Ceres’ rotation solution under the gravitational torque of the Sun

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ABSTRACT
Available observations of the shape of Ceres show it as a rotationally symmetric oblate spheroid. However, deviations from axisymmetry even at the level of observational accuracy may show significant effects on its rotational dynamics. These presumed deviations can be accounted for in a purely analytical way by means of perturbation theory. In our approach, the spherical rotor is taken as the unperturbed part of the motion instead of the more common torque-free motion or uniaxial body approaches. This alternative allows us to compute an analytical solution for the rotation of Ceres under the gravitational pull of the Sun by proceeding with a successive elimination of the different angles, which only involves quadratures of straightforward computation.

Key words: methods: analytical – celestial mechanics – minor planets, asteroids: individual: Ceres.

1 INTRODUCTION
The protoplanet Ceres is one of the targets of NASA’s ongoing Dawn mission,1 which is expected to provide crucial information on the history of the Solar system (Russell et al. 2006). An accurate determination of Ceres’ shape is essential for the determination of its internal structure (Bills & Nimmo 2010) and, therefore, for understanding its evolution. Thus, Castillo-Rogez & McCord (2010) demand an accuracy better than 0.5 km while the accuracy derived from actual observations is limited to about 2 km. What is even worse, some discrepancies are found when comparing the available shape measurements from different authors (Carr et al. 2008; Thomas et al. 2005; McCord & Sotin 2005; Millis et al. 1989).

Available observations on the shape of Ceres show it as a rotationally symmetric, almost spherical, oblate spheroid. However, deviations from axisymmetry could happen to Ceres at the level of observational accuracy, and it is known that small deviations from axisymmetry may show non-negligible effects on the rotational dynamics of rigid bodies (Souchay, Folgueira & Bouquillon 2003). Indeed, we check that a small departure from axisymmetry of a few hundreds of metres in the length of the intermediate axis would have, by far, a larger effect on the rotation of Ceres than the small perturbation of its torque-free rotation produced by the coupling with its orbital dynamics about the Sun.

When studying the rotational motion of Ceres, all these inconsistencies in the values of the physical parameters should be taken into account. Therefore, it seems desirable to have a rotation solution available for Ceres that may assume a small triaxiality and which handles its physical parameters in a pure analytical way.

Although the rotation of celestial bodies is usually described with the Euler angles of rotation, precession and nutation, the use of action and angle variables may be much more advantageous in the computation of perturbation theories. Besides, as demonstrated by Sadov (1970a,b, 1984) and Kinoshita (1972), action and angle variables useful for dealing with perturbed rigid-body motion are conveniently derived from the Hamiltonian of the torque-free motion when formulated in Andoyer variables (Andoyer 1923; Deprit 1967).

The computation of a preliminary transformation from Andoyer to action and angle variables may be unavoidable in the computation of the perturbed rotation of a general, triaxial rigid body. Therefore, Kinoshita’s set of action and angle variables is repeatedly used in the computation of rotation theories (see Kinoshita 1972, 1977; Escapa, Getino & Ferrándiz 2002, for instance). However, this preliminary transformation to action and angle variables may be avoided in specific cases. One case is the torque-free motion of axisymmetric bodies, for which case Andoyer variables are directly action and angle variables. Therefore, perturbations of the torque-free motion of axisymmetric bodies are efficiently approached in Andoyer variables (Getino, Escapa & Miguel 2010; Lara, Fukushima & Ferrer 2010). In addition, when the deviation from axisymmetry is small, the triaxiality itself can be treated as a perturbation of the axisymmetric case (Touma & Wisdom 1994) and being grouped with other perturbations (Henrard & Moons 1978; Zanardi 1986; Getino & Ferrándiz 1991, 1995).

The extreme case that can be tackled with Andoyer variables occurs for almost spherical bodies, where a departure from sphericity

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can be considered a perturbation of a spherical rotor and, therefore, can be handled in the same way as other perturbations of the torque-free motion (Ferrándiz & Sansaturio 1989; Ferrándiz, Sansaturio & Caballero 1993). This is exactly the case of Ceres, where, on one side, the effect of its oblateness on rotation can be taken as a perturbation of the spherical rotor. However, we may further assume that Ceres has a very small triaxiality which is below the accuracy of the actual measurements. Therefore, the perturbed rotation of a triaxial Ceres under the gravitational torque of the Sun can be studied in Andoyer variables by following the straightforward perturbation approach proposed by Ferrer & Lara (2010).

We formulate the perturbed problem in Andoyer variables, and consider the orbital motion of Ceres about the Sun to be purely Keplerian. The resulting Hamiltonian is of three degrees of freedom and time dependent. The frequencies of the motion are obtained after the reduction of the Hamiltonian to a function of only momenta. This reduction is obtained by a chain of canonical transformations computed by the Lie series approach (Hori 1966; Deprit 1969; Campbell & Jefferys 1970), which is very well suited to Hamiltonian problems (Farago & Laskar 2009; Noelyes, Dufay & Lemaitre 2010). In this way, the time and periodic terms are successively eliminated from the Hamiltonian up to a certain order. The reduction is carried out in a purely analytical way, although the ordering of the Hamiltonian is tailored to the actual values of Ceres.

