Magnetar oscillations in the presence of a crust

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ABSTRACT

We study the axisymmetric perturbations of neutron stars endowed with a strong magnetic field (magnetars), considering the coupled oscillations of the fluid core with the solid crust. We recover discrete oscillations based mainly on the crust and a continuum in the core. We also confirm the presence of ‘discrete Alfvén modes’ in the gap between two contiguous continua and, in addition, we can resolve some of them also inside the continua. Our results can explain both the lower and the higher observed quasi-periodical oscillations (QPOs) in SGR 1806–20 and SGR 1900+14 and put constraints on the mass, radius and crust thickness of the two magnetars.

Key words: asteroseismology – MHD – stars: magnetic field – stars: neutron – stars: oscillations – gamma rays: general.

1 INTRODUCTION

Over the last few years, a number of observational discoveries have brought magnetars (ultramagnetized isolated neutron stars) to the forefront of researchers’ attention. These extreme objects comprise the anomalous X-ray pulsars (AXPs; 10 objects) and the soft gamma-ray repeaters (SGRs; five objects), which are observationally very similar classes in many respects. They are all slowly rotating X-ray pulsars with spin periods clustered in a narrow range \( (P \sim 2–12 \text{ s}) \), relatively large period derivatives \( (P \sim 10^{-13}–10^{-10} \text{ s s}^{-1}) \), spin-down ages of \( 10^3–10^7 \text{ yr} \), and magnetic fields, as inferred from \( G \sim 30^{14}–30^{22} \) (2011) doi:10.1111/j.1365-2966.2011.18602.x

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The decaying, oscillating tail that follows the spike displays many tens of cycles at the neutron star spin rate. This is interpreted as being due to a ‘trapped fireball’ which remains anchored inside the magnetosphere and cools down in a few minutes. The total energy released in this tail is \( 10^{34} \text{ erg} \) in all three events detected so far.

A power spectrum analysis of the high time resolution data from the 2004 December 27 event of SGR1806–20, observed with the X-Ray Timing Explorer (RXTE), led to the discovery of fast quasi-periodic oscillations (QPOs) in the X-ray flux of the decaying tail of SGR (Israel et al. 2005). QPOs with different frequencies were detected, some of which were active simultaneously and displayed highly significant QPO signals at about 18, 26, 30, 93, 150, 625 and 1840 Hz (Watts & Strohmayer 2006). A re-analysis of the decaying tail data from the 1998 giant flare of another magnetar, SGR 1900+14, revealed QPOs around frequencies of 28, 54, 84 and 155 Hz (Strohmayer & Watts 2006). Hints for a signal at \( \sim 43 \text{ Hz} \) in the 1979 March event from SGR 0526–66 were reported as early as 1983 (Barat et al. 1983). All QPO signals show large amplitude variations with time and especially with the phase of the stars’ rotational modulation.

QPOs have also been argued to provide independent evidence for superstrong magnetic fields in SGRs (Vietri, Stella & Israel 2007). Numerous explanations have been proposed for the origin of the QPOs including the torsional oscillations of the crust alone or even as global seismic vibration modes of magnetars (Piro 2005; Samuelsson & Andersson 2007; Sotani, Kokkotas & Stergioulas 2007a; Steiner & Watts 2009). Moreover, Levin (2006) argued that the QPOs may be driven by the global mode of the magnetohydrodynamic (MHD) fluid core of the neutron star and its crust, rather than by the mechanical mode of the crust. Following this idea (Sotani, Kokkotas & Stergioulas 2008; Cerdá-Durán, Stergioulas & Font 2009; Colaiuda, Beyer & Kokkotas 2009) recently made two-dimensional numerical simulations (both linear and non-linear) and...
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2 DESCRIPTION OF THE PROBLEM

Here we give a brief description of the equations and the boundary conditions that we used in our study. First, we consider a spherically symmetric and static star, described by the TOV equations and the line element:

\[ ds^2 = -e^{2\Phi(r)}dr^2 + e^{2\Lambda(r)}d\theta^2 + r^2(d\phi^2 + \sin^2 \theta d\phi^2). \]

