Faraday conversion and rotation in uniformly magnetized relativistic plasmas

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ABSTRACT

We provide precise fitting formulae for Faraday conversion and rotation coefficients in uniformly magnetized relativistic plasmas. The formulae are immediately applicable to rotation measure and circular polarization (CP) production in jets and hot accretion flows. We show the recipe and results for arbitrary isotropic particle distributions, in particular thermal and power law. The exact Faraday conversion coefficient is found to approach zero with the increasing particle energy. The non-linear corrections of Faraday conversion and rotation coefficients are found essential for reliable CP interpretation of Sagittarius A*.

Key words: plasmas – polarization – radiation mechanisms: general – radiative transfer – Galaxy: centre.

1 INTRODUCTION

Cyclosynchrotron emission, also called magneto-bremsstrahlung emission, is one of the most important radiative mechanisms in astrophysics. It is believed to produce radio emission in the centres of active galactic nuclei (AGNs) and low-luminosity AGNs [such as the Galactic Centre (GC)]. The polarized nature of the cyclosynchrotron emission is of increasing interest for radio observers. With the help of polarization, one can understand the magnetic field structure in radio sources. The basic theory of the emission and propagation of polarized light has been established (e.g. Legg & Westfold 1968; Pacholczyk 1970; Jones & O’Dell 1977). Particles in cold plasmas emit cyclotron radiation, which is circularly polarized. When linearly polarized light propagates through cold magnetized plasmas, it undergoes Faraday rotation. In turn, relativistic plasmas emit synchrotron radiation, which is linearly polarized. Light traversing relativistic plasmas undergoes both Faraday conversion and Faraday rotation.

In simple theory, the strength of the Faraday rotation effect is proportional to $\lambda^2 n_e B \cdot \delta l$, where $\lambda$ is the photon wavelength, $n_e$ is the electron density, $B$ is the magnetic field vector and $\delta l$ is the displacement along the line of sight. However, Trubnikov (1958) and Melrose (1997a) have shown that, in a general case, Faraday rotation of plasmas depends also on the Lorentz factor of electrons, $\gamma$. Faraday rotation weakens with the increase in $\gamma$ as $\ln \gamma^2$. The electric vector position angle (EVPA) of linearly polarized light will be preserved better, if the electrons are relativistic. Thus, we can infer the intrinsic EVPA of a synchrotron-emitting source. The electrons in the vicinity of the black hole Sagittarius A* (Sgr A*) in the GC are often modelled by a relativistic Maxwellian (thermal) distribution with temperature $>10^{16}$ K (e.g. Yuan, Quataert & Narayan 2003). When Faraday rotation is strong and non-uniform across the beam, then beam depolarization can diminish the resultant linear polarization (LP) fraction (Bower et al. 2005). The effect of large Lorentz factors on Faraday rotation measure near Sgr A* must be considered to explain the detected linearly polarized fluxes in submillimetre to near-infrared (NIR) bands (Quataert & Gruzinov 2000). Previous work (e.g. Ginzburg 1964; Sazonov 1969; Melrose 1997b) also highlighted Faraday conversion, or generalized Faraday rotation. This quantity describes the interconversion between linearly polarized and circularly polarized light. The Faraday conversion effect is normally expected (Homan et al. 2009) to convert emitted linearly polarized light into circularly polarized light during propagation in a magnetized medium. This is the likely cause of the observed high circular polarization (CP) fraction of the Sgr A* spectrum in the radio band (Bower et al. 2002) and submillimetre band (Munoz et al. 2011).

The strength of Faraday conversion was found proportional to $\lambda^3 n_e B^2 \sin^2 \theta \delta l$, where $\theta$ is the angle between $B$ and $\delta l$ (or $k$, see in Fig. 1). Additional suggested proportionality to the electron temperature, $T_e$, makes Faraday conversion reach very large values in relativistic plasmas. However, this proportionality ceases at very high $T_e$ and Faraday conversion measure approaches zero (Shcherbakov 2008). A detectable CP fraction can be generated near Sgr A* in the submillimetre band (e.g. Ballantyne, Ozel & Psaltis 2007), but a precise treatment

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of Faraday rotation and conversion is essential (Shcherbakov, Penna & McKinney 2010). A non-zero CP fraction is already detected with the Submillimeter Array (Munoz et al. 2011) at 230 and 345 GHz. This by itself points in the direction of a very hot radiatively inefficient accretion flow (RIAF). The origin of CP near Sgr A* and jets was quantified by Beckert (2003) and Shcherbakov et al. (2010). Yet more detailed and accurate calculations are needed to quantify the CP for the GC supermassive black hole and other radio sources.

Huang et al. (2009a) was the first to incorporate Faraday rotation and conversion within a general relativistic polarized radiative transfer framework, though with some approximations. Shcherbakov & Huang (2011, hereinafter Paper I) provided a method to accurately calculate cyclosynchrotron absorption and Faraday conversion/rotation for electrons in an isotropic thermal distribution and outlined the exact procedure for polarized radiative transfer in General Relativity. Precise Faraday rotation and conversion coefficients were computed earlier for thermal plasmas in Shcherbakov (2008). An important cornerstone in computing propagation effects is linear approximation, in which only the first non-zero terms in series expansion in the $\Omega_0/\omega$ ratio are taken as corresponding quantities (e.g. formula 47). It was also derived therein and found consistent with the result provided by Melrose (1997b). The precise values of Faraday conversion coefficients in Shcherbakov (2008) and Paper I match the linear approximation from the cold to weakly-relativistic regimes of thermal plasmas. In a relativistic regime, Faraday conversion largely deviated from the linear approximation, because it breaks at a finite ratio of $\Omega_0/\omega$. Here $\Omega_0 = eB/(mc)$ is the cyclotron frequency and $\omega$ is the radiation frequency. Due to the lack of full investigation of various electron distributions, an important question was left unanswered: is thermal distribution special or such behaviour of Faraday conversion is generic?

In this paper, we expand the computations of Faraday conversion and rotation coefficients to non-thermal particle distributions. We compute the absorption coefficients as well with the same unique method. We find solutions of the wave equation and natural modes from the cold limit to ultrarelativistic limit. Our formulae are precise at all reasonable particle $\gamma$ values. We find a large discrepancy, if the linear approximations to Faraday conversion and rotation are used, thus justifying the need for precise computations. To be practical, we adopt $\delta$-function energy distribution of electrons and provide the fitting formulae, which can then be integrated over any isotropic distribution of particles. We also provide a public code in MATHEMATICA v8 to numerically compute the integrals.

This paper is organized as follows. We derive dielectric tensor and dispersion relations for uniformly magnetized relativistic plasmas with an isotropic monoenergetic particle distribution in Section 2. The properties of natural modes are investigated in Section 3. In Section 4, we provide simplified formulæ and generalize to arbitrary electron energy distributions. Once its polarization prediction is described, in Section 5, we show that the polarized spectrum changes a lot when linear approximations are used for Sgr A* modelling.

2 RESPONSE TENSOR AND EIGENMODES OF UNIFORMLY MAGNETIZED RELATIVISTIC PLASMAS WITH $\delta$-FUNCTION ENERGY DISTRIBUTION

2.1 Geometry and definitions

Let us define a coordinate system in three-dimensional flat space with $e_1$ along the major-axis of the synchrotron radiation ellipse, $e_2$ along the minor-axis and $e_3$ towards the observer (see Fig. 1). Vector $e_3$ is perpendicular to $B$. This coordinate system is right-handed, that is, the observer finds a counterclockwise rotation from $e_1$ to $e_2$. We define Minkowski space-time with basis $e_\mu$, so that $e_0 = (-1, 0, 0, 0)$ and $e_j = (0, e_j), j = 1, 2, 3$. Unlike in Paper I, here, we perform all derivations in spatial basis ($e_1, e_2, e_3$), instead of ($\hat{e}_1, \hat{e}_2, \hat{e}_3$). In the latter basis, $\hat{e}_1 = e_1$ and $\hat{e}_3 \parallel B$. We set the speed of light to unity, $c = 1$. 

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Normalized vectors of electric field in a transverse wave are
\[ \hat{E}_1 = e_1 e^{i(kX)} \quad \text{and} \quad \hat{E}_2 = e_2 e^{i(kX)}, \tag{1} \]
where \( k_\mu = \omega(-1, 0, 0, 1)^T \) is the covariant photon momentum, \( X^\mu \) are four-coordinates and \( kX \) represents the inner product \( k \cdot X = k_\mu X^\mu \). The projections of the electric field \( E \) of an arbitrary wave along these unit vectors have complex amplitudes \( A^{1,2} \), or real amplitudes \( A^{1,2} \) and phases \( \delta^{1,2} \), so that
\[ E = E_1 + E_2 = \hat{A}^1 \hat{E}_1 + \hat{A}^2 \hat{E}_2 = A^1 e^{i\delta^1} \hat{E}_1 + A^2 e^{i\delta^2} \hat{E}_2. \tag{2} \]

The tensor of intensity is defined as
\[ I_{ij} = \langle E_i \cdot \text{conj}[E_j]\rangle, \tag{3} \]
where \( \text{conj}[\ldots] \) denotes complex conjugate and \( \langle \ldots \rangle \) represents the average over the wave ensemble.