The rotation theory is provided in an algorithmic way. The actual motion is recovered after propagating the secular terms and undoing the canonical transformations that allow the periodic effects to be recovered. Comparisons with numerical integrations show that the theory is accurate up to the first order in the ratio of the free rotation rate to the mean orbital motion. Thus, after six Ceres’ orbital periods, the rotation state is recovered within one tenth of an arcsec. Specifically, the analytical theory suffers from a secular trend that remains below a few mas per orbital period for any angle, with periodic oscillations of the same order.

2 HAMILTONIAN FORMULATION

The rotational motion of a rigid body has been customarily studied in a body-fixed frame since the times of Euler. The body frame is related to an inertial frame by a set of rotations that are commonly based on the Euler angles. Alternatively, the motion can be studied from a Hamiltonian perspective. The latter requires dealing with the conjugate momenta, contrary to velocities, to the angular coordinates that describe the attitude of the rigid body in the inertial frame.

The inertial and body-fixed frames are related in a straightforward way by means of Euler angles. However, the use of Euler angles and their conjugate momenta complicates the Hamiltonian formulation unnecessarily, and the canonical set of Andoyer coordinates is usually preferred.

Andoyer variables \((\lambda, \mu, \nu, \Lambda, M, N)\) can be seen as two sets of Euler angles. First, the body-fixed frame is referred to an intermediate plane perpendicular to the angular momentum vector, the invariant plane, by means of two rotations. The first is a rotation about the \(z\)-axis of the body frame and amplitude \(\nu\) defined by the \(x\)-axis of the body frame and the axis materialized by the intersection of the body’s equatorial plane and the invariant plane. The second rotation is about this last axis, with an amplitude \(J = \arccos N/M\) defined by the angle between the angular momentum vector and its projection on the \(z\)-axis of the body frame.

Therefore, the invariant and inertial planes are related by means of three new rotations. First, a rotation about the angular momentum vector of amplitude \(\mu\) defined by the intersection of the body’s equatorial plane and the invariant plane on one side, and the intersection of the invariant plane and the inertial plane on the other. Then, a rotation about the axis defined by the intersection of the invariant and inertial planes of amplitude \(I = \arccos \Lambda/M\). Finally, a rotation about the axis defined by the projection of the angular momentum vector on the \(z\)-axis of the inertial frame, whose amplitude \(\lambda\) is the angle between the axis defined by the intersection of the invariant and inertial planes, and the \(x\)-axis of the inertial plane.

The Hamiltonian of the rigid body rotation is
\[
H = T + V
\]
where \(T\) is the kinetic energy and \(V\) the disturbing potential due to external torques. When using Andoyer variables, the kinetic energy is
\[
T = \frac{1}{2} \left( \frac{\sin^2 \nu}{A} + \frac{\cos^2 \nu}{B} \right) (M^2 - N^2) + \frac{1}{2} \frac{N^2}{C},
\]
(1)
where \(A \leq B \leq C\) are the rigid body’s principal moments of inertia.

After formulating the potential \(V\) in Andoyer variables, the equations of motion are obtained from the Hamiltonian equations
\[
\frac{d(\lambda, \mu, \nu)}{dt} = \frac{\partial H}{\partial(\Lambda, M, N)},
\]
\[
\frac{d(\Lambda, M, N)}{dt} = -\frac{\partial H}{\partial(\lambda, \mu, \nu)}.
\]
(2)

2.1 External torque

In our model we consider the torque exerted on a rigid body of mass \(m\) by the gravitational attraction of a heavy distant body of mass \(m_1\). We assume that the disturbing body is far enough to be taken as a mass point, and limit to the MacCullagh’s (1840) potential
\[
V = -\frac{G}{2r^3} \left( A + B + C - 3D \right),
\]
where \(G\) is the gravitational constant, \(r\) is the distance between the centres of mass of the perturbing and perturbed bodies, and \(D = A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2\) is the moment of inertia of the rigid body with respect to an axis in the direction of the line joining its centre of mass with the perturber, of direction cosines \(\gamma_1, \gamma_2\) and \(\gamma_3\). Note that the mass-point assumption fits quite well to the case of the Sun, whose oblateness coefficient is very small (~2 × 10⁻⁷; cf. Damiani et al. 2011).

We further assume that the non-sphericity of the rigid body does not affect its orbital motion about the distant body, which is therefore Keplerian. Note that the main orbital effect that would prevent the orbit to be Keplerian arises from the planetary perturbations, which have been neglected in advance when choosing a two-body model. Thus,
\[
r = \frac{a(1 - e \cos f)}{1 + e \cos f},
\]
where \(a\) is the orbit semimajor axis, \(e\) is the eccentricity, \(\eta = \sqrt{1 - e^2}\) and \(f\) is the true anomaly. Then, the rotational motion is studied separately and we may drop the Keplerian term from the Hamiltonian because it does not produce any effect on the rotation. Therefore, we only deal with the potential
\[
V = -\frac{G}{2r^3} \left[ (C - B)(1 - 3\gamma_3^2) - (B - A)(1 - 3\gamma_1^2) \right],
\]
(3)
where we made use of the constraint \(\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1\).
Because the orbital motion has been assumed to be Keplerian, it is convenient to choose the orbital plane as the inertial reference frame. The orbital reference frame is then related to the body frame through the direction cosines by means of the rotations

\[
\begin{pmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3
\end{pmatrix} = R_3(\nu) R_1(J) R_3(\mu) R_1(I) R_3(\theta)
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix},
\]

where \( \theta = \lambda - \theta_c \), \( \theta_c \) is the usual polar coordinate of the orbital motion \( \theta = f + \omega \) and \( \omega \) is the argument of the periapsis. If we finally assume that the inertial \( x \) direction is pointing to the periapsis, then \( \theta = f \) and \( \theta_c = \lambda - f \).