Although here we used an ultrastrong magnetic field \( B = 4 \times 10^{15} \) G, we neglect the induced deformation due to magnetic pressure, since it is still small because the magnetic energy \( E_m \) is a few orders of magnitude smaller than the gravitational binding energy \( E_g \). Typically, \( E_m/E_g \approx \sim 10^{-4}B/(10^{15}G) \); see Colaiuda et al. (2008) and Haskell et al. (2008) for results on magnetar's deformation. Axial and polar perturbations do not couple to each other and could be evolved independently, because we consider pure axisymmetric perturbations on a spherically symmetric star. Note that this assumption is not valid in a non-axisymmetric background; for example, as we mentioned before, a strong magnetic field will slightly deform the star away from sphericity and in this case there is a coupling of axial and polar oscillations. We have chosen to ignore the effects of smaller deformations since although they will introduce a tiny coupling between the polar and axial modes, in practice the properties of the two spectra will not be affected either qualitatively or quantitatively. Other effects, such as resistivity, can potentially have more significant contributions (see Lander & Jones 2010).

In the following we consider only torsional oscillations, which are of axial type and do not induce density variations in spherical stars: in this way, the radiative part of the metric describing the gravitational field does not vary significantly. For this reason the frequency of torsional oscillations is determined with satisfactory accuracy even when the metric perturbations were completely ignored by setting \( \delta g_{\mu\nu} = 0 \), i.e. working in the Cowling approximation (Cowling 1941). In Sotani et al. (2007a) the MHD oscillations of the equilibrium model were derived by transforming the perturbed linearized equation of motion and the perturbed magnetic induction equations into 2D-wave equation for the displacement function \( \mathcal{Y}(t, r, \theta) \) which is related to the contravariant component of the perturbed four-velocity \( \delta u^\theta \) via

\[ \delta u^\theta = e^{-\Phi} \frac{\partial \mathcal{Y}}{\partial t} \]

see Sotani et al. (2007a) for an analytical derivation. The 2D-wave equation for the displacement \( \mathcal{Y}(t, r, \theta) \) has the following form:

\[ A_{10} \frac{\partial^2 \mathcal{Y}}{\partial t^2} = A_{20} \frac{\partial^2 \mathcal{Y}}{\partial r^2} + A_{11} \frac{\partial^2 \mathcal{Y}}{\partial r \partial \theta} + A_{02} \frac{\partial^2 \mathcal{Y}}{\partial \theta^2} + A_{10} \frac{\partial \mathcal{Y}}{\partial r} + A_{01} \frac{\partial \mathcal{Y}}{\partial \theta}. \]

All coefficients \( A_{e}, A_{20}, A_{11}, A_{02}, A_{10} \) and \( A_{01} \) depend only on \( r \) and \( \theta \); see Sotani et al. (2008). In Colaiuda et al. (2009) a coordinate transformation of the form

\[ X = \pm \sqrt{\gamma} \sin \theta \quad \text{and} \quad Y = \pm \sqrt{\gamma} \cos \theta, \]

is used to transform the 2D-wave equations into a 1D-wave equation in the part of the star where the shear modulus was zero (core). The final form of this equation is

\[ A_{10} \frac{\partial^2 \mathcal{Y}}{\partial t^2} = \tilde{A}_{10} \frac{\partial^2 \mathcal{Y}}{\partial Y^2} + \tilde{A}_{11} \frac{\partial \mathcal{Y}}{\partial Y}, \]

where

\[ \tilde{A}_{10} = \frac{1}{2\pi^2} a_1 a_1^2, \]

\[ \tilde{A}_{11} = \frac{X}{2\pi^2} a_1 \left( \frac{2}{r} a_1 e^{2\Lambda} - 4\pi j_1 e^{2\Lambda} - \frac{a_1^2}{2} \right). \]

Here \( a_1(r) \) and \( j_1(r) \) are the radial components of the electromagnetic four-potential and the four-current and the prime indicates derivative with respect to \( r \). Note the function \( a_1(r) \) is dimensionless.

The distribution of the magnetic field inside the star can be found by solving the Grad–Shafranov equation (Grad & Rubin 1958; Shafranov 1966):

\[ a_i e^{-2\Lambda} + (\Phi' - \Lambda' e^{-2\Lambda}) a_i - \frac{2}{r} a_i = -4\pi j_i, \]

with the appropriate boundary and initial conditions; see Colaiuda et al. (2008) for details. As discussed in Colaiuda et al. (2009), \( a_1(r) \) shows a maximum inside the star and then the transformation (4) is not any more one to one. Therefore, after this maximum we choose the minus sign in equation (4).

