \delta_y$, at $t_M = 14.75$.

![Figure B1. The top panel shows the scintillation covariance calculated for the true Moon’s image at $t_M = 7.5$ d and its rescaled difference with the ellipse model. The low panel plots the maximum and minimum model errors $E = |B_{\text{ell}}(r) - B_{\text{Moon}}(r)| / B_{\text{Moon}}(0)$ as a function of Moon’s age $t_M$.](https://academic.oup.com/mnras/article-abstract/404/3/1186/1049024/0)
B2 Matrix model

In this model, we approximate the power spectrum of Moon’s image \(|\lambda_{\text{Moon}}(f)|^2\) by a fourth-order polynomial in \(f\) fitted at every pixel of the frequency plane. The normalized dimensionless spatial frequency \(f' = f/\xi\) is used, so the model does not depend on the distance \(z\) or Moon’s diameter \(\theta\). The spectra of real images are calculated in the same way as above, but on the 4000\(^2\) pixel grid and with different padding ratio (Moon’s diameter 670 pixels or 1/6 of the grid size). Only the central 400\(^2\) pixels of the spectra are retained, meaning that the details smaller than 10 pixels in the original images are smoothed out.

We tried first to fit the polynomials over the full range of Moon’s phases \(\tau_m\) from 2.4 to 24.6, but found that these polynomials fit the measured spectra too strongly. The difference between the two measured covariances calculated with the matrix model and with real images does not exceed \(\pm 3\)% of the variance of the signal in the time interval of \(\pm 6\) d around full Moon, 8.5 < \(\tau_m\) < 21. The maximum difference in covariances between matrix and ellipse models is 8.7 %.

B3 Numerical details

Calculation of the WFs is the most time-consuming part of the OTP restoration. We refresh the set of WFs every hour, considering that the change during this interval is small. The Moon’s power spectrum for appropriate phase is calculated on a fixed grid (256\(^2\) pixels for ellipse model or 400\(^2\) pixels for matrix model) in the \(f'\) space. A grid of 100 points in \(\xi\) from 0.3 m to 10 km with uniform logarithmic sampling is defined. For each \(\xi\), the frequency sampling in \(m^{-1}\) is found, the detector filter \(P_{\text{det}}(f)\) is calculated and multiplied by \(P_{\text{Moon}}\), turbulence spectrum and the normalization coefficient. If the wind speed and direction are known, the \(P_{\text{wind}}\) is included as well.

The WFs for the set of baselines are computed from the scintillation spectrum by FT. The angle between the vertically oriented baseline and the \(x\)-axis (perpendicular to Moon’s terminator) is a sum of the parallactic angle and the position angle of the illuminated Moon side. We account for this angle in advance by selecting the \(f_1\) axis to be parallel to the baseline and rotating Moon’s spectrum accordingly. The scintillation spectrum is averaged over \(f_1\) and the FT is done in one dimension.

APPENDIX C: STATISTICAL ERRORS OF COVARIANCES

The covariances are measured with certain statistical errors related to the properties of the scintillation signal. Here, we show that these errors are dominated by the slow scintillation produced in high atmospheric layers and that covariances at all baselines have strongly correlated errors. The terminology becomes confused when we talk about covariances of covariance errors, i.e. fourth statistical moments of the signal.

Consider the estimate of covariance \(\hat{B}_{ij}\) between detectors \(i\) and \(j\) obtained by averaging the signals over time \(T\). The statistical error of this estimate is related to the covariance between normalized intensity fluctuations \(\xi_t\) with a time lag \(t\),

\[
\hat{B}_{ij}(t) = \langle \xi_i(t') \xi_j(t') \rangle + \xi_t(t) \xi_t(t). \tag{C1}
\]

The signal variance is \(\sigma^2 = B_{ij}(0)\). Textbooks give formulae for calculating the variances of statistical estimates. For example, equation (8.95) of Bendat & Piersol (1986) reads

\[
\text{Var}[\hat{B}_{ij}] = \frac{1}{T} \int_{-T}^{T} dt \left(1 - |t|/T\right) \times [B_{ij}(t) \hat{B}_{ij}(t) + B_{ij}(t) \hat{B}_{ij}(-t)]. \tag{C2}
\]