Radiative transfer in a uniform medium is described by the equation (Sazonov 1969)
\[ \frac{d}{d\xi} I_{ij}(\xi) = \varepsilon_{ij} + i(\alpha^{*\mu} \pi_m \mu - \alpha^{*\mu} I_{im}), \tag{4} \]
where \( \xi \) is the distance along the ray, \( \varepsilon_{ij} \) is the tensor of spontaneous emission and \( \alpha^{*\mu} \) is the tensor of wave propagation, or the response tensor. Equation (4) can be rewritten in a more familiar form (Paper I):
\[ \frac{d\mathbf{S}}{d\xi} = \begin{pmatrix} \varepsilon_t \\ \varepsilon_q \\ 0 \\ \varepsilon_V \end{pmatrix} - \begin{pmatrix} \eta_t & \eta_q & 0 & \eta_V \\ \eta_q & \eta_t & \rho_V & 0 \\ 0 & -\rho_V & \eta_t & \rho_q \\ \eta_V & 0 & -\rho_q & \eta_t \end{pmatrix} S \tag{5} \]

with the polarization vector
\[ S = (I, Q, U, V)^T \tag{6} \]
being the vector of Stokes parameters. The intensities \( S \) and integrated polarized fluxes can be directly observed. Here \( \varepsilon_t, \varepsilon_q \) and \( \varepsilon_V \) are the emission coefficients,
\[ \eta_t = \text{Im}(\alpha^{22} + \alpha^{11})/v, \quad \eta_q = \text{Im}(\alpha^{11} - \alpha^{22})/v, \quad \eta_V = 2\text{Re}(\alpha^{12})/v, \tag{7} \]
are the absorption coefficients, and
\[ \rho_V = 2\text{Im}(\alpha^{12})/v, \quad \rho_q = \text{Re}(\alpha^{22} - \alpha^{11})/v, \tag{8} \]
where \( v = \omega/(2\pi) \) is the frequency. In the following, we will concentrate mainly on the Faraday rotation coefficient \( \rho_V \) and Faraday conversion coefficient \( \rho_q \), which are generally called propagation coefficients. They directly influence the observed polarized fluxes.

### 2.2 Response tensor and dispersion relations

We start with the formula for the response tensor for isotropic electrons:
\[ \alpha^{\mu\nu}(k) = -\frac{e^2}{m_ec} \int d^3 p \frac{d f(y)}{d^3 p} \vec{U}^\mu \vec{U}^\nu + \frac{ie^2 \omega}{m_ec} \int d^3 p \int_0^\infty d\xi \xi (\xi - \varepsilon) \left[ \frac{\partial^2}{\partial S_\mu \partial S_\nu} \int d^4 p \left( -\frac{1}{y} \frac{d f(y)}{d^4 p} e^{-iR(\xi u + S)} \right) \right]_{R = 0}, \tag{9} \]

where \( \vec{U}^\mu = (1, 0, 0, 0), \) \( U^\mu \) is 4-velocity of electrons in the observer’s Minkowskian frame, \( p \) is dimensionless 3-momentum defined as \( p = \sqrt{y^2 - 1} \) and \( f(y) \) is the energy distribution function of electrons. It is normalized to the number density of electrons \( n_e \) as
\[ \int f(y) d^3 p = n_e. \tag{10} \]

Tensor \( R^{\mu\nu}(\xi) \) describes how the velocity of electrons changes with proper time \( \xi \), \( R(\xi u + S) \) represents the difference in phase \( (kX) \) of the electron and \( S_\mu \) is an auxiliary variable. Formula (9) coincides with equation (2.3.11) in Melrose (2008) and with equation (19) in Melrose (1997a), except for \( R^{\mu\nu}(\xi) \), because we defined a different coordinate system. Equation (35) in Paper I offers a similar expression derived in the three-dimensional space with basis \( (\hat{e}_1, \hat{e}_2, \hat{e}_3) \). The derivation of equation (9) and the related definitions can be found in Appendices A and B.

The 4-vectors in expression (9) can be split into temporal and spatial parts as
\[ R_\mu = (-\omega \xi, R_\bot), \quad U_\mu = (\gamma, u^\bot), \quad S_\mu = (s_0, s_\bot). \tag{11} \]
The phase then becomes $-i R(\xi, \nu) + S \nu = i a_0 \xi \gamma - i R_j \nu^j + s_0 \nu + s_\nu = i (a_0 - i s_0) \nu - i (R_j + i s_j) \nu^j$. The momentum integral in the response tensor is

$$
\int d^3 p \left[ -\frac{1}{\gamma} \frac{df(\gamma)}{d\gamma} e^{i \omega_\nu (\xi - i s_0) \gamma \nu - i (R_j + i s_j) \nu^j} \right] = \int_{\gamma_{\nu \min}}^{\gamma_{\nu \max}} d\gamma \frac{|d\nu|}{|\nu|} \frac{1}{\gamma} \frac{df(\gamma)}{d\gamma} \int_0^{2 \pi} e^{-i R \nu \cos \theta} \sin \theta d\theta d\phi_p
$$

where

$$
I(\xi, S) = - \frac{I(\gamma, \xi, S)}{\sin \left( \frac{|R + is| \sqrt{\gamma^2 - 1}}{|R + is|} \right)} \sin \left( \frac{|R + is| \sqrt{\gamma^2 - 1}}{|R + is|} \right)
$$

and

$$
A(\gamma, \xi, S) = 4 \pi \frac{\sin \left( \frac{|R + is| \sqrt{\gamma^2 - 1}}{|R + is|} \right)}{|R + is|} e^{i \omega_\nu (\xi - i s_0) \gamma \nu - i (R_j + i s_j) \nu^j}.
$$

The absolute values are taken as $|R + is| = \sqrt{(R^j + is^j)(R_j + is_j)}$ and $|\nu| = \sqrt{\nu^j \nu_j}$. Note that for any distribution with $f(\gamma_{\min} A(\gamma_{\min}) \to 0$ and $f(\gamma_{\max} A(\gamma_{\max}) \to 0$, the second term in the last expression of equation (12) vanishes.

As the first step of the calculation for an arbitrary distribution of electrons, we use a $\delta$-function as the distribution:

$$
f(\gamma) = \delta(\gamma - \gamma_0).
$$

Then $I(\xi, S)$ becomes $I(\gamma_0, \xi, S)$ as

$$
I(\gamma_0, \xi, S) = e^{i \omega_\nu (\xi - i s_0) \gamma \nu - i (R_j + i s_j) \nu^j} \frac{\sin \left( \frac{|R + is| \sqrt{\gamma^2 - 1}}{|R + is|} \right)}{|R + is|} + \frac{\gamma_0}{\gamma} \cos \left( \frac{|R + is| \sqrt{\gamma^2 - 1}}{|R + is|} \right)
$$

where

$$
p_0 = \sqrt{\gamma_0 - 1}
$$

is the dimensionless momentum.

Now we apply the differential operator $\partial^2 / (\partial S_\mu \partial S_\nu)$ to $I(\gamma_0, \xi, S)$, set $S_\mu = 0$, and only choose $(\mu, \nu) = (1, 2)$ to isolate the transverse wave component. The final expression of the $2 \times 2$ response tensor is

$$
\alpha^{ij}(k, \gamma_0) = \frac{4 \pi e^2}{m c} \int_0^\infty d(\omega_\nu) e^{i \omega_\nu \gamma_0} \left[ i^{ij}(\xi) (\xi_0 \Pi_1 + \Pi_2 - \tilde{T}^{ij}(\xi) (\xi_0 \Pi_2 + \Pi_1) \right],
$$

where

$$
\Pi_1 = \frac{\sin (R(\xi) p_0)}{R^3(\xi)} - \frac{\cos (R(\xi) p_0)}{R^3(\xi)},
Pi_2 = \frac{3 \sin (R(\xi) p_0)}{R^3(\xi)} - \frac{3 \cos (R(\xi) p_0)}{R^3(\xi)} - \frac{p_0^2 \sin (R(\xi) p_0)}{R^3(\xi)},
Pi_3 = \frac{\gamma_0}{R(\xi)},
Pi_4 = \frac{\gamma_0}{R(\xi)} - \frac{\gamma_0 p_0}{R^2(\xi)},
$$

and

$$
R(\xi) = \sqrt{R^2(\xi) R_0(\xi)} = \sqrt{\frac{a_0^2 \sin^2 \theta}{x_0^2} [2 - 2 \cos (\Omega_0 \xi)] + \cos^2 \theta a_0^2 \xi^2}],
$$

$$
i^{ij}(\xi) = \left( \begin{array}{cc}
\cos (\Omega_0 \xi) & - \cos \theta \sin (\Omega_0 \xi) \\
\cos \theta \sin (\Omega_0 \xi) & \sin^2 \theta + \cos^2 \theta \cos (\Omega_0 \xi) \\
\end{array} \right),
$$

$$
\tilde{T}^{ij}(\xi) = R^2(\xi) \tilde{R}(\xi) = \frac{\omega^2 \sin \theta}{\Omega_0^2} \left( \begin{array}{cc}
(1 - \cos (\Omega_0 \xi))^2 & - \cos \theta (\sin (\Omega_0 \xi) - \Omega_0 \xi) (1 - \cos (\Omega_0 \xi)) \\
- \cos \theta (\sin (\Omega_0 \xi) - \Omega_0 \xi) (1 - \cos (\Omega_0 \xi)) & \cos^2 \theta (\sin (\Omega_0 \xi) - \Omega_0 \xi)^2 \\
\end{array} \right).
$$

The differentiation of $A(\gamma; \xi, S)$ yields a response tensor boundary term as

$$
a^{ij}(k, \gamma_{\min \max}) = \frac{4 \pi e^2}{m c} \int_0^\infty d(\omega_\nu) e^{i \omega_\nu / \gamma_{\min \max}} \left[ i^{ij}(\xi) \Pi_1 - \tilde{T}^{ij}(\xi) \Pi_2 \right].
$$

Expression (17) is needed for distributions confined by cut-off Lorentz factors.