In this paper we extend our previous work incorporating a solid crust in the magnetar model. The presence of the crust is accompanied by a non-zero shear modulus \( \mu \) in the coefficients of the wave equation (3). The presence of the shear modulus \( \mu \) does not allow for dimensional reduction and the transformation of equation (3) leads again to a 2D-wave equation, in the new coordinates \( X \) and \( Y \). The new equation has the following form:

\[ A_{10} \frac{\partial^2 \mathcal{Y}}{\partial t^2} = \tilde{A}_{10} \frac{\partial^2 \mathcal{Y}}{\partial X^2} + \tilde{A}_{11} \frac{\partial^2 \mathcal{Y}}{\partial Y \partial X} + \tilde{A}_{10} \frac{\partial \mathcal{Y}}{\partial Y} \]

\[ + \tilde{A}_{11} \frac{\partial \mathcal{Y}}{\partial X} + \tilde{A}_{10} \frac{\partial \mathcal{Y}}{\partial \theta}, \]

where the coefficients are

\[ \tilde{A}_{10} = \frac{1}{4\pi^2} a_1 a_1^2 + \mu \pi^2 \left( \frac{a_1^2 \cos^2 \theta}{4a_1} \right) \]

\[ + a_1 e^{4 \cos^2 \theta}, \]

\[ + a_1 e^{-4 \sin^2 \theta}, \]

\[ + a_1 e^{-4 \cos^2 \theta}, \]

\[ + a_1 e^{4 \cos^2 \theta}, \]

\[ + a_1 e^{4 \sin^2 \theta}. \]

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\[ A_{20} = \mu \tau^4 \left( \frac{a_i^2}{4a_1} \sin^2 \theta + \frac{a e^\lambda \cos^2 \theta}{r^2} \right), \]

(12)

\[ A_{11} = \mu \tau^4 \left( \frac{a_i^2}{2a_1} - 2 \frac{a e^\lambda}{r^2} \right) \sin \theta \cos \theta, \]

(13)

\[ A_{01} = \frac{X}{2 \tau^2} a_1 \left( \frac{2}{r^2} a_1 e^{2\lambda} - 4 \pi J_i e^{\lambda} - \frac{a_i^2}{2} \right), \]

(14)

\[ + \mu \frac{a_i^2}{2 \sqrt{a_1}}, \]

(15)

\[ + \mu \frac{\tau^4}{\sqrt{a_1}} \left( \frac{a_i^2}{a_1} - 2 \frac{a_i}{r} - \frac{a e^\lambda}{r^2} - 2 \pi J_i \right) \cos \theta, \]

(16)

\[ \Lambda_{10} = \left[ \frac{\mu}{\sqrt{a_1}} \left( - \frac{a_i^2}{4a_1} + 2 \frac{a_i}{r} - 2 \pi J_i \right) \sin \theta \right. \]

(17)

\[ - 3 \mu \sqrt{a_1 e^\lambda \cos \theta \cos \theta} + \mu' \frac{a_i^2}{2 \sqrt{a_1}} \cos \theta \right] \tau^4. \]

(18)

Note that for \( \mu = 0 \) one recovers equation (5). The value of \( \mu \) in the following is calculated from \( v_s \), the shear velocity (see Schumaker & Thorne 1983):

\[ v_s = (\mu/\rho)^{1/2} \approx 1 \times 10^8 \text{ cm s}^{-1}. \]

(19)

### 2.1 Boundary conditions

In order to solve equations (5) and (9), appropriate boundary conditions must be imposed. Actually, the boundary conditions in spherical coordinates were reported in Sotani et al. (2007a).

Since the wave equation is given in the \((X, Y)\) coordinates, the aforementioned boundary conditions translate as follows:

(i) regularity at the centre: \( \mathcal{Y}(X, Y) = 0 \);

(ii) no traction on the surface for the open lines:

\[ X \frac{\partial \mathcal{Y}}{\partial X} + Y \frac{\partial \mathcal{Y}}{\partial Y} = 0; \]

(20)

(iii) axisymmetry

\[ Y \frac{\partial \mathcal{Y}}{\partial X} = 0 \quad \text{at} \quad X = 0; \]

(21)

(iv) equatorial plane symmetry for \( \ell = 2 \) initial data, i.e.

\[ Y \frac{\partial \mathcal{Y}}{\partial X} = 0 \quad \text{at} \quad Y = 0; \]

(22)

(v) equatorial plane symmetry for \( \ell = 3 \) initial data, i.e.

\[ \mathcal{Y}(X, Y) = 0 \quad \text{at} \quad Y = 0; \]

(23)

(vi) continuity of the radial function \( a_1 \) when the sign is switched in the coordinate transformation (4) at \( a_1(r) = 0 \).