In the following, we assume that the averaging time \(T\) is much longer than the signal correlation time. The autocovariances \(B_{ij}\) and \(\hat{B}_{ij}\) are, of course, equal, while \(B_{ij}(t) = B_{ij}(-t)\). This simplifies equation (C2) to

\[
\text{Var}[\hat{B}_{ij}] = \frac{1}{T} \int_{-\infty}^{\infty} dt \left[B_{ij}^2(t) + B_{ij}(t)B_{ij}(-t)\right] = \frac{\sigma^4}{T} \tau_{ij}. \tag{C3}
\]

Here we define the time constant \(\tau_{ij}\) as

\[
\tau_{ij} = \sigma^4 \int_{-\infty}^{\infty} dt \left[B_{ij}^2(t) + B_{ij}(t)B_{ij}(-t)\right]. \tag{C4}
\]

These time constants depend on the baselines. The time constant for zero baseline \(\tau_0 = \tau_s\) is a useful characteristic of the signal variation in one detector. The approximation (C3) is valid for \(T \gg \tau_0\). The relative error of the variance measurement is equal to \(\sqrt{\tau_0}/T\).

For calculating the error of the reconstructed profile, we also need to know the correlation between the errors at pairs of baselines. The scintillation signal contains an important low-frequency component, therefore all covariance errors, even those involving different detector pairs, are correlated.

Let \(B = \hat{B}_{ij}\) and \(B' = \hat{B}_{ij}\) be two measured covariances, where some indices may coincide. The signals \(\xi_i\) are Gaussian, so the fourth moment is expressed by a combination of the second moments,

\[
\langle B B' \rangle = \langle \xi_i \xi_j \rangle \langle \xi_i \xi_j \rangle + \langle \xi_i \xi_j \rangle \langle \xi_i \xi_j \rangle + \langle \xi_i \xi_j \rangle \langle \xi_i \xi_j \rangle. \tag{C5}
\]

The correlation (we do not say covariance to avoid confusion) between two errors is

\[
\text{Cov}[B B'] = \langle B \Delta B' \rangle = \langle B B' \rangle - \langle B \rangle \langle B' \rangle = \langle \xi_i \xi_j \rangle \langle \xi_i \xi_j \rangle + \langle \xi_i \xi_j \rangle \langle \xi_i \xi_j \rangle. \tag{C6}
\]

Continuing the analogy with Bendat & Piersol (1986), the estimates obtained over a finite time \(T \gg \tau_0\) will have the errors correlation

\[
\text{Cov}[B B'] = \frac{1}{T} \int_{-\infty}^{\infty} dt \left[ B_{ij}(t) B_{ij}(t) + B_{ij}(t) B_{ij}(t) \right] = \sigma^4 \tau_{ijkl}/T. \tag{C7}
\]

The formula (C7) shows that the correlation between the errors of covariances measured at two baselines depends on the temporal covariances at four baselines corresponding to all possible pair wise combinations of the four detectors involved (where some may coincide).

The temporal covariances \(B_{ij}(t)\) can be estimated from the data itself, as done by Hickson et al. (2009) (see Fig. 4). Alternatively, a model of turbulence and wind profiles can be used to get an idea of the expected errors. One such model (double-exponential \(C_2(h)\) profile, wind speed 20 m s\(^{-1}\) at 45\(^\circ\) angle to the baseline) leads to \(\tau_0 = 0.14\ s\). All \(\tau_0\) are longer than 0.085 s and \(\tau_{ijkl}\) are longer than 0.05 s, showing the strong correlation between measurement errors on all baselines. For accumulation time \(T = 60\ s\), the relative error of the variance estimate is \(\sqrt{\tau_0}/T = 0.05\).

It is clear that the errors of the measured covariances depend on the scintillation produced by all layers jointly. Scintillation from high layers is slow and will dominate the measurement errors, even if we are interested only in measuring the low-altitude turbulence with LuSci.

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