### 2.2 Ceres’ dynamical characteristics

Ceres’ shape is assumed to be axisymmetric to the accuracy of available observations. Thus, in accordance to Carry et al. (2008) and Thomas et al. (2005), the polar radius is about 30 km shorter than the major and minor equatorial radii. However, as highlighted by Castillo-Rogez & McCord (2010), non-negligible differences of several km in its mean radius are found in the works of Carry et al. (2008) and Thomas et al. (2005), despite both works claiming a similar measurement uncertainty of about 2 km.

Following Castillo-Rogez & McCord (2010), we take the values of Thomas et al. (2005): \( a = 487.3 \) km for the equatorial radius and \( c = 454.7 \) km for the polar radius, because they are based on a wider range of measurements. Also, we take the mass \( m = 4.730 \times 10^{19} M_\odot \approx 9.41 \times 10^{20} \) kg determined by Kováčević & Kazmanoski (2007) during Ceres’ encounter with asteroid (5303) Parijskij. With respect to Ceres’ orbital motion about the Sun, we take the values of 413 382.587 km for the semimajor axis, 0.078 for the eccentricity and 1680.5 d for the orbital period as provided by JPL (http://ssd.jpl.nasa.gov/), which we assume constant in our simplification of the Keplerian motion.

An analytical solution for the rotation of an axisymmetric, oblate model is relatively simple, at least at the first order computed by Lara et al. (2010), where the disturbing body was assumed to be in circular motion. However, a small deviation from axisymmetry, even at the level of the measurements accuracy of about 2 km, may introduce significant changes in the rotation.

Thus, for a constant density model we compute the moments of inertia

\[
A = \frac{1}{5} m (a^2 + c^2) = 8.36010 \times 10^{25} \text{kg km}^2,
\]

\[
C = \frac{2}{5} m a^2 = 8.93804 \times 10^{25} \text{kg km}^2,
\]

for an axisymmetric oblate spheroid. However, we may assume a small departure from axisymmetry by taking a minor equatorial radius \( b = 487.2 \) km, a difference of 100 m with the accepted value which is clearly below the actual accuracy of the data. This almost negligible triaxiality introduces small changes in the moments of inertia, giving

\[
A = \frac{m}{5} (b^2 + c^2) = 8.35826 \times 10^{25} \text{kg km}^2,
\]

\[
B = \frac{m}{5} (a^2 + c^2) = 8.36010 \times 10^{25} \text{kg km}^2,
\]

\[
C = \frac{m}{5} (a^2 + b^2) = 8.93621 \times 10^{25} \text{kg km}^2,
\]

with differences of less than 0.02 per cent with respect to the previous case. However small these changes may seem, they are enough to introduce noticeable effects in the time evolution of all the Andoyer variables.

To illustrate the different behaviour of the axisymmetric and triaxial models, we propagate the rotation of Ceres for both cases. We assume circular motion about the Sun, and take the initial inclination angles \( J = 10^{-4} \) rad and \( I = 3^\circ \). The solution is computed by the numerical integration of the Hamilton equations of the perturbed model, and the differences between the Andoyer variables for each case are shown in Fig. 1. The important differences in \( \mu \), which grow at a rate slightly higher than half a degree per rotational period, are clearly shown in Fig. 1. Abscissas are scaled by the rotational period of Ceres of 9.074 h (cf. Chamberlain, Sykes & Esquerdo 2007).

The elliptic motion of Ceres about the Sun also introduces small differences in the rotation with respect to the circular motion approximation. Because of the small eccentricity of the orbit of Ceres, these differences are of a higher order than those introduced by the triaxiality character and are only apparent after a long-term propagation. As shown in Fig. 2 the deviation between both solutions grows at a linear rate of about \( 10^{-7} \) rad times orbital period for \( \nu \) and \( \lambda \), and twice this value for \( \mu \). This linear deviation is modulated by periodic oscillations with about 1 arcsec semi-amplitude for all angles.

Therefore, an analytical solution describing the rotation of Ceres should include both a possible small triaxiality and the eccentricity of its orbit about the Sun.

### 3 THE PERTURBED SPHERICAL ROTOR

We note that the kinetic energy (equation 1) can be reordered as proposed by Andoyer (1923)

\[
T = \frac{M^2}{2C} \left[ 1 + \alpha \sin^2 J (1 - \beta \cos 2\nu) \right],
\]

with the inertia coefficients

\[
\alpha = 1 - \frac{1}{2} \left( \frac{C}{A} + \frac{C}{B} \right),
\]

\[
\beta = \frac{1}{2\alpha} \left( \frac{C}{A} - \frac{C}{B} \right) = \frac{1 - \frac{C}{A}}{\frac{A}{A} + \frac{B}{B} - \frac{C}{C}}.
\]
where $\beta$ is bounded by the limit values $\beta = 0$ for $A = B$, the oblate spheroid, and $\beta = 1$ for $B = C$, the prolate spheroid. Hence, it is usually called the triaxiality coefficient.