An additional boundary condition is needed at the base of the fluid–core interface: this condition in spherical coordinates is given by

\[ \partial_r \mathcal{Y} = \left[ 1 + \frac{(2\ell - 1)(2\ell + 3)}{3(\ell^2 + \ell - 1)} \frac{v_s^2}{v_A^2} \right] \partial_r \mathcal{Y}, \]

(24)

where \( v_s = B/(4\pi \rho)^{1/2} \) is the Alfvén velocity; see Sotani et al. (2007a) and Glampedakis, Samuelsson & Andersson (2006). In the \((X, Y)\) coordinates the previous condition becomes

\[ \left[ X \frac{\partial \mathcal{Y}}{\partial X} + Y \frac{\partial \mathcal{Y}}{\partial Y} \right] = \left[ 1 + \frac{(2\ell - 1)(2\ell + 3)}{3(\ell^2 + \ell - 1)} \frac{v_s^2}{v_A^2} \right] \partial_r \mathcal{Y}. \]

(25)

Although in this two-dimensional study we have not used decomposition in spherical harmonics, we kept the relation (25) in order to be able, during the mode recycling, to infer in the best possible way the frequency of the eigenfunction that corresponds to a specific angular index \( \ell \). For the initial runs we have used the more generic formula:

\[ \partial_r \mathcal{Y} = \left[ 1 + v_s^2/v_A^2 \right] \partial_r \mathcal{Y}, \]

(26)

which has also been used by Gabler et al. (2011) to make out a general picture of the spectrum. In general, neither the spectrum nor the eigenfunctions depend critically on this choice.

### 2.2 Numerical method

We used two representative equations of state (EOSs): the APR (Akmal, Pandharipande & Ravenhall 1998) and the WFF (Wiringa, Fiks & Fabrocini 1988) for various mass models and different values of the magnetic field. At the end the ones that seem to fit better the observational data are the ones with a mass of 1.4 \( M_\odot \) while the value of the magnetic field strength which is in accordance with independent estimations was \( B_p = 4 \times 10^{15} \text{ G} \) on the pole.

We used a numerical equidistant grid 60 \( \times \) 60 in the \((X, Y)\) coordinates, setting \( X_{\text{max}} = \sqrt{a_1 \text{ max}} \cdot \text{sin} \theta \) and \( Y_{\text{max}} = \sqrt{a_1 \text{ max}} \cdot \cos \theta \) and varying \( \theta \) from 0 to \( \pi/2 \). The accuracy of the code was tested, performing a simulation with a 90 \( \times \) 90 equidistant grid: the results show that the frequencies are not significantly influenced by a change in the number of grid points. The stability of the scheme was also checked: our simulation lasted 2 s without showing any signs of instability. The base of crust (\( X_{\text{crust}}, Y_{\text{crust}} \)) is taken at \( \rho = 2.4 \times 10^{14} \text{ g cm}^{-3} \), according to the crust model by Negele & Vautherin (1973) (NV). In this way the study is divided into two evolution problems coupled via the interface condition (25).

First, we evolve the one-dimensional wave equation (5) till the base of the crust (\( X_{\text{crust}}, Y_{\text{crust}} \)). Special care should be taken at the point where the coordinate transformation (4) is changing sign, i.e. at \( a_1 \text{ max} \). In practice, one evolves along ‘magnetic strings’ throughout the core until the base of the crust. Then in the crust, we evolve the two-dimensional wave equation (9), and the oscillating ‘magnetic strings’ of the core are replaced by a ‘magneto-elastic membrane’.

As a test run for our model, we reproduce the results in Sotani et al. (2007a) for crust oscillations. In this paper, the authors study the torsional oscillations of a non-magnetized star. They also consider, as a first approximation, the no-traction condition in the core-crust interface, i.e. instead of the condition (24) they require that \( \mathcal{Y} \) has to be continuous through the interface i.e. \( \mathcal{Y}_{\text{crust}} = \mathcal{Y}_{\text{core}} \). Here, we cannot set the magnetic field equal to zero since our coordinate system, \( X, Y \) is based on the magnetic field strength, i.e. on \( a_1(r) \). However, it was shown in Sotani et al. (2007a) that the influence of the magnetic field on the frequencies becomes important only if \( B > 10^{14} \text{ G} \), thus we can choose a very low magnetic field for magnetars, e.g. \( B = 10^{14} \text{ G} \), and perform test runs in order to reproduce the results of Sotani et al. (2007a). Using this magnetic field and the no-traction condition on the interface, we find the results reported in

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Sotani et al. (2007a). This test verified that the code can reproduce earlier results for the torsional oscillations of the crust.