The dielectric tensor

$$
\varepsilon^{ij} = \delta^{ij} + \frac{4 \pi e^2}{\omega^2} a^{ij}
$$

leads to the wave equation

$$
\left( k^2 c^2 - \delta^{ij} - \varepsilon^{ij} \right) \left( \begin{array}{c}
\hat{E}_1 \\
\hat{E}_2
\end{array} \right) = 0.
$$

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The corresponding wave dispersion relation is
\[ k_x^2 c^2 = \omega^2 + 2\pi c \left[ \alpha^{11} + \alpha^{22} \pm \sqrt{ (\alpha^{11} - \alpha^{22})^2 + 4\alpha^{12}\alpha^{21} } \right], \]
where \( \alpha^{21} = -\alpha^{12} \) due to the Onsager principle (Landau & Lifshits 1980). Note that \( k_x^2 \) are real, when only \( \alpha^{ij} \) has the Hermitian part, while \( k_x^2 \) become complex, if \( \alpha^{ij} \) has both the Hermitian and the anti-Hermitian parts. The approximate relation \( k_x^2 c^2 \approx \omega^2 + (2\pi c)\text{Re}(\alpha^{11} + \alpha^{22}) \) helps to determine if the waves can propagate through a medium. For example, the number density \( n_e < 10^7 \) was estimated near Sgr A* (Yuan et al. 2003), so that \( k_x \) are dominated by their real parts for \( \omega > 100 \text{ MHz} \) and the waves can propagate.

3 NUMERICAL CALCULATION OF THE RESPONSE TENSOR AND THE ELLIPTICAL NATURE OF NATURAL MODES

3.1 Numerical calculation of the response tensor

We substitute the components of the response tensor from equation (16) and perform an integration over \( \xi \) in a complex plane in \textsc{mathematica} v8. The source code written in \textsc{mathematica} v8 can be found at http://astroman.org/Faraday_conversion/. We then find the propagation coefficients \( \rho_\nu \) and \( \rho_\sigma \) and absorption coefficients \( \eta_\nu, \eta_\sigma \) and \( \eta_\nu \) according to relations (7) and (8). We analogously compute the boundary terms by substituting the components of \( \alpha^{ij} \) from equation (17). Just like in Paper I, we do not perform the integration over \( \xi \) along the real axis. To accelerate the convergence, we integrate in a complex plane along the ray originating at \( \xi = 0 \) at a positive angle \( \psi \in (0, \pi/2) \) to the real axis. Angle \( \psi \) needs to be small enough in order to avoid crossing the branch points of \( \alpha^{ij} \). These branches are produced by zeros of \( \text{Re}(\xi) \). We integrate the full complex expressions to find the response tensor \( \alpha^{ij} \) and the boundary term. If \( \gamma_0 \) is small, then computations of anti-Hermitian parts involve substantial cancellations with the values of integrals close to zero. Thus, it is hard to reach good accuracy for absorptivity calculations with the method chosen, whereas the corresponding Hermitian parts easily converge. When \( \gamma_0 \) is larger than \( \sim 10 \), the values of the Hermitian-related part and anti-Hermitian-related part become comparable and all integrals converge.

We choose the fiducial model with \( \Omega_e/\omega = 10^{-4} \) and \( \theta = 45^\circ \) and plot in Fig. 2 propagation coefficients and absorption coefficients as functions of the Lorentz factor. In the left-hand panel, \( 2\text{Im}(\alpha^{12}) \propto \rho_\nu \) and \( -\text{Re}(\alpha^{11} - \alpha^{22}) \propto \rho_\sigma \), multiplied by \( (m_e c)/(4\pi e^2) \), are shown in the red and purple dotted lines, respectively. They both monotonically increase as \( \gamma_0 \) increases from 1 to \( \sim 60 \). As \( \gamma_0 \) increases further, the profile of \( \rho_\nu \) becomes flatter, while \( \rho_\sigma \) reaches its peak and decreases to negative values.

In the right-hand panel, \( \text{Im}(\alpha^{11} - \alpha^{22}) \propto \eta_\sigma \) and \( 2\text{Re}(\alpha^{12}) \propto \eta_\nu \), divided by \( (4\pi e^2/m_e c) \), are shown in the green and blue dotted lines, respectively. We also show \( (\omega/2\pi)\eta_\sigma \) and \( (\omega/2\pi)\eta_\nu \) as the long-dashed and dashed black lines, respectively. Here \( \eta_\sigma \) and \( \eta_\nu \) correspond to absorption and propagation coefficients for the fiducial model with parameters \( \Omega_e/\omega = 10^{-4} \) and \( \theta = 45^\circ \) for the monoenergetic electron distribution.
the integrals in Sazonov (1969)\(^1\)
\[
\eta'_Q = -\frac{v_B \sin \theta}{v} \sqrt{3e^2} \frac{\partial\gamma'}{\partial \gamma'} \int_1^\infty \frac{d\gamma'}{\gamma'^2} \left[ \frac{\delta(\gamma' - \gamma)}{\gamma'^2} \right] \frac{v}{v'_y} K_{2/3} \left( \frac{v}{v'_y} \right),
\]
\[
\eta'_\nu = -\frac{v_B \cos \theta}{v} \sqrt{3emc} \frac{\partial\gamma'}{\partial \gamma'} \int_1^\infty \frac{d\gamma'}{\gamma'^2} \left[ \frac{\delta(\gamma' - \gamma)}{\gamma'^2} \right] \frac{v}{v'_\nu} K_{1/3} \left( \frac{v}{v'_\nu} \right) + \int_{v'_\nu}^{\infty} dz K_{1/3}(z),
\]
where \(v_B\) is the cyclotron frequency, \(v_y = (3eB \sin \theta \gamma^2)/(4\pi mc)\) is the characteristic frequency and \(K_\alpha\) is the modified Bessel function of the second kind of order \(\alpha\). We integrate by parts to deal with the differential of the \(\delta\)-function. Note that our absorptivities deviate a lot from the correct values, when \(\gamma_0\) is small. As mentioned earlier, this is due to large cancellations of the parts of the integral, so that absorptivities cannot easily converge. When \(\gamma_0 > 30\), they coincide with the approximate expressions.

Despite inaccuracy at low Lorentz factors, the calculation clearly shows an important property of plasma absorption. The absorption in the \(Q\)-component is smaller, compared to that in the \(V\)-component, \(\eta_Q < |\eta_V|\), at low Lorentz factors, while \(\eta_Q > |\eta_V|\) at high Lorentz factors. This shows that the radiation mechanism changes from a CP-dominated cyclotron to LP-dominated synchrotron as the Lorentz factor increases. The traditional approximations do not exhibit this property, because they assume high Lorentz factors.

### 3.2 The axial radios and natural modes

The corresponding eigenvectors from the wave equation are \((\tilde{T}^+, i)^T\) and \((\tilde{T}^-, i)^T\), respectively, where
\[
\tilde{T}^+ = \frac{\alpha_{11} - \alpha_{22} + \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}^2\alpha_{21}^2}}{2i\kappa_{12}},
\]
\[
\tilde{T}^- = \frac{\alpha_{11} - \alpha_{22} - \sqrt{(\alpha_{11} - \alpha_{22})^2 + 4\alpha_{12}^2\alpha_{21}^2}}{2i\kappa_{12}}.
\]
These \(\tilde{T}^+\) and \(\tilde{T}^-\) are complex axial ratios of the transverse wave. They obey the relation
\[
\tilde{T}^+ \tilde{T}^- = -1.
\]

Polar decomposition into real amplitudes \(T^+(\geq 0)\) and phases \(\varphi^\pm\) reads
\[
\tilde{T}^+ = T^+ e^{\varphi^+} \quad \text{and} \quad \tilde{T}^- = T^- e^{\varphi^-},
\]
so that
\[
T^+ T^- = 1 \quad \text{and} \quad \varphi^+ + \varphi^- = -\pi.
\]

These two wave eigenvalues define two natural wave modes as
\[
\hat{E}_+ = \frac{\tilde{T}^+ \hat{E}_1 + i\hat{E}_2}{\sqrt{1 + \tilde{T}^+ \text{conj}[\tilde{T}^+]}} = \frac{T^+ e^{\varphi^+} \hat{E}_1 + i\hat{E}_2}{\sqrt{1 + (T^+)^2}},
\]
\[
\hat{E}_- = \frac{\tilde{T}^- \hat{E}_1 + i\hat{E}_2}{\sqrt{1 + \tilde{T}^- \text{conj}[\tilde{T}^-]}} = \frac{T^- e^{\varphi^-} \hat{E}_1 + i\hat{E}_2}{\sqrt{1 + (T^-)^2}}.
\]

In practice, \(\varphi^+ < -\pi/2\) and \(\varphi^- > -\pi/2\). Thus, the electric vector \(\hat{E}_+\) rotates counterclockwise and the electric vector \(\hat{E}_-\) rotates clockwise, as seen by the observer. Note that
\[
\hat{E}_+ \cdot \text{conj}[\hat{E}_+] = \frac{e^{\varphi^+ - \varphi^-} + 1}{T^+ + T^-} \begin{cases} 0, & \text{if } \varphi^+ = -\pi \\ \neq 0, & \text{otherwise}, \end{cases}
\]
that is, \(\hat{E}_+\) and \(\hat{E}_-\) are not perpendicular to each other, unless the anti-Hermitian part, absorption, can be neglected.