Contrarily, the coefficient $\alpha$ can take any positive value depending on the physical characteristics of the rigid body. However, for almost spherical bodies $\alpha$ will be small. In fact, its value is of just a few thousandths for the Earth or Mars, and tenths of thousandths for the Moon.

### 3.1 Splitting kinetic energy

Andoyer’s ordering is not the only way of expressing the kinetic energy of the torque-free motion as a perturbation of the axisymmetric case. For instance, Touma & Wisdom (1994) propose the form

$$
T = \frac{M^2}{2B} \left[ 1 - \left( 1 - \frac{B}{C} \right) \cos^2 J + \left( \frac{B}{A} - 1 \right) \sin^2 J \sin^2 \nu \right],
$$

but other re-orderings also produce the desired effect:

$$
T = \frac{M^2}{2C} \left[ 1 + \left( \frac{C}{B} - 1 \right) \sin^2 J + \left( \frac{C}{A} - \frac{B}{C} \right) \sin^2 J \sin^2 \nu \right],
$$

$$
T = \frac{M^2}{2A} \left[ 1 - \left( 1 - \frac{A}{C} \right) \cos^2 J - \left( 1 - \frac{A}{B} \right) \sin^2 J \cos^2 \nu \right],
$$

$$
T = \frac{M^2}{2C} \left[ 1 + \left( \frac{C}{A} - 1 \right) \sin^2 J - \left( \frac{C}{A} - \frac{B}{C} \right) \sin^2 J \cos^2 \nu \right],
$$

the last one giving rise to Andoyer’s expression.

Ceres is not as spherical as the Earth or Mars, but its $\alpha$ value is still small. Indeed, assuming a minor equatorial axis 2 km smaller than the major one, we get $\alpha = 0.067$ and $\beta = 0.035$. Therefore $\alpha \beta = 0.002 \approx O(a^2)$, and the torque-free motion of Ceres can be approached as a perturbation problem: the so-called perturbed spherical rotor (Ferrer & Lara 2010) in which the zero-order term is the spherical rotor of energy $T = M^2/(2C)$, the axisymmetric part is a first-order perturbation and the triaxiality is a second-order effect.

### 3.2 Splitting the potential

The MacCullagh potential (equation 3) can be rewritten as

$$
\mathcal{V} = -\frac{M^2}{2C} \frac{n^2}{M^2/C^2} \frac{a^3}{r^3} \times \left[ \left( 1 - \frac{B}{C} \right) \left( 1 - 3\gamma_1^2 \right) - \left( \frac{B}{C} - \frac{A}{C} \right) \left( 1 - 3\gamma_2^2 \right) \right].
$$

Therefore, its perturbation order over the spherical rotor is given by the ratio of the mean orbital motion $n$ and the free rotation $M/C$ of the spherical rotor. For Ceres we can take $M/C \approx 2 \times 10^{-4}$ rad s$^{-1}$, its observed rotation rate, and $n \approx 4 \times 10^{-8}$. Consequently, $n/(M/C) = O(a^3)$, and the perturbation due to the external torque exerted by the Sun will appear at least at the sixth order in our perturbation scheme.

Moreover, we find it convenient to split the MacCullagh potential into the sum $\mathcal{V} = V_1 + V_2$, which separates the terms that are free from $\nu$

$$
V_1 = -\frac{M^2}{2C} \delta \frac{n^2}{M^2/C^2} \frac{a^3}{r^3} \left( 1 - 3\gamma_1^2 \right),
$$

where

$$
\delta = \frac{1}{2} \left( 2 - \frac{B}{C} - \frac{A}{C} \right),
$$

corresponding to perturbations of the axisymmetric problem $B = A$, from those terms

$$
V_2 = +\frac{M^2}{2C} \varepsilon \frac{n^2}{M^2/C^2} \frac{a^3}{r^3} \left[ \left( 1 - 3\gamma_1^2 \right) + \frac{1}{2} \left( 1 - 3\gamma_2^2 \right) \right],
$$

where

$$
\varepsilon = \frac{B}{C} - \frac{A}{C},
$$

which carry the perturbation of the triaxial case. Note that $V_2/V_1 = O(\epsilon/\delta)$, which is of the order of $\alpha$ or higher in our assumptions for Ceres of a small triaxiality. Therefore, $V_2$ is of higher order than $V_1$.

Finally, we note that, because of the non-circular Keplerian motion, the time appears explicitly through the true anomaly $t \equiv f(t)$. Therefore, in our Hamiltonian treatment we find it convenient to use the extended phase space formulation by introducing a variable $\tau$ and its conjugate momentum $T$ so that

$$
\frac{d\tau}{dt} = \frac{\partial H}{\partial T} = 1, \quad \frac{dT}{dt} = -\frac{\partial H}{\partial \tau} = -\frac{\partial H}{\partial f} \frac{df}{dt}.
$$