In a similar way we can isolate the continua found in Colaiuda et al. (2009) and trace its changes due to the presence of the crust. One of the characteristic properties of the continua is the scaling of the frequencies that seem to form the edges of the continua. That is, the edges of the various continua follow the rule that has been found for odd type of initial data in Colaiuda et al. (2009), i.e.

\[ f_{L} \approx (n+1)f_{L0} \quad \text{and} \quad f_{U} \approx (n+1)f_{U0}, \]

where \( f_{L} \) stands for the lower frequency of the continua with an eigenfunction located mainly near the polar axis, and \( f_{U} \) stands for the upper frequency of the continua with maximum amplitude near the last open magnetic field line. These scaling laws for the continua can be easily spotted in Figs 1 and 2.

In this coupled core–crust problem apart from the continua and the crust modes, we found the imprints of a new family of discrete modes, a detailed description of these modes is presented in the following section.

Finally, we made some additional numerical tests, i.e. we set initial data in order to excite only the crust and to study the propagation of the energy in the system. We pick up two points, one in the core and one in the crust, and we follow the evolution of the perturbation during the time of our simulation, performing an FFT at different times and comparing the amplitude of their peaks. We found that the energy quickly flows from the crust towards the core, exciting the Alfvén continua as seen by Levin (2007) and by Gabler et al. (2011). In the same way, the perturbations of the core excite the crust: for this reason it is more appropriate to define the modes generated by the continua, global modes, i.e. modes that involve both crust and core oscillations. The conclusion of this test is in agreement with earlier suggestions (Glampedakis et al. 2006) claiming that when the magnetic field permeates the whole the star, the perturbations cannot be confined in the crust. Another notable result is that even when we associated a mode to the crust, the actual eigenfunction was not confined in the crust but it was extended throughout the star; see Fig. 3. In addition, we find that when a crustal mode is embedded in the continua, its eigenfunction shows the same structure of the continua that hosts it. Contrary, when a crust frequency is found in the gap between the continua, its eigenfunction shows a structure more similar to the one of the fundamental crustal frequencies; compare the upper panel of Fig. 3 and the lower panel of Fig. 4.

Figure 1. Identification of the frequencies of SGR 1806−20. We show that a stellar model APR with mass \( M = 1.4\ M_{\odot} \) and radius \( R = 11.57\ km \) can explain all the frequencies observed. Upper panel: the magnetic field strength is \( 4 \times 10^{15}\ G \). Lower panel: the magnetic field strength is \( 2 \times 10^{15}\ G \). In both panels the blue bands represent the continuous spectrum of the core oscillations.

Figure 2. Identification of the frequencies of SGR 1900+14. Since just a few frequencies were identified for this SGR, we show that a stellar model APR (upper panel) with mass \( M = 1.4\ M_{\odot} \), radius \( R = 11.57\ km \) and magnetic field \( B = 4.25 \times 10^{15}\ G \) can explain all the frequencies observed, as well as a stellar model WFF with mass \( M = 1.4\ M_{\odot} \), radius \( R = 10.91\ km \) and magnetic field \( B = 4 \times 10^{15}\ G \) (lower panel). In both the panels the blue bands represent the continuous spectrum of the core oscillations.
3 RESULTS AND DISCUSSION

The scope of this paper is not only to understand if the oscillations are global modes (i.e. combined crust–core oscillations), pure crustal modes or pure core modes, but also to identify the magnetar models that can fit the observed QPOs. This can potentially lead to a better understanding of the order in which the various modes were excited, leaving their imprints in the signal. Moreover, a proper identification of the observed QPOs will unveil the details of the magnetar structure.

3.1 Analysis of the spectrum

As we already pointed out and as it is shown in Figs 1 and 2, the spectrum of the oscillations is composed of three different types of modes as follows.