### 3.3 Properties of radiation in natural modes

Let us decompose the natural modes along \(\hat{E}_1\) and \(\hat{E}_2\) as
\[
E^+_1 = \frac{A^+ T^+ e^{\varphi^+}}{\sqrt{1 + (T^+)^2}} \hat{E}_1 \quad \text{and} \quad E^+_2 = \frac{iA^+}{\sqrt{1 + (T^+)^2}} \hat{E}_2.
\]

\(^1\) The sign of \(\eta_V\) is opposite to that in Sazonov (1969), if the current IAU/IEEE definition of the sign of CP is followed.

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We define the Stokes parameters for natural modes as

\[ I_0^\pm = \langle E_1^\pm \cdot \text{conj} [E_1^\pm] \rangle + \langle E_2^\pm \cdot \text{conj} [E_2^\pm] \rangle = (A^\pm)^2, \]

\[ Q_0^\pm = \langle E_1^\pm \cdot \text{conj} [E_1^\pm] \rangle - \langle E_2^\pm \cdot \text{conj} [E_2^\pm] \rangle = (A^\pm)^2 \frac{(T^\pm)^2 - 1}{1 + (T^\pm)^2}, \]

\[ U_0^\pm = \langle E_1^\pm \cdot \text{conj} [E_2^\pm] \rangle + \langle E_2^\pm \cdot \text{conj} [E_1^\pm] \rangle = (A^\pm)^2 \frac{2T^\pm \sin \varphi^\pm}{1 + (T^\pm)^2}, \]

\[ V_0^\pm = i \left( \langle E_1^\pm \cdot \text{conj} [E_2^\pm] \rangle - \langle E_2^\pm \cdot \text{conj} [E_1^\pm] \rangle \right) = (A^\pm)^2 \frac{2T^\pm \cos \varphi^\pm}{1 + (T^\pm)^2}. \]

(29)

Note that \( X_0 \neq X_0^+ + X_0^- \) for \( X_0 = I_0, Q_0, U_0, V_0 \), since \( \langle \dot{E}_s \cdot \text{conj} [\dot{E}_s] \rangle \neq 0 \) unless \( \varphi^+ = -\pi \). These Stokes parameters correspond to elliptically polarized radiation with ellipticity \( \beta \):

\[ \beta^\pm = \frac{1}{2} \sin^{-1} \frac{V_0^\pm}{I_0^\pm} = \frac{1}{2} \sin^{-1} \left( \frac{2T^\pm \cos \varphi^\pm}{1 + (T^\pm)^2} \right) \]

(30)

and the EVPAs

\[ \chi^\pm = \frac{1}{2} \tan^{-1} \frac{U_0^\pm}{Q_0^\pm} = \frac{1}{2} \tan^{-1} \left( \frac{2T^\pm \sin \varphi^\pm}{(T^\pm)^2 - 1} \right). \]

(31)

The relations between the axial ratios and the Stokes parameters for each mode are

\[ \frac{(R_{\text{CP}}^+)^2}{(R_{\text{LP}}^+)^2} = \left( \frac{V_0^+}{Q_0^+} \right)^2 + \left( \frac{U_0^+}{Q_0^+} \right)^2 = \frac{(R_{\text{CP}}^-)^2}{(R_{\text{LP}}^-)^2} = \left( \frac{V_0^-}{Q_0^-} \right)^2 + \left( \frac{U_0^-}{Q_0^-} \right)^2 = \left( \frac{2}{|T^+ + T^-|^2} \right)^2 |\text{Re}(\tilde{T}^+)| |\text{Re}(\tilde{T}^-)|. \]

(32)

We define a special quantity

\[ Z^\pm = \frac{R_{\text{CP}}^\pm}{R_{\text{LP}}^\pm} \frac{\rho_0}{\rho_\nu}, \]

which effectively measures how well the normal modes can be described, if we ignore absorption. We show amplitudes \( T^\pm(\geq 0) \), phases \( \varphi^\pm \), ellipticities \( \beta^\pm \) and EVPAs \( \chi^\pm \) in Fig. 3. All quantities with the superscript \( (\pm) \) are shown in the solid lines and those with the superscript \( (\mp) \) are shown in the dashed lines.

### 3.3.1 Cold plasma limit

In cold plasmas, the anti-Hermitian part of the response tensor can be neglected compared to the Hermitian part. In this case, \( \varphi^+ = -\pi \), \( \varphi^- = 0 \) (see the grey lines in the top left-hand panel of Fig. 3), \( U_0^+ = U_0^- = 0 \) and \( |\rho_\nu| \gg |\rho_\nu| \). The axial ratios are

\[ \tilde{T}^+ = \frac{1 - \sqrt{1 + (\rho_\nu/\rho_\nu)^2}}{\rho_\nu/\rho_\nu} = -T^+ \approx -1. \]

\[ \tilde{T}^- = \frac{1 + \sqrt{1 + (\rho_\nu/\rho_\nu)^2}}{\rho_\nu/\rho_\nu} = T^- \approx 1. \]

(34)

They are consistent with equation (4.6) in Melrose (1997b). The corresponding ellipticities \( \beta \) and EVPAs are

\[ \beta^+ = -\beta^- = \frac{1}{2} \sin^{-1} \left( -\frac{2}{T^+ + T^-} \right) \approx -\pi/4. \]

\[ \chi^+ = -\pi/2, \quad \chi^- = 0. \]

(35)

Two natural modes are circularly polarized. They are orthogonal with the major- and minor-axes aligned with \( e_1 \) and \( e_2 \), respectively. The relation between the Stokes parameters and the axial ratios becomes

\[ \frac{R_{\text{CP}}^+}{R_{\text{LP}}^+} = \frac{R_{\text{CP}}^-}{R_{\text{LP}}^-} = \frac{V_0^-}{Q_0^-} = \frac{2}{T^+ + T^-} = \frac{\rho_\nu}{\rho_\nu}, \quad \text{that is,} \quad Z^\pm = 1. \]

(36)

This means the ratio of circular to linear radiation intensities in cold plasma eigenmodes equals the ratio of Faraday rotation to Faraday conversion coefficients. The total emission is dispersionless in the sense that the term \( (\rho \times p) \) in the transfer of polarized radiation in Landi Degl’Innocenti & Landolfi (2004) vanishes (also see Huang et al. 2009a for similar results with different definitions of the axial ratios). As Faraday rotation is much stronger than Faraday conversion, the circularly polarized intensity is much larger than the linearly polarized intensity in eigenmodes. These are the well-known properties of ‘cyclotron’/cold plasma regimes.

In the limit \( \omega^2 \gg a_{\mu\nu} \), the refractive indices of the two modes are

\[ n_\pm = \frac{k c}{\omega} \sqrt{1 + \frac{2\pi c}{\omega^2} \left( a_{11}^0 + a_{22}^0 \pm \frac{\omega}{2\pi} \rho_\nu \right)} \approx 1 + \frac{2\pi c}{\omega^2} \left( a_{11}^1 + a_{22}^1 \pm \frac{\omega}{2\pi} \rho_\nu \right). \]

(37)

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which corresponds to standard rotation by $\theta_f$ of the EVPA plane, Faraday rotation effect, as
\[
\frac{d\theta_f}{dt} = \frac{\omega}{2c} (n_+ - n_-) \approx \frac{1}{2} \rho_v.
\]  

(38)

3.3.2 Ultrarelativistic plasma limit

In plasmas with very high Lorentz factors, $\rho_Q$ changes its sign to be negative. Two natural modes also change their polarization property from circular to linear. We find an interesting result that the quantity $Z^\pm$ is approaching another constant. This means the proportion of linear to circular radiation in each eigenmode can still be easily measured by the proportion of Faraday conversion to rotation.

As it appears, $\eta_0 \approx -2\rho_Q$ in this limit and $|\eta_v| \ll |\rho_v| \ll |\rho_Q|$. So, the complex axial ratios become

\[
\mathcal{T}^\pm = \frac{\rho_Q}{\rho_v} \left[ -1 + i \left( \frac{\eta_0}{\rho_Q} \right) \mp \sqrt{\left[ -1 + i \left( \frac{\eta_0}{\rho_Q} \right) \right]^2 - \left[ \frac{\eta_v}{\rho_Q} + i \left( \frac{\rho_v}{\rho_Q} \right) \right]^2} \right]
\]

\[
\approx \frac{\rho_Q}{\rho_v} \left[ 1 + 2i \pm \sqrt{(1 + 2i)^2 + (\rho_v/\rho_Q)^2} \right] \approx \frac{\rho_Q}{\rho_v} \left[ 1 + 2i \pm (1 + 2i) \left[ 1 + \frac{1}{2} \left( \frac{\rho_v}{\rho_Q} \right)^2 \right] \right].
\]  

(39)
We derive
\[
\tilde{T}^+ \rightarrow \frac{\rho_0}{\rho_v} (2 + 4i), \quad T^+ \rightarrow \infty, \quad \varphi^+ \rightarrow -\pi + \tan^{-1}(2) \approx -2, \\
\tilde{T}^- \rightarrow -\frac{\rho_v}{\rho_0} (0.1 - 0.2i), \quad T^- \rightarrow 0, \quad \varphi^- = -\pi - \varphi^+ \rightarrow -\tan^{-1}(2) \approx -1, \\
\beta^+ \rightarrow 0, \quad \chi^+ \rightarrow 0, \quad \chi^- \rightarrow -\frac{\pi}{2},
\]
and
\[
\frac{R^+_{CP}}{R^-_{CP}} = \frac{2 \rho^+_{CP}}{|T^+ + T^-|} \sqrt{\text{Re}(\tilde{T}^+)\text{Re}(\tilde{T}^-)} \rightarrow \frac{2 \rho^+_{CP}}{4.47 \rho_0 / \rho_v} \sqrt{\frac{\rho^+_{CP}}{\rho_v}} \approx -0.2 \frac{\rho^+_{CP}}{\rho_0}.
\]
that is, \( Z^+ \rightarrow -0.2. \)

Note that the total intensity cannot be calculated by simply adding intensities in the two modes, because they are not orthogonal.