### 3.3 Hamiltonian ordering

The Hamiltonian of the perturbed motion is then formulated as

$$
H = \sum_{i=0}^{\infty} (\sigma^i/t^i) H_i
$$

where the formal parameter $\sigma$ is used to manifest the relative importance of each term in the Hamiltonian. In consequence of our previous discussion, $\alpha = O(\sigma)$ and $n/(M/C) = O(\sigma^3)$; moreover
we consider $\delta = O(\sigma)$ and, therefore, $\varepsilon = O(\sigma^2)$. So,

$$H_0 = \frac{1}{2} M^a / C,$$

$$H_t = H_0 \alpha s^2,$$

$$H_z = \frac{T - H_0 (\alpha \beta) s^2 \cos 2\nu}{2!},$$

$$H_0 = 0,$$

$$H_1 = 0,$$

$$H_6 = 0,$$

$$H_t = \frac{1}{7!} V_1$$

$$= -H_0 \delta \left( \frac{n}{M/C} \right)^{2} \frac{a_1 \alpha^3}{r^3} \frac{1}{8} \{ (4 - 6s^2) \}
\times \left[ 2 - 3x^2 + 3c^2 \cos 2\nu \right] - 12c_1x_1s_1 [2c_1 \cos \mu
+ (1 - c_1) \cos (2\theta - \mu) - (1 + c_1) \cos (2\theta + \mu)]
+ 3x_1^2 + (1 - c_1)^2 \cos (2\theta - 2\mu) + 2s_1^2 \cos 2\mu
+ (1 + c_1)^2 \cos (2\theta + 2\mu)] \},$$

$$H_8 = \frac{1}{8!} V_2$$

$$= -H_0 \delta \left( \frac{n}{M/C} \right)^{2} \frac{a_1 \alpha^3}{r^3} \frac{3}{32} \{ 6s^2 x^2 \} \cos (2\theta - 2\nu)
+ \cos (2\theta + 2\nu) \} - 4 \left( 1 - 3c_1^2 \right) \cos 2\nu + (1 + c_1)^2
\times \left( [2 - 3x^2 + 3c^2 \cos 2\nu] - 12c_1x_1s_1 [2c_1 \cos \mu
+ (1 - c_1) \cos (2\theta - \mu) - (1 + c_1) \cos (2\theta + \mu)]
+ 3x_1^2 + (1 - c_1)^2 \cos (2\theta - 2\mu) + 2s_1^2 \cos 2\mu
+ (1 + c_1)^2 \cos (2\theta + 2\mu)] \},$$

where $s_x$ and $c_x$ are abbreviations for $\sin x$ and $\cos x$, respectively, that are used when $x$ is a function of only momenta.

An approximate solution to the Hamiltonian flow defined by equation (4) is computed by perturbation theory using the Lie series approach (Hori 1966; Deprit 1969; Campbell & Jefferys 1970). Note that our model neglects Jupiter disturbances, which may be as high as one hundredth of those of the Sun, depending on the Jupiter–Ceres configuration. Therefore, Jupiter’s effects should appear between the eighth and ninth order. Consequently, at most an eighth-order theory could be consistent with the assumptions of our model.

3.4 Secular motion

We perform three consecutive canonical transformations: first

$$(\mu, \nu, \lambda, M, N, \Lambda) \xrightarrow{\tau} (\mu', \nu', \lambda', M', N', \Lambda')$$

for eliminating the angle $\mu$ from the original Hamiltonian. Then

$$(\mu'', \nu'', \lambda'', M'', N'', \Lambda'') \xrightarrow{\tau} (\mu''', \nu''', \lambda''', M''', N''', \Lambda''')$$

eliminates $\nu$ from the transformed Hamiltonian. Finally,

$$(\mu''', \nu''', \lambda''', M''', N''', \Lambda''') \xrightarrow{\tau} (\mu^{\prime\prime\prime}, \nu^{\prime\prime\prime}, \lambda^{\prime\prime\prime}, M^{\prime\prime\prime}, N^{\prime\prime\prime}, \Lambda^{\prime\prime\prime})$$

eliminates $\lambda$ and the time variable $\tau$.

We arrive at a secular Hamiltonian

$$K = \sum_{i=0} \left( \sigma / i! \right) K_i$$

in which $K_{\text{even}} = 0$ and

$$K_0 = \frac{1}{2} M^a / C,$$

$$K_1 = K_0 \alpha s^2,$$

$$K_i = K_0 \beta^2 \left( \frac{1}{2} \right)^i, 3 \beta^2 \left( \frac{1}{2} \right)^i, 5 \beta^2 \left( \frac{1}{2} \right)^i,$$

$$K_i \beta^2 \left( \frac{1}{2} \right)^i$$

from which the triple primes have been dropped without risk of confusion. As expected, this Hamiltonian is the same as the one of the unperturbed case given in Appendix B of Ferrer & Lara (2010), except for the last summand of the term $K_1$ given above, which is related to the gravitational torque of the disturbing body.

The momenta $M, N$ and $\Lambda$ are constant, on average, and the secular frequencies result from the corresponding Hamilton equations. Thus, scaling these frequencies by the free rotation frequency, we get

$$\frac{\omega_0}{M/C} = 1 - \delta \left( \frac{n}{M/C} \right)^{2} \frac{3}{32} \left[ \frac{c^2 + (1 - 6c^2)}{c^2 + (1 - 3c^2)} \right] + \alpha
\times \left[ 1 - \frac{\beta^2}{2} \left( \frac{1 + 1}{c^2} \right) - \frac{\beta^4}{2} \left( \frac{9 + 11}{c^2} \right) - \frac{\beta^6}{2} \left( \frac{78 + 107}{c^2} \right) - \frac{\beta^8}{2} \left( \frac{144 + 130}{c^2} \right) - \frac{\beta^{10}}{2} \left( \frac{70 + 27}{c^2} \right) \right].$$