Crustal modes: they are associated with the crust and they produce a discrete spectrum. We found that, when the magnetic field increases, some modes are absorbed by the continua, compare upper panel of Fig. 5, where $B \approx 10^{15} \text{G}$, with the lower panel of the same figure, where $B \approx 6 \times 10^{14} \text{G}$: it is clear that some modes disappear. The strong coupling with the crust leads to an effective transfer of energy from the crust to the core, especially when the crust frequency is embedded in the continua (see, for example, the 66-Hz frequency in the upper panel of Fig. 1): from a combined analysis of the peak amplitude of a crust frequency embedded in the continua and of the peak amplitude of the edge of the host-continua, we found that the energy lost by the crust frequency is stored in the edges of the continua. In addition, we found that the fundamental crustal frequency does not scale with the increase or decrease of the magnetic field: this behaviour assures us that the crustal frequencies are not scaling with the magnetic field unlike the Alfvén modes. For magnetic field $B > 10^{15} \text{G}$, the relation among the fundamental frequency and the overtones is given by

$$f_{\text{crust}}^\ell = (\ell - 1) f_{\text{crust}}^\ell = \frac{2}{\ell} \text{ for } \ell > 2 \text{ and } n = 0.$$ \hspace{1cm} (28)

For a magnetic field $B < 4 \times 10^{14} \text{G}$, the law found by Samuelsson & Andersson (2007) is recovered:

$$f_{\text{crust}}^\ell = \sqrt{(\ell - 1)(\ell + 2)} \frac{v_s}{2\pi R} \text{ for } n = 0.$$ \hspace{1cm} (29)

For intermediate magnetic fields, a proper scaling cannot be found. The problem of the scaling of the crustal mode was treated also by Gabler et al. (2011), where the authors discuss the appearance or disappearance of the crustal mode in relation to the magnetic field strength.

Continuous spectrum: the continuous spectrum is generated by the presence of a strong magnetic field ($B > 10^{15} \text{G}$). It is generated in the magnetized fluid present in the core but in the case of strong magnetic field it extends and, through the coupling with the crust, its oscillations also penetrate the crust, giving rise to global modes. When the magnetic field increases, the continuous spectrum becomes more and more energetic, absorbing also some crust modes. In addition, the edges of the continua scale with the magnetic field, since $f = B/\sqrt{4\pi \rho L}$, where $L = L(X)$ is the length..
Figure 5. Upper panel: the crustal frequencies of an APR$_{14}$ model for a weak magnetic field $B \simeq 10^{14}$ G. Lower panel: for a weak magnetic field $B \simeq 6 \times 10^{14}$ G: some of the frequencies disappear as the magnetic field increases.

Figure 6. Typical eigenfunctions of a ‘discrete Alfvén mode’ for the model WFF$_{14}$ (SGR 1900+14).

of the ‘magnetic strings’ along which the perturbation propagates (compare the upper and lower panel of Fig. 1). The frequencies of the continua show the scaling described by equation (27). Note that the continuous spectrum always shows gaps in its structure for the model APR (see Fig. 1), while the model WFF does not have a gap between the fourth and fifth continua that overlap (see Fig. 2): the presence/absence of gaps seems then to be a characteristic of a particular model rather than a general one, as pointed out in van Hoven & Levin (2011).

Discrete Alfvén modes: these modes, as the crustal modes, are discrete but we identify them as Alfvén modes, because they scale with the magnetic field as the frequencies of the continua. Their structure is similar to the one of the crustal mode (compare the upper panel of Figs 6 and 7 with Fig. 3) but their scaling follows the scaling observed in the continua:

$$f_n^{(D)} \approx (n + 1) f_0^{(D)}.$$  (30)

Note that those modes have not been observed in the case of the absence of a crust (see Colaiuda et al. 2009). For this reason, they can be interpreted as an effect of the coupling between the fluid core and a solid crust (see van Hoven & Levin 2011).

3.2 Identification of the QPOs

From the variety of magnetars models that we have examined in order to fit to the observational data only a few provide oscillation frequencies that are in agreement with the observed QPOs. More specifically, we compared the data that our numerical code produced with the frequencies from the timing analysis of the SGR 1806–20 and SGR 1900+14 shown in Strohmayer & Watts (2006). In this paper, the authors identify several QPOs, of different duration, in the tail of the two events. In particular, for the more recent event, SGR 1806–20, the identified frequencies are 18, 26, 30, 92, 150, 625 and 1840 Hz while for the SGR 1900+14 they found the following frequencies: 28, 53, 84 and 155 Hz. A more recent study by Hambaryan et al. (2011), based on predictions by Colaiuda et al. (2009), confirms the earlier results by using a different analysis technique and, in addition, reveals at least three new frequencies for the SGR 1806–20: 16, 21 and 36 Hz.