3.3.3 Intermediate regime
In plasmas with intermediate Lorentz factors (about \( \gamma_0 \sim 10^1 - 10^2 \) for the fiducial model with \( \Omega_{d/w} = 10^{-4} \)), the properties of the natural modes change gradually from those in the cold limit to those in the ultrarelativistic limit.

As \( \gamma_0 \) increases, \( T^+ \) decreases to 0, while \( T^- \) approaches \( \infty \), ellipticities \( |\beta^\pm| \) decrease to 0 and EVPAs \( \chi^\pm \) deviate from \(-90^\circ \) (or \( 0^\circ \)). At a special value of \( \gamma'_0 \), the Faraday conversion coefficient \(-\text{Re}(\alpha^{11} - \alpha^{22})\) changes its sign, as shown in Fig. 2. The Lorentz factor for the fiducial model is \( \gamma'_0 \approx 100 \). Flips of \( T^\pm, \beta^\pm \) and \( \chi^\pm \) between two modes help to preserve the handedness of the modes, that is, the wave with \( \vec{E}_\parallel \) always has a counterclockwise rotation, while the wave described by \( \vec{E}_\perp \) has a clockwise rotation. The value of \( Z^\pm \) decreases in the intermediate regime from 1 to \(-0.2\).

4 PROPAGATION AND ABSORPTION COEFFICIENTS FOR ISOTROPIC PLASMAS WITH ARBITRARY ENERGY DISTRIBUTIONS

4.1 Integration over the particle distribution function
The natural form of the response tensor is related to \( I(\xi, S) \) as shown by equation (12). With the aid of
\[
I(\xi, S) = \int f(\gamma) I(\gamma; \xi, S) \, d\gamma,
\]
one has
\[
\alpha^{ij}(k) = \left\{ \begin{array}{ll}
\frac{i e^2 \omega}{m_e c} \int_0^{2\pi} d\xi \int_0^\infty \frac{d\xi}{\xi} I(\xi, S) \frac{\partial^2}{\partial S_\mu \partial S_\nu} I(\xi, S) - \left[ f(\gamma) \frac{\partial^2}{\partial S_\mu \partial S_\nu} A(\gamma; \xi, S) \right]_{\gamma = \gamma_{\max}} \mu = 1, 2 \\
= \int_{\gamma_{\min}}^{\gamma_{\max}} f(\gamma) \alpha^{ij}(k, \gamma) \, d\gamma - f(\gamma_{\max}) \alpha_{\max}^{ij}(k, \gamma_{\max}) + f(\gamma_{\min}) \alpha_{\min}^{ij}(k, \gamma_{\min}).
\end{array} \right.
\]
The isotropic distribution function \( f(\gamma) \) is normalized as \( \int f(\gamma) \, d\gamma = n_e \). In this section, we compute the propagation and absorption coefficients for thermal and power-law particle distributions.

4.1.1 Thermal distribution
The thermal distribution is
\[
f(\gamma) = \frac{n_e}{4\pi \Theta_e K_0(\Theta_e^{-1})} e^{-\gamma/\Theta_e}, \quad 1 \leq \gamma < +\infty,
\]
where \( \Theta_e = k_B T_e / (m_e c^2) \) is the dimensionless particle temperature. It is normalized to the number density of electrons \( n_e \) as
\[
\int f(\gamma) \, d^3 p = 4\pi \int_1^{+\infty} \gamma \sqrt{\gamma^2 - 1} f(\gamma) \, d\gamma = n_e.
\]
In this case, \( f(\gamma_{\max} \rightarrow +\infty) \rightarrow 0 \), so that the response tensor becomes
\[
\alpha^{ij}(k) = \frac{n_e}{4\pi \Theta_e K_0(\Theta_e^{-1})} \left[ \int_1^{+\infty} e^{-\gamma/\Theta_e} \alpha^{ij}(k, \gamma) \, d\gamma + e^{-1/\Theta_e} \alpha_{\max}^{ij}(k, 1) \right] = \frac{n_e}{4\pi \Theta_e K_0(\Theta_e^{-1})} \int_1^{+\infty} e^{-\gamma/\Theta_e} \alpha^{ij}(k, \gamma) \, d\gamma,
\]
since \( \alpha_{\min}^{ij}(k, 1) \rightarrow 0 \). The propagation and absorption coefficients for \( n_e = 1 \) are shown in the thick dotted lines in Fig. 4.
Figure 4. Propagation and absorption coefficients for thermal energy distribution. Left-hand panel: Faraday rotation and conversion. Linear approximations by Melrose (1997b) are shown in the black dashed lines. Right-hand panel: absorption coefficients. Traditional approximations are shown in the black solid lines.

In the left-hand panel, we show in the grey dashed lines the linear approximations to propagation coefficients elaborated in Melrose (1997b) and Shcherbakov (2008) for $n_e = 1$. The related formulae are:

\[
\rho_{Q,\text{lin}} = \frac{2\pi e^2 \Omega_0}{m_e c\omega} \left[ K_1(\Theta_1^{-1}) \sin^2 \theta + 6\Theta_2 \sin^2 \theta \right],
\]

\[
\rho_{\varphi,\text{lin}} = \frac{4\pi e^2 \Omega_0}{m_e c\omega} \frac{K_0(\Theta_2^{-1})}{K_1(\Theta_1^{-1})} \cos \theta.
\]

We scale $2\text{Im}(\alpha^{22})$ and $(\omega/2\pi)\rho_{Q,\text{lin}}$ by a factor of 300 for a better layout together with $\text{Re}(\alpha^{11})$ and $(\omega/2\pi)\rho_{Q,\text{lin}}$. Note that $\rho_{\varphi,\text{lin}}$ is a good approximation for a Faraday rotation coefficient at any temperature, while $\rho_{Q,\text{lin}}$ is a good approximation for a Faraday conversion coefficient only at low temperatures, not at high temperatures. The peak of Faraday conversion for monoenergetic particles leads to a similar peak at about $10^{11}$ K for thermal distribution. Faraday conversion is much lower than the linear approximation predicts, if the temperature rises. We will show in the last section that the linear approximations lead to wrong predictions of CP from Sgr A*, the difference being a factor of several.

The black long-dashed and dashed lines in the right-hand panel show $(\omega/2\pi)\eta'_Q$ and $(\omega/2\pi)\eta'_\varphi$, for $n_e = 1$, respectively. The absorption coefficients $\eta'_Q$ and $\eta'_\varphi$ are calculated according to equation (21) by substituting the thermal distribution function. Similar to the case of monoenergetic particles, our thermal calculations match the approximations well at high temperatures, while they become inaccurate at low temperatures. In practice, one can just adopt the simple traditional approximations for synchrotron absorption coefficients.

We show $T^\pm$ and phases $\delta^\pm$, $\chi^\pm$ and $Z^\pm$ in Fig. 5. In general, they are similar to those for the $\delta$-function distribution discussed in Section 3.3. Note that the EVPAs ($\chi$) deviates by as much as $\sim 20^\circ$ from 0° (or $-90^\circ$) at $T_e \approx 10^{11}$ K.

### 4.1.2 Power-law distribution

The number density per unit Lorentz factor is

\[
N(\gamma) = \frac{b - 1}{\gamma_{\text{min}} - \gamma_{\text{max}}} n_e \gamma^{-b}, \quad \gamma_{\text{min}} < \gamma < \gamma_{\text{max}} \quad \text{and} \quad b > 1,
\]

for a power-law particle distribution, where $b$ is the energy spectral index, set greater than 1 as an example. Thus, the distribution function is

\[
f(\gamma) = \frac{N(\gamma)}{4\pi\gamma^2 |v|} = \frac{n_e}{4\pi} \frac{b - 1}{\gamma_{\text{min}}^{1-b} - \gamma_{\text{max}}^{1-b}} \frac{\gamma^{-(b+1)}}{\sqrt{\gamma^2 - 1}}, \quad \gamma_{\text{min}} < \gamma < \gamma_{\text{max}}.
\]

We set $\gamma_{\text{max}} = +\infty$ in the computations below for $b > 1$. Then, $f(\gamma_{\text{max}} \rightarrow +\infty) \rightarrow 0$ and the response tensor becomes

\[
\alpha^{ij}(k) = \frac{n_e (b - 1)}{4\pi \gamma_{\text{min}}^{1-b}} \left[ \int_{\gamma_{\text{min}}}^{+\infty} \frac{\gamma^{-b}}{\sqrt{\gamma^2 - 1}} \alpha^{ij}(k, \gamma) d\gamma + \frac{\gamma_{\text{min}}^{b-1}}{\sqrt{\gamma_{\text{min}}^2 - 1}} \alpha_{ij}^{00}(k, \gamma_{\text{min}}) \right].
\]

Note that a Faraday conversion coefficient in the ‘linear’ regime is actually proportional to $(\Omega_e/\omega)^2$.