$$\frac{\omega_0}{M/C} = \delta \left( \frac{n}{M/C} \right)^{2} \frac{3}{32} \left( 1 - 3c^2 \right) c_j - \alpha c_j
\times \left[ 1 - \frac{\beta^2}{2} \left( \frac{3 + 1}{c^2} \right) - \frac{\beta^4}{2} \left( \frac{51 + 22}{c^2} \right) - \frac{\beta^6}{2} \left( \frac{192 + 195}{c^2} \right) - \frac{\beta^8}{2} \left( \frac{112 + 45}{c^2} \right) \right].$$

$$\frac{\omega_0}{M/C} = \delta \left( \frac{n}{M/C} \right)^{2} \frac{3}{32} \left( 1 - 3c^2 \right) c_j.$$
body (Lara et al. 2010) remain also in the triaxial case (see also Chernous’ko 1963). Thus, the terms related to the gravity-gradient perturbation vanish from equations (6)–(8) and the terms that remain are related to the free rigid body when formulated as a perturbed spherical rotor (cf. Ferrer & Lara 2010).

The short-period effects are recovered after undoing the transformation equations of each canonical transformation that has been performed. The corresponding equations can be computed from the generating functions in the Appendix.

4 NUMERICAL VALIDATION

To test the validity of the approximate analytical solutions, we run several examples and compare them with the numerical integration of the equations of motion, equations (2). For the initial values of $I$ and $J$ we take $I_0 = 3^\circ$, the accepted value of the tilt (Thomas et al. 2005), and $J_0 = 10^{-4}$ rad ($\sim 20$ arcsec), a speculative value that assumes a small difference between the figure and angular momentum axes, which are usually assumed to match within the precision of existing observations (Drummond & Christou 2008). In addition, we take a value $b = 485.3$ for the intermediate axis of Ceres, that would be possible within the accuracy of actual observations.

For our computations we use internal units. The time is measured in units of the rotational period of Ceres when expressed in Eulerian periods ($T(\alpha) = 1$), where $T$ is the rotational period of Ceres, and the units of length and mass are chosen so that $(2\pi/T) C = 1$ which scales the Andoyer momenta to the order of $1$.

We take the initial conditions $f_0 = v_0 = \nu_0 = 0.1, M_0 = 1, N_0 = M_0 \cos I_0$ and $\Lambda_0 = M_0 \cos f_0$. Using the transformation equations built with the generating functions given in the Appendix, the initial conditions in the new (triple primes) variables are

\[
\mu'''' = \mu_0 - 1.74265 \times 10^{-6}
\]

\[
v'''' = v_0 - 6.02266 \times 10^{-7}
\]

\[
\lambda'''' = \lambda_0 + 2.34811 \times 10^{-6}
\]

\[
M'''' = M_0 - 1.95807 \times 10^{-10}
\]

\[
N'''' = N_0 - 2.42546 \times 10^{-10}
\]

\[
\Lambda'''' = \Lambda_0 + 3.15743 \times 10^{-8}.
\]

We, then, propagate the secular equations

\[
\mu'''' = \mu''' + \omega_\mu t,
\]

\[
v'''' = v''' + \omega_v t,
\]

\[
\lambda'''' = \lambda''' + \omega_\lambda t
\]

for six Ceres’ orbital periods. After propagation, we select a large number of states for different times, and compare with a direct numerical integration of the equations of motion. As expected, the dominant terms in the errors are long- and short-periodic terms that are missed in the averaging procedure, and which mask the much lesser secular errors. As shown in Fig. 3, the long-period errors are clearly related to the orbital period of Ceres about the Sun. It should be noted that, although short-periodic errors in the angular variables may be important, they allow for predictions in the long term within an accuracy of about $\pm 1^\circ$ for $\mu$ and $\nu$, and about 2 arcsec for $\lambda$.

The analytical solution is not limited to the secular terms, and the short-and-long period terms may be recovered from the transformation equations. Thus, for each of the secular states we undo the three canonical transformations, and compare these transformed states with corresponding ones obtained from the numerical integration of the non-averaged equations of motion for the initial conditions $f = 0$ and $\mu_0, v_0, \nu_0, M_0, N_0, \Lambda_0$. Results are presented in Fig. 4, where we note that the momenta are recovered except for a small secular trend and periodic errors. Errors in $M$ and $N$ remain with very small values of $\sim 10^{-13}$ and only the case of $\Lambda$ is presented for the momenta. The angular variables predicted by the full analytical theory show a secular error of about 15 mas per orbital period for $\mu$ and $\nu$, that falls below the mas per orbital period in the case of $\lambda$, which is modulated by short period effects of very small amplitude.

The origin of the short- and medium-periodic errors is related to the truncation order of the analytical theory transformation equations. Thus, we see in the Appendix that the generating function...
$W_\nu$ of the transformation from primes to original variables depends on circular functions of the short-periodic arguments $\mu$ and $2\mu$, and the medium-periodic argument $2\nu$. Higher harmonics of $\nu$ also appear in the generating function $W_\nu$ of the transformation from second primes to primes variables, but since they are multiplied by corresponding powers of the sine of the angle $J$, their contribution should be negligible for Ceres. Therefore, only short-periodic errors with a period close to the rotation period or half of it are expected to be apparent, as well as medium-periodic terms with a period close to half the nutation period. As presented in Figs 5–7, that is exactly what we find.