The new results pose extra challenge, since already the previous studies of pure torsional oscillations in magnetars could not explain all observed frequencies, in particular the lower ones (18, 26, 29 Hz), mainly because of the small spacing among them. If the two new frequencies (16 and 21 Hz) are added then explaining via crust oscillations alone is impossible. On the other hand, the Alfvén continua in the core (Cerdá-Durán et al. 2009; Colaiuda et al. 2009) offer better chances for an explanation but still both calculations did not take into account the presence of a crust.

Here we show that our modelling of magnetar dynamics can explain all the observed frequencies, and inversely via this explanation we can constrain the parameters of the magnetar (radius, mass, equation of state and magnetic field strength). In Figs 1 and 2 we graphically show our findings, which cover three types of oscillations, the Alfvén continuum in the core (Cerdá-Durán et al. 2009; Colaiuda et al. 2009) offer better chances for an explanation but still both calculations did not take into account the presence of a crust.

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the eigenfunctions recycling program already used in Gaertig & Kokkotas (2009) (see there for more details).

A model with the EOS APR for the core and the NV model for the crust, with mass \( M = 1.4 M_{\odot} \) and radius \( R = 11.57 \text{ km} \), could fit the new data discovered by Hambaryan et al. (2011) as well as the previous data for the SGR 1806–20 (Watts & Strohmayer 2007). Here, we focus on the first five frequencies, i.e. 16, 18, 22, 26, 29 Hz, since they are the more difficult to be explained, the higher ones can be explained in various ways, as multiples of the lower ones or as polar modes (see Sotani & Kokkotas 2009).

In the upper panel of Fig. 8, the eigenfunction of the 16-Hz frequency, i.e. the first edge of the continua (see Fig. 1), is shown. The perturbation involves both core and crust and it can be classified as a global oscillation, with the crust and the core oscillating together due to the strong coupling induced by the magnetic field. The perturbation seems to be localized near the magnetic axis and just outside the region with the closed lines. In the bottom panel of Fig. 8, the eigenfunction of the second edge of the first continuum, the 18-Hz frequency, is shown: its structure is similar to the one of the 16-Hz frequency, but it shows a broader spread and an intense oscillation.

The 22-Hz mode has a discrete nature, as we have already pointed out in the previous section; see the upper panel in Fig. 3. Studying the time evolution of the perturbation, as explained in the previous section, we find that the core is excited after some time, when the crust successfully transfers its energy to it. It seems to be a fossil of crust mode found for a weaker magnetic field and it is then reasonable to classify it as crustal frequency. The next frequency found, at 26 Hz, see Fig. 7, is similar in structure to the crustal one but, analysing its time evolution, we find that it originates at the crust–core interface and its nature is discrete. In addition, although it is a discrete frequency, it scales with the magnetic field: it has then all the characteristics of a discrete Alfvén mode.

Finally, at a frequency of \( \approx 30 \text{ Hz} \), we observe excitation mainly of the core region but we also notice significant excitation in the crust and thus this frequency can be identified as a global oscillation, that is localized mainly in the core and then forces the crust to oscillate violently.

Note that the higher frequencies are located closer to the magnetic axis than the lower ones. This behaviour was already seen in Colaiuda et al. (2009) and Cerdá-Durán et al. (2009). Note also that the part of the star that seems not to be excited corresponds to the closed magnetic field lines. As we have already noted in Colaiuda et al. (2009), the closed magnetic lines have a significant smaller amplitude than the open magnetic field lines and cannot be observed via oscillations or thermal surface phenomena, since they are confined in the interior of the star.