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We show the results of the numerical integration for $b = 2.5$ in Fig. 6. In the left-hand panel, we also show the approximations for propagation coefficients given by Sazonov (1969) for $n_e = 1$ in the grey dashed lines. The related formulae are

$$\rho_{Q,\text{appr}} = 8.5 \times 10^{-3} \left( b - 2 \right)^{-1} \left( \frac{\omega}{\Omega_0 \sin \theta \gamma_{\min}^{1-b}} \right)^{(b-2)/2} - 1 \left( b - 1 \right) \frac{\Omega_0}{\omega} \sin \theta \gamma_{\min}^{1-b} \frac{2\pi}{\omega},$$

$$\rho_{V,\text{appr}} = 1.7 \times 10^{-2} \ln \gamma_{\min} \left( b + 1 \right) \frac{\Omega_0}{\omega} \gamma_{\min}^{1-b} \frac{2\pi}{\omega} \cos \theta \frac{2\pi}{\omega}. \tag{51}$$

[Similar formulae can be also found in Jones & O'Dell (1977).]

We scale $2\text{Im}(\alpha_{12}^2)$ and $(\omega/2\pi)\rho_{V,\text{appr}}$ by a factor of 150 for a better layout. Here $\rho_{V,\text{appr}}$ is a good approximation of a Faraday rotation coefficient only for large $\gamma_{\min}$. It significantly underestimates Faraday rotation, if the cut-off Lorentz factor is low. The approximation $\rho_{Q,\text{appr}}$ works well for a Faraday conversion coefficient, if $\gamma_{\min} < 100$. It accurately describes the exact behaviour including the peak.

In the right-hand panel of Fig. 6, the long-dashed and dashed black lines show $(\omega/2\pi)\eta_{Q,\text{appr}}^1$ and $(\omega/2\pi)\eta_{Q,\text{appr}}^2$, respectively, for $n_e = 1$. The approximate absorption coefficients $\eta_{Q,\text{appr}}^1$ and $\eta_{Q,\text{appr}}^2$ are computed based on equation (21) by substituting the power-law distribution function. In this case, our calculations of $\text{Im}(\alpha_{11}^2 - \alpha_{22}^2)$ and $2\text{Re}(\alpha_{12}^2)$ match their traditional approximations well for all $\gamma_{\min}$. This is because the particles with high $\gamma$ play a major role in the power-law distribution compared to the thermal, so that the inaccuracy at low values of $\gamma$ is concealed. Similar to the case of thermal distribution, one can adopt simple traditional approximations for synchrotron absorption. We also show approximations for absorption coefficients given by Sazonov (1969) for $n_e = 1$, in the grey long-dashed and dashed lines.

---

**Figure 5.** Properties of two natural modes in plasmas with thermal energy distribution. Amplitudes $T^\pm$ (black) and phases $\varphi^\pm$ (grey) of complex axial ratios, ellipticities $\beta^\pm$, EVPAs $\chi^\pm$ and $Z^+ = Z^-$ (equation 33) are shown in the four panels (from top left-hand panel to bottom left-hand panel), respectively.
4.2 New approximate formulae for Faraday conversion/rotation coefficients

We have shown that the traditional synchrotron approximations to the absorption coefficients are accurate and practical. On the contrary, simple linear approximations of the propagation coefficients have large errors at high electron energies. Although Sazonov (1969) also provided the integral expressions for the propagation coefficients in equation (2.3), and Jones & O’Dell (1977) in their equation (C16) as well, the Faraday rotation \[ \eta_\delta \] therein and especially Faraday conversion \[ \eta_\beta \] therein are inaccurate for high energies of electrons.

We devise a new set of approximate formulae for the complex response tensor in a plasma with monoenergetic particle distribution. The goal is to provide simple relations for the accurate evaluation of propagation coefficients. We provide the numerical code in MATHEMATICA v8 for full evaluation at http://astroman.org/Faraday_conversion/, but we encourage the readers to use the simplified formulae for practical applications. These simplified formulae for Faraday rotation and Faraday conversion are computed for plasmas with \( \delta \)-function energy distribution. They can be easily integrated over the arbitrary energy distribution. Similar approximate formulae for thermal particle distributions were computed in Shcherbakov (2008).

4.2.1 Computations for monoenergetic particle distribution

It is non-trivial that good approximations to Faraday rotation/conversion coefficients exist in a three-dimensional parameter space of \( \Omega_0/\omega \), \( \theta \) and \( \gamma \). However, we manage to find the formulae accurate to within 10 per cent at most reasonable combinations of parameters.

We define an auxiliary quantity

\[
X_A = \sqrt{\frac{\sqrt{2} \sin \theta \Omega_0}{10^{-4} \omega}}
\]

\[(53)\]
Figure 7. Properties of two natural modes for the power-law particle distribution. Amplitudes $T^\pm$ (black) and phases $\psi^\pm$ (grey) of complex axial ratios, ellipticities $\beta^\pm$, EVPAs $\chi^\pm$ and $Z^+ = Z^-$ (equation 33) are shown in the four panels (top left-hand panel to bottom left-hand panel), respectively, for $b = 2.5$. Lines for the ($^+$) mode are solid and those for the ($^-$) mode are dashed. The same quantities for the power-law distribution with $b = 1.5$ are shown in green and cyan.

and introduce four new expressions: $H_X$, $H_B$, $g_X$ and $g_B$ to approximate $-\text{Re}(\alpha^{11} - \alpha^{22})$, $-\text{Re}(\alpha_B^{11} - \alpha_B^{22})$, $2\text{Im}(\alpha^{12})$ and $2\text{Im}(\alpha_B^{12})$, respectively. The formulae are

\[
H_X(\gamma_0) = \begin{cases} 
 9.29 \times 10^{-9} \sqrt{1 - \gamma_0^{-1}(X_A\gamma_0)^{0.036}}, & X_A\gamma_0 < 40, \\
-0.002 \exp \left[-\frac{(\ln X_A\gamma_0 - 4.2137)^2}{0.5429}\right] + 0.00083 \exp \left[-\frac{(\ln X_A\gamma_0 - 4.2137)^2}{0.2121}\right], & X_A\gamma_0 \geq 40,
\end{cases}
\]

\[
H_B(\gamma_0) = \begin{cases} 
 4.67 \times 10^{-9}(1 - \gamma_0^{-1})^{3/2}(X_A\gamma_0)^{0.34}, & X_A\gamma_0 < 40, \\
0.864 - 2.082(\ln X_A\gamma_0)^2 + 0.175(\ln X_A\gamma_0)^4 - 0.00626(\ln X_A\gamma_0)^6 + 1.0175 \times 10^{-5}(\ln X_A\gamma_0)^8 - 7.686 \times 10^{-6}(\ln X_A\gamma_0)^{10} & X_A\gamma_0 \geq 40,
\end{cases}
\]

\[
g_X(\gamma_0) = 1 - 0.4 \exp \left[-\frac{(\ln X_A\gamma_0 - 9.21)^2}{11.93}\right] - 0.05 \exp \left[-\frac{(\ln X_A\gamma_0 - 5.76)^2}{1.33}\right] + 0.075 \exp \left[-\frac{(\ln X_A\gamma_0 - 4.03)^2}{0.65}\right],
\]

\[
g_B(\gamma_0) = 1 - 0.0045(X_A\gamma_0)^{0.52}.
\]
\[ \rho_Q(\gamma) = \frac{2\pi}{\omega} \text{Re}(\alpha^{11} - \alpha^{22}) = \frac{8\pi^2e^2}{m_e\omega} \dot{\chi}_\alpha H_x(\gamma), \]

\[ \rho_Q(\gamma) = \frac{-2\pi}{\omega} \text{Re}(\alpha^{11} - \alpha^{22}) = \frac{8\pi^2e^2}{m_e\omega} H_y(\gamma), \]

\[ \rho_V(\gamma) = \frac{2\pi}{\omega} \text{Im}(\alpha^{12}) = \frac{8\pi^2e^2\Omega_0 \cos \theta}{m_e\omega^2} \left[ \gamma_0 \ln \left( \frac{1 + p_0/\gamma_0}{1 - p_0/\gamma_0} \right) - 2p_0 \right] \text{g}_x(\gamma), \]

\[ \rho_V(\gamma) = \frac{2\pi}{\omega} \text{Im}(\alpha^{12}) = \frac{8\pi^2e^2\Omega_0 \cos \theta}{m_e\omega^2} \left[ \gamma_0 \ln \left( \frac{1 + p_0/\gamma_0}{1 - p_0/\gamma_0} \right) - 2p_0 \right] \text{g}_y(\gamma). \]

Those for the arbitrary particle distribution \( f(\gamma) \) are calculated as

\[ \rho_Q = \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} f(\gamma)\rho_Q(\gamma) \, d\gamma - f(\gamma_{\text{max}})\rho_Q(\gamma_{\text{max}}) - f(\gamma_{\text{min}})\rho_Q(\gamma_{\text{min}}), \]

\[ \rho_V = \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} f(\gamma)\rho_V(\gamma) \, d\gamma - f(\gamma_{\text{max}})\rho_V(\gamma_{\text{max}}) - f(\gamma_{\text{min}})\rho_V(\gamma_{\text{min}}). \]