Thus, the top plot of Fig. 5 shows periodic oscillations in the errors of $\Lambda$ with a period approximately equal to half the rotational period; almost identical behaviour has been found for the other momenta and corresponding plots are not presented. Errors in $\lambda$ also occur with the same period, but their amplitude is very small and, as shown in the bottom plot of Fig. 5, these short-periodic errors are dominated by the linear trend of the secular errors. In contrast, the higher amplitude of the errors in $\mu$ and $\nu$ have almost the same period as the rotation period, as illustrated with Fig. 6.

Fig. 7 shows medium-period errors that modulate the short-periodics with approximately half the period of the nutational motion. We only find these errors to be apparent in $\mu$ and $\nu$, and do not present plots for other variables. Note that long-periodic errors whose period is related to the orbital motion of Ceres about the Sun also start to be apparent in Fig. 7.

5 PLANETOLOGICAL IMPLICATIONS

We note that the pair of variables $(\nu, J)$ describe the rotation of the angular momentum vector around the body’s axis of maxima inertia, so, except for short-period effects, the frequency $\omega_0$ is closely related to the wobble. Moreover, assuming that Ceres’ rotation happens very close to the figure axis, terms of the order of $J^2$ may be neglected, hence the expressions (Kinoshita 1977)

$\phi = \lambda + \frac{J}{\sin I} \sin \mu,$

$\theta = I + J \cos \mu,$

$\psi = \nu + \mu - J \cot I \sin \mu,$

establish a first-order relation between Andoyer variables and Euler’s angles. Then, on average

$\langle \dot{\phi} \rangle_\mu = \langle \dot{\lambda} \rangle_\mu,$

$\langle \dot{\theta} \rangle_\mu = \langle I \rangle_\mu,$

$\langle \dot{\psi} \rangle_\mu = \langle \nu \rangle_\mu + \langle \mu \rangle_\mu,$

where dots mean differentiation with respect to time. Therefore, we find that $\omega_0 \approx \langle \dot{\lambda} \rangle_\mu$ carries the secular terms of the precession, while, on average, $\omega_0 \approx \langle \dot{\psi} \rangle_\mu = \omega_0$ accounts for most of the rotation.

To clearly show the influence of the triaxiality of Ceres in the secular frequencies of the motion, we calculate the corresponding periods by allowing the intermediate momentum of inertia $B$ to vary between both extreme values $A$ and $C$. Results are presented in Fig. 8, where we note that the period of Ceres’ proper rotation varies almost linearly with $\beta$ (top plot of Fig. 8) with a maximum value of 8.977 h for $\beta = 1$, accounting for most of the full observed rotation of 9.074 h, and with a minimum value of 8.485 h for the oblate case $\beta = 0$. The wobble’s period evolves non-linearly in a different time-scale, reaching a minimum of 5.445 d for $\beta = 0$, and a maximum of 34.849 when $\beta = 1$ (centre plot of Fig. 8). Finally, the precessional motion evolves in a totally different time-scale of thousands of centuries, as presented in the bottom plot of Fig. 8, with an apparent linear proportionality to the triaxiality coefficient.

Note, however, that the validity of this analysis is constrained by the model itself. As our theory has been built on the assumptions of the perturbed spherical rotor, the discussion of curves presented above only makes sense for a range of $\beta$ values consistent with the model.

In addition, Fig. 9 shows the dependence of the whole Ceres’ rotation on its possible triaxiality. The differences are of the order of the gravity-gradient perturbation and evolve linearly with $\beta$. 

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solution directly. Alternatively, if the spin and wobble periods are determined from Dawn observations, they may be used to solve the rotation solution for the triaxiality coefficient, thus providing a more accurate constraint on the Ceres' moments of inertia.

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CONCLUSIONS

It may be that Ceres is not actually as axisymmetrical as supposed, and that it is affected by a very small triaxiality. With the current exactness of available observations a conclusion regarding this cannot be reached. As this article shows, minute deviations in axisymmetry affect the rotational movement of Ceres. The study of uncertainties regarding the shape of Ceres along with the improvement suggested in this article, and based on an analysis of the rotation, has two characteristics. On the one hand, it is carried out in a purely analytical way that is convenient for data handling. On the other hand, the perturbed spherical rotor approach eases the computation of the Ceres’ rotational state directly in Andoyer variables.

Some of the scientific objectives of NASA’s Dawn mission are to determine the volume, shape, spin state and mass of Ceres. Observation will start approximately a month prior to a capture at Ceres. Dawn will use the on-board High Resolution Stereo Camera Image Composite to observe the landmarks on Ceres and determine its rotational property. Since the proposed theory for Ceres’ rotation is purely analytical, the new data, which are expected to improve on current data by several orders of magnitude, will enter the rotation
APPENDIX A: LIE SERIES TRANSFORMATIONS

The three canonical transformations carried out to eliminate from the Hamiltonian first \( \mu \), then \( \nu \), and finally \( \lambda \) together with the time, are derived from three generating functions of the type

\[
W = \sum_{i \geq 0} \frac{1}{i!} W_{i+1}.
\]

The corresponding transformation equations are computed from the so-called ‘Lie triangle’ that is successively constructed based on the recurrence

\[
x_{i,j} = x_{i+1,j+1} + \sum_{0 \leq k \leq i} \binom{i}{k} \left\{ x_{i-k,j}; W_{i+k-j} \right\},
\]

where \( x_{0,0} = x, x_{i,0} = 0 \) for \( i > 0 \), \( x \) stands for any of the Andoyer variables, and the operator \( \{ \} \) is the Poisson bracket. Full details on the application of Lie series to perturbation theory can be found in Deprit (1969).