Concerning the SGR 1900+14, an unique model that can fit better the observed frequencies cannot be found because of the paucity of the observations. For this reason, we find that two models can fit the identified QPOs for the SGR 1900+14: the EOS APR with a mass \( M = 1.4 M_{\odot} \), radius \( R = 11.57 \text{ km} \) and a magnetic field strength \( B = 4.25 \times 10^{15} \text{ G} \), as well as the EOS WFF with mass and radius respectively \( M = 1.4 M_{\odot} \) and \( R = 10.91 \text{ km} \), and a magnetic field strength \( B = 4 \times 10^{15} \text{ G} \). We focus our study on the identification of the first three frequencies observed: 28, 53 and 84 Hz. In the case of APR\(_{14}\) with a magnetic field strength \( B = 4.25 \times 10^{15} \text{ G} \), all the three frequencies can be identified as discrete Alfvén modes;
see the upper panel of Fig. 2. The oscillations involve both crust and core. In contrast, in the case of model WFF$_{14}$ (see the lower panel of Fig. 2), the frequencies can be identified as global modes (the 28 and 54 Hz) and as crustal mode (the 84 Hz). In particular, the 28-Hz frequency is identified as the first edge of the second continuum while the 54 Hz is identified as the second edge of the third continuum (see Fig. 2): both of these frequencies are global modes, with excitations that involve both the crust and the core. The 84 Hz is a crust frequency: its nature is discrete and the oscillations, although initially confined in the crust, expand rapidly in the core. Also in this model, we find a discrete Alfvén mode at around 26 Hz.

Note that the EOS used for the crust in all the models that we present in this paper is the NV. As it is already pointed out in Sotani, Kokkotas & Stergioulas (2007b), the use of the EOS proposed by Douchin & Haensel (2001) for the crust does not alter the results significantly. Still since it produces a thinner crust, it creates difficulties in the simulations since it demands a finer grid in the crust in order to evolve properly the perturbations inside it.

The results presented in this section partially agree with the ones found by Lee (2008), where a similar problem of a star with solid crust and fluid core both permeated by a magnetic field was studied in Newtonian theory as an eigenvalue problem. Without time evolutions only discrete Alfvén frequencies could be found and thus no continuous spectrum appears. However, the discrete Alfvén frequencies that we have found here agree qualitatively with the ones described by Lee.

It is worth mentioning that we have also analysed alternative combinations of magnetic field configuration, consisting of both poloidal and toroidal magnetic fields. We found that the presence of a poloidal component in the background magnetic field shifts the spectrum towards lower values. This extra parameter complicates significantly the procedure of identifying the magnetar model that will better fit the observed frequencies.

Note that a more accurate treatment of the shear modulus than the one used in equation (19) may change quantitatively the crustal modes. In addition, the presence of a superfluid component in the inner crust and/or in the core could affect the frequencies significantly, as discussed in Andersson, Glampedakis & Samuelsson (2009).

4 CONCLUSIONS

We studied the torsional oscillations of a magnetar in a general relativistic framework, consisting of a relativistic neutron star with a solid crust and a fluid core. The core and the crust are coupled by the strong magnetic field and this coupling is defined by the appropriate boundary conditions. We find that the presence of a crust and its coupling with the core partially alter the earlier picture presented in Colaiuda et al. (2009).

In particular, the presence of a crust makes the spectrum denser and thus offers an explanation of the nature of all the observed frequencies. We can distinguish, using the eigenfunction, between global modes (i.e. modes for which both the crust and the core oscillate), crust modes and discrete Alfvén modes (i.e. modes that have a discrete nature but that scale with the magnetic field). Unlike the information given in the paper by van Hoven & Levin (2011), we find that also in the case of intermediate magnetic field ($B < 10^{15}$ G), the oscillations are not confined to the core but the crust is also excited. We find a family of modes, the discrete Alfvén modes, in the gaps between two contiguous continua (as found by van Hoven & Levin 2011) and, in addition, we can resolve them also inside the continua.

We can identify all the frequencies observed in SGR 1806–20 and SGR 1900+14 and this allows us to put some constraints on radius, mass, crust thickness and magnetic field strength. In particular, the frequencies observed in SGR 1806–20 can be fitted uniquely with the APR stellar model with a mass $M = 1.4 M_\odot$, a radius $R = 11.57 \text{ km}$, a compactness $M/R = 0.178$ and a crust thickness $\Delta r/R = 0.099$. The magnetic field strength at the pole is $B = 4 \times 10^{15}$ G. In contrast, the SGR 1900+14 cannot be strongly constrained because of the limited information from the observations, i.e. just three QPOs were identified. For this reason, the SGR 1900+14 cannot be reproduced uniquely by a single model, as it has been done for the SGR 1806–20, since at least two stellar models could reproduce its observed frequencies with a good accuracy: the APR stellar model with a mass $M = 1.4 M_\odot$, a radius $R = 10.91 \text{ km}$, a compactness $M/R = 0.189$, a crust thickness $\Delta r/R = 0.085$ and a magnetic field strength $B = 4 \times 10^{15}$ G. Note that, in all the models, the crust is described by the NV equation of state.

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