The above formulae are accurate, with errors of several per cent for \( \delta \)-distribution in general. As a test, we choose \( f(\gamma) \) in thermal distribution and calculate \((\omega/2\pi)\rho_Q^T\) and \((\omega/2\pi)\rho_V^T\) by them for a wide range of temperatures. We then calculate \((\omega/2\pi)\rho_Q^R\) and \((\omega/2\pi)\rho_V^R\) by contour integrate formulae for thermal distribution, which is discussed in the Section 4.2.2. The errors of \( \rho_Q \) are defined as \( |(\omega/2\pi)\rho_Q^T - (\omega/2\pi)\rho_Q^R|/|(\omega/2\pi)\rho_Q^T| \) and those of \( \rho_V \), similarly, as \( |(\omega/2\pi)\rho_V^T - (\omega/2\pi)\rho_V^R|/|(\omega/2\pi)\rho_V^T| \), that is, error of 1 means the coefficient is accurate within a factor of 2. We choose nine pairs of the parameters \((\frac{\Omega_0}{\omega}, \theta)\), such as \((10^{-2}, 1^\circ), (10^{-2}, 45^\circ), (10^{-2}, 89^\circ), (10^{-4}, 1^\circ), (10^{-4}, 45^\circ), (10^{-4}, 89^\circ), (10^{-5}, 1^\circ), (10^{-5}, 45^\circ), (10^{-5}, 89^\circ)\), as examples represented by different symbols (cross, square and triangle) and different colours (blue, green and red). As shown in Fig. 8, the errors are less than 1 (100 per cent) in general. For high temperatures at which synchrotron emission is effective (>10^10 K), the errors are as good as within 0.1 (10 per cent).
and $i^i(\xi)$ and $\tilde{i}^i(\xi)$ are given in Section 2.2. See the corresponding derivation in Appendix C, which closely follows Trubnikov (1958), Melrose (1997a) and Swanson (2003). The integrals for thermal $\alpha^i(k)$ in equation (57) are easier to compute numerically. However, such simplification can only be done for thermal distribution. The detailed computations and discussion of equation (57) can be found in Shcherbakov (2008). Good fittings for thermal distribution, accurate within 10 per cent except with large $\Omega_{\omega}/\omega$ and large $\theta$, are also provided therein. We will provide their expressions for electrons in the present notation:\footnote{Note that the basis vectors in Shcherbakov (2008) are different, thus a different sign of $\rho_Q$.}

$$X_e = T_e \sqrt{\frac{2}{3} \sin \theta \left( \frac{10}{\rho_0 \omega} \right)},$$

$$\rho_{V,\text{th}} = \frac{4\pi^2 \Omega_0}{mc^2} \frac{K_0(T_e^{-1})}{K_2(T_e^{-1})} \cos \theta g(X_e),$$

$$\rho_{Q,\text{th}} = \frac{2\pi^2 \Omega_0}{mc^2} \left( \frac{K_1(T_e^{-1})}{K_2(T_e^{-1})} + 6T_e \right) \sin^2 \theta h(X_e),$$

with approximate multipliers

$$g(X_e) = 1 - 0.11 \ln(1 + 0.035 X_e),$$

$$h(X_e) = 2.011 \exp \left( - \frac{X_e^{0.135}}{4.7} \right) - \cos \left( \frac{X_e}{2} \right) \exp \left( - \frac{X_e^{1.2}}{2.73} \right) - 0.011 \exp \left( - \frac{X_e}{47.2} \right).$$

5 APPLICATIONS

We computed the response tensor in uniformly magnetized relativistic plasmas with isotropic particle distributions. We found Faraday conversion, Faraday rotation and absorption coefficients by numerical integration in the complex plane. We then discussed the properties of natural modes of cyclo-synchrotron radiation and presented the results for specific electron energy distributions. We provided accurate practical fitting formulae for Faraday conversion and rotation. The method of complex plane integration allows to calculate both absorption and propagation coefficients consistently. In practice, formulae in Sazonov (1969) are good enough for absorption coefficients. Therefore, we focus on improving the calculations of Faraday conversion and rotation coefficients. Faraday conversion and rotation coefficients can be generally expressed as $\rho_Q \propto \hbar(\gamma, \lambda B, \theta) n_e B^2 \lambda^3$ and $\rho_V \propto g(\gamma, \lambda B, \theta) n_e B \lambda^2$, where $h$ and $g$ are functions of the electron Lorentz factor $\gamma$, product of wavelength $\lambda$ and magnetic field $B = |B|$, and angle $\theta$ between wavevector $k$ and magnetic field $B$.

Previous work (e.g. Melrose 1997b; Quataert & Gruzinov 2000) has shown that $g \approx 1$, if $\gamma \approx 1$ for typical $\lambda, B, \theta$, but $g \ll 1$ if $\gamma \gg 1$. This means the Faraday rotation becomes less important, when the electrons become relativistic. Therefore, the direction of the LP plane changes little. Our calculations suggest the function $g$ and the Faraday rotation coefficients were previously computed precisely, but the function $h$ and Faraday conversion coefficients were not. The function $h$ grows from small to intermediate $\gamma$, but $h$ steeply decreases, if $\gamma$ grows more. This means the Faraday conversion also becomes less important when the electrons become relativistic, although the peak Lorentz factor depends on the frequency ratio $\Omega_{\omega}/\omega$ and the particle distribution. In summary, as the absorption coefficients at a given frequency $\omega$ also decrease with $\gamma$, ultrarelativistic electrons interact less with radiation field.

We have shown that the Faraday conversion coefficient should be computed more precisely. Let us now demonstrate that imprecise estimates of propagation effects result in large error in polarized simulated fluxes for accretion on to compact objects. We compare polarized spectra for two cases: accurate general relativistic polarized radiative transfer (Paper I) and assuming that Faraday conversion and rotation coefficients are given by linear approximations. We test two types of dynamical models for Sgr A* accretion to prove the case. Both models assume thermal electron distribution.

First, we adopt the analytic model from Huang, Takahashi & Shen (2009b). This model is established for Sgr A* based on the RIAF solution with different temperatures in ions and electrons. It reasonably fits the Sgr A* polarized millimetre/submillimetre spectrum. In the left-hand panel of Fig. 9, we plot the accurate ratio $\rho_Q/\rho_V$ (solid lines) and the ratio $\rho_{Q,\text{lin}}/\rho_{V,\text{lin}}$ in linear approximation (dashed lines) as functions of electron temperature $T_e$ for $\Omega_{\omega}/\omega = 10^{-4}$ and different $k - B$ angles. In the right-hand panel, we plot simulated circular and linearly polarized spectra computed with accurate Faraday conversion and rotation coefficients (solid lines) and the results for the same dynamical model for assumed linear approximations of propagation coefficients (dashed lines). The inclination angles of the disc $i = 75^\circ$ (black) and $i = 90^\circ$ (red) are chosen.

The frequency ratio $\Omega_{\omega}/\omega \sim 10^{-4}$ corresponds to the submillimetre band close to the event horizon of Sgr A*. There the electron temperature is $T_e \sim 10^{10.5} - 10^{11.5}$ K. The Faraday conversion becomes strong ($\rho_Q/\rho_V > 1$) at large angles $\theta > 45^\circ$. One can also note from
Figure 9. Left-hand panel: examples of accurate ratios of $\rho_\Omega/\rho_V$ (solid line) and their linear approximations $\rho_{\Omega,\text{lin}}/\rho_{V,\text{lin}}$ (dashed line) for thermal electrons with $\Omega_0/\omega = 10^{-4}$ and different $\theta$. Right-hand panel: simulated circularly polarized and linearly polarized fluxes based on accurate Faraday conversion/rotation (solid line) and their linear approximations (dashed line) for the Sgr A* accretion model in Huang et al. (2009b) with inclination angles of 75° (black) and 90° (red).

Figure 10. Simulated LP (left-hand panel) and CP (right-hand panel) fractions for Sgr A* for various prescriptions of Faraday rotation and Faraday conversion: for linear approximations $\rho_{\Omega,\text{lin}}/\rho_{V,\text{lin}}$ (dashed red) and for accurate $\rho_V$ and $\rho_\Omega$ (solid green). We employ the best-fitting dynamical model from Shcherbakov et al. (2010) with dimensionless spin $a_*=0.9$.

The right-hand panel of Fig. 4 that the $\eta_V$ absorption coefficient is less than 1 per cent of $\eta_I$. Therefore, emissivity in $V$ is equally weak according to Kirchhoff’s Law. Thus, Faraday conversion plays a major role in the production of CP. However, $\rho_\Omega/\rho_V$ reaches a peak at a specific temperature around $10^{11}$ K and then decreases again, while $\rho_{\Omega,\text{lin}}/\rho_{V,\text{lin}}$ monotonically increases to much greater than 1. Therefore, the amplitudes of CP predicted for accurate ($\rho_\Omega, \rho_V$) are less than half of those predicted for approximate ($\rho_{\Omega,\text{lin}}, \rho_{V,\text{lin}}$), although accurate and simplified propagation coefficients predict similar LP.

Changes in simulated polarized fluxes are shown in Fig. 10 for the best-fitting Sgr A* accretion model from Shcherbakov et al. (2010). This model is inspired by three-dimensional general relativistic magneto hydrodynamic simulations. The structures of magnetic field, velocity and density fields are taken directly from simulations. All free parameters are adjusted to achieve the best fit. Similarly to the aforementioned analytic model, CP in the submillimetre band is significantly lower, when the precise Faraday conversion and rotation coefficients are adopted for the same dynamical model. Thus, substantially lower predicted CP fractions, when the precise propagation coefficients are adopted, is a generic model-independent result. Our calculation showed that linear approximations of Faraday conversion/rotation are invalid for electrons with such high energy. They significantly overestimate the CP for the relevant range of the ratio $\Omega_0/\omega$. When fitting polarized observations and predicting polarized spectra, one should adopt the accurate Faraday conversion and rotation coefficients computed in Section 2.2 or the simplified fitting formulae provided in Section 4.2.