For the convenience of interested readers, we provide the three necessary generating functions.

Elimination of \( \mu \)
Transformation \((\mu, \nu, \lambda, M, N, \Lambda) \to (\mu', \nu', \lambda', M', N', \Lambda')\)

We get \( W_{\mu,i} = 0 \) for \( i = 1, \ldots, 6 \) and,

\[
\frac{W_{\mu,5}}{71M^3} = \frac{3}{32} \frac{n^2}{(M/C)^2} \frac{a^3}{r^3} \frac{d}{\beta} \left\{ s_j^5 \left[ (1 - c_j)^2 \sin(2\theta - 2\mu) - 2s_j \sin 2\mu - (1 + c_j)^2 \sin(2\theta + 2\mu) \right] - 8c_j s_j s_k \sin(2\theta - \mu) - 2c_k \sin \mu + (1 + c_j) \sin(2\theta + \mu) \right\}
\]

\[
\frac{W_{\mu,8}}{81M^4} = \frac{3}{128} \frac{n^2}{(M/C)^2} \frac{a^3}{r^3} \times \left\{ - e \left\{ (1 - c_j)^2 \sin(2\theta + 2\mu - 2\nu) + 2s_j \sin(2\theta - 2\nu) - (1 - c_j)^2 \sin(2\theta - 2\mu + 2\nu) + (1 + c_j)^2 \sin(2\theta + 2\mu + 2\nu) + 2s_j \sin(2\theta + 2\nu) - (1 - c_j)^2 \sin(2\theta - 2\nu) - 2c_k \sin(2\theta - \mu) + (1 + c_j) \sin(2\theta + \mu) - 8s_j s_k (1 - c_j) \sin(2\theta - \mu - 2\nu) - 2c_k \sin(2\theta - 2\nu) - 2c_k \sin(2\theta + 2\nu) - (1 - c_j)^2 \sin(2\theta + 2\mu) + 2s_j \sin 2\mu - (1 - c_j)^2 \sin(2\theta - 2\mu) + 8s_j s_k c_j \sin(2\theta - \mu) + (1 - c_j) \sin(2\theta + \mu) + (1 + c_j) \sin(2\theta + \mu) \right\} \right\}
\]

where we alleviated notation by omitting primes.

Elimination of \( \nu \)
For the transformation

\((\mu', \nu', \lambda', M', N', \Lambda') \to (\mu'', \nu'', \lambda'', M'', N'', \Lambda'')\)

we get the generating function of coefficients

\[
W_{\nu,1} = \frac{\beta}{2} s_j^2 \frac{c_j}{r} \sin 2\nu
\]

\[
W_{\nu,2} = -\frac{\beta^2}{2} s_j^2 \frac{c_j}{r} \sin 4\nu
\]

\[
W_{\nu,3} = \frac{\beta^3}{2} \left[ \left( \frac{29 - 2c_j}{c_j} + \frac{5}{c_j} \right) s_j^2 \sin 2\nu + \frac{s_j^2}{c_j} \sin 6\nu \right]
\]

\[
W_{\nu,4} = -\frac{3\beta^4}{2} \left[ \left( \frac{9 - 2c_j}{c_j} + \frac{9}{c_j} \right) s_j^2 \sin 4\nu + \frac{5s_j^2}{8c_j} \sin 8\nu \right]
\]

\[
W_{\nu,5} = \frac{3\beta^5}{2} \left[ \left( 219 - 24 \frac{c_j}{c_j} + 70 \frac{c_j}{c_j} - 48 \frac{c_j}{c_j} + 39 \frac{c_j}{c_j} \right) s_j^2 \sin 2\nu + \left( 87 - \frac{30}{c_j} + \frac{199}{c_j} \right) s_j^2 \sin 6\nu + \frac{7}{c_j} \sin 10\nu \right]
\]

\[
W_{\nu,6} = -\frac{15\beta^6}{2} \left[ \left( 2361 - 900 \frac{c_j}{c_j} + 4630 \frac{c_j}{c_j} - 3716 \frac{c_j}{c_j} + 3769 \frac{c_j}{c_j} \right) s_j^2 \sin 4\nu + \left( 16 - \frac{7}{c_j} + \frac{62}{c_j} \right) s_j^2 \sin 8\nu + \frac{21}{c_j} \frac{s_j^2}{c_j} \sin 12\nu \right]
\]

from which the double primes have been omitted.

Elimination of \( \tau \) and \( \lambda \)
Transformation

\((\mu'', \nu'', \lambda'', M'', N'', \Lambda'') \to (\mu''', \nu''', \lambda''', M''', N'''', \Lambda''')\)

We get \( W_{\tau,j} = 0 \) for \( j = 1, \ldots, 4 \) and \( 6 \) and, omitting triple primes,

\[
W_{\tau,5} = \frac{n}{51M/C} \frac{\delta}{16\pi c_j} \left( 1 - 3c_j \right) \left\{ -6 \sin(2\tau + \lambda - t + e \sin f) \right\}
\]

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