The validity of linear approximations of Faraday conversion and rotation coefficients depends not only on the energy of electrons, but also on the frequency ratio $\Omega_0/\omega$. The observational frequency in the NIR is $\sim 10^3$ times larger than that in the submillimetre. The typical frequency ratio $\Omega_0/\omega$ is around $10^{-6} - 10^{-8}$, which yields the values of ($\rho_\Omega, \rho_V$) similar to those of ($\rho_{\Omega,\text{lin}}, \rho_{V,\text{lin}}$) for $\gamma < 50$. Thus, the linear approximations of Faraday conversion and rotation coefficients still adequately describe the corresponding effects in the NIR.
Observations of submillimetre CP from Sgr A* are one of today’s big interests and challenges. CP fraction is only about 1 per cent in the submillimetre, but was already detected at several frequencies (Munoz et al. 2011). Still, there is substantial spread between different Sgr A* models (Shcherbakov et al. 2010) in circularly polarized fluxes at frequencies where circularly polarized flux was not yet measured, for example, 88, 145 and 690 GHz. Observations at these frequencies can help to further discriminate between models of various types, which have different black hole spins.

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APPENDIX A: DEFINITIONS AND DERIVATIONS

Here we summarize all the definitions and derivations related to the response tensor. They are similar to those in Melrose (1997a); 1997b), except for the signature of the metric tensor and the geometry of the coordinate system. We list all related tensors which are different from those in Melrose’s work.

The metric tensor in Minkowski space–time: \( g^{\mu\nu} = \text{diag}[-1, 1, 1, 1] \).

Wave vector: \( k^\mu = \omega (1, 0, 0, 1)^T \).

Tensor of magnetostatic field:

\[
F_0^{\mu\nu} = B f^{\mu\nu}, \quad B = \left( \frac{1}{2} F_0^{\mu\nu} F_0^{\nu\mu} \right)^{1/2}, \quad f^{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \cos \theta & \sin \theta & 0 \\
0 & -\cos \theta & 0 & 0 \\
0 & -\sin \theta & 0 & 0
\end{pmatrix}.
\]
Auxiliary tensor for electron velocity perturbations:

\[ i^{\mu
u}(\tau) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \cos \Omega_0 \tau & -\cos \theta \sin \Omega_0 \tau & -\sin \theta \sin \Omega_0 \tau \\ 0 & \cos \theta \sin \Omega_0 \tau & \sin^2 \theta + \cos^2 \theta \cos \Omega_0 \tau & -\sin \theta \cos \theta + \sin \theta \cos \theta \cos \Omega_0 \tau \\ 0 & \sin \theta \sin \Omega_0 \tau & -\sin \theta \cos \theta + \sin \theta \cos \theta \cos \Omega_0 \tau & \cos^2 \theta + \sin^2 \theta \cos \Omega_0 \tau \end{pmatrix}. \]

Auxiliary tensor for electron position perturbations:

\[ T^{\mu
u}(\tau) = i^{\mu
u}(\tau) - i^{\mu
u}(0) = \begin{pmatrix} \tau & 0 & 0 & 0 \\ 0 & \sin \Omega_0 \tau & \cos \theta \sin \Omega_0 \tau & \sin \theta \sin \Omega_0 \tau \\ 0 & \cos \theta \sin \Omega_0 \tau & \sin^2 \theta + \cos^2 \theta \cos \Omega_0 \tau & -\sin \theta \cos \theta + \sin \theta \cos \theta \cos \Omega_0 \tau \\ 0 & \sin \theta \sin \Omega_0 \tau & -\sin \theta \cos \theta + \sin \theta \cos \theta \cos \Omega_0 \tau & \cos^2 \theta + \sin^2 \theta \cos \Omega_0 \tau \end{pmatrix}. \]

In six-dimensional phase space: electron velocity is \( v \), electron momentum is \( m_p \) and distribution function of electrons is \( f(p) \).

In eight-dimensional phase space: Lorentz factor \( \gamma = 1/\sqrt{1-|v|^2} \), velocity of electron \( U^\mu = (\gamma, v) \), momentum \( P^\mu = (\gamma m_e, m_e v, m_e v) \) and distribution function of electrons \( f(P) = 2m_e \delta(P^2 + m_e^2) f(p) \).

The general form of the response tensor for an arbitrary isotropic distribution of electrons is

\[ \alpha^{\mu
u}(k) = i e^2 \int d^4 P(\tau) \int_0^\infty \tilde{U}^\mu(\tau) e^{ik\cdot(X(t)-X(\tau))} k U(\tau - \xi) \left[ g^{\mu\nu} - \frac{k^\mu U^\nu(\tau-\xi)}{k U(\tau-\xi)} \right] i^\nu(\tau - \xi) \frac{\partial f(\gamma)}{\partial P^\rho(0)}. \]

**APPENDIX B: DERIVATION OF THE RESPONSE TENSOR FOR MONOENERGETIC PARTICLE DISTRIBUTION**

Let us introduce \( \bar{U}^\mu = [1, 0, 0, 0] \) - 4-velocity in the rest frame of plasmas. Then,

\[ \alpha^{\mu\nu}(k) = i e^2 \int d^4 P(\tau) \times \int_0^\infty \tilde{U}^\mu(\tau) e^{ik\cdot(X(t)-X(\tau))} k U(\tau - \xi) \left[ g^{\mu\nu} - \frac{k^\mu U^\nu(\tau-\xi)}{k U(\tau-\xi)} \right] i^\nu(\tau - \xi) \frac{2m_e \delta(P^2 + m_e^2) m_e^4}{m_e^4} \frac{d\tilde{P}(\gamma)}{d\gamma} \left. \right|_{\gamma = 1} \]

\[ = i e^2 \int d^4 P(\tau) \times \int_0^\infty \tilde{U}^\mu(\tau) e^{ik\cdot(X(t)-X(\tau))} k U(\tau - \xi) \left[ g^{\mu\nu} - \frac{k^\mu U^\nu(\tau-\xi)}{k U(\tau-\xi)} \right] i^\nu(\tau - \xi) \frac{2m_e \delta(P^2 + m_e^2) m_e^4}{m_e^4} \frac{d\tilde{P}(\gamma)}{d\gamma} \int_0^\infty \tilde{U}^\sigma(\tau - \xi) \tilde{U}^\rho(\tau - \xi) \frac{d\tilde{P}(\gamma)}{d\gamma} \left. \right|_{\gamma = 1} \]

\[ = \frac{e^2}{m_e} \int d^4 P(\tau) \left[ 2m_e \delta(P^2 + m_e^2) m_e^4 \frac{d\tilde{P}(\gamma)}{d\gamma} \right] \frac{1}{\gamma} \frac{d\tilde{P}(\gamma)}{d\gamma} \int_0^\infty \tilde{U}^\mu(\tau - \xi) \tilde{U}^\nu(\tau - \xi) \frac{d\tilde{P}(\gamma)}{d\gamma} \left. \right|_{\gamma = 1} \]

which coincides with equation (9). We denoted \( R^\mu(\xi) = k^\rho T_{\rho\mu}(\xi) \) and introduced an auxiliary variable \( S_\eta \).

**APPENDIX C: DERIVATION OF THE RESPONSE TENSOR FOR THERMAL PARTICLE DISTRIBUTION**

Substituting the thermal distribution into equation (12), we obtain

\[ I(\xi, s) = \frac{n_e}{\Theta_3 K_2(\Theta_3)} \int_1^\infty e^{-i(\theta - i\omega t + s \omega t)} \left[ (i \omega \xi + s_0) \sin \left( \frac{|R + is||p|}{|R + is|} \right) \right] dy. \]
where $|p| = \sqrt{\gamma^2 - 1}$. We can simplify the above formula as follows:

$$
\Theta_e K_z(\Theta_e^{-1}) I(\xi_s) = \left[-\frac{\text{i} \omega \xi + s_0}{\Theta_e^{-1} - \text{i} \omega \xi + s_0} - 1\right]\int_0^{+\infty} \frac{\sin(|R + \text{i}s||p|)e^{-\left(\Theta_e^{-1} - \text{i} \omega \xi + s_0\right)\gamma}}{|R + \text{i}s|} \, \frac{\text{d}z}{|p|}
$$

$$
= \frac{1}{\Theta_e^{-1} - \text{i} \omega \xi + s_0} \int_0^{+\infty} e^{-\left(\Theta_e^{-1} - \text{i} \omega \xi + s_0\right)\gamma} \cos(|R + \text{i}s||p|) \, \frac{\text{d}z}{|p|}
$$

$$
= \frac{1}{2(\Theta_e^{-1} - \text{i} \omega \xi + s_0)} \int_{-\infty}^{+\infty} e^{-\left(\Theta_e^{-1} - \text{i} \omega \xi + s_0\right)\gamma} \cdot |R + \text{i}s||p| \, \frac{\text{d}z}{|p|}
$$

$$
= \frac{1}{2(\Theta_e^{-1} - \text{i} \omega \xi + s_0)} \int_{-\infty}^{+\infty} e^{-\left(\Theta_e^{-1} - \text{i} \omega \xi + s_0\right)\gamma} \cosh z + |R + \text{i}s| \sinh z \, \frac{\text{d}z}{|p|}
$$

Following equation (9) and using

$$
\frac{\partial}{\partial \Theta_e^{-1} - \text{i} \omega \xi + s_0} \int_{-\infty}^{+\infty} e^{-\left(\Theta_e^{-1} - \text{i} \omega \xi + s_0\right)\gamma} \cosh z + |R + \text{i}s| \sinh z \, \frac{\text{d}z}{|p|}
$$

we have

$$
\alpha^{ij}(k) = \frac{\text{i} n e^2}{m c \Theta_e^{-2} K_z(\Theta_e^{-1})} \int_0^{+\infty} \text{d}(\omega \xi) \left[i^{ij}(\xi) \frac{K_z(\Theta_e^{-1})}{R^2(\xi)} - \hat{I}^{ij}(\xi) \frac{K_z(\Theta_e^{-1})}{R^2(\xi)}\right],
$$

which is equation (57).

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