Effects of differential rotation and meridional circulation in solar oscillations of high degree $l$

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ABSTRACT

Helioseismic measurements at high degree $l$ are sensitive to mode couplings induced by large-scale flows, with the largest contribution coming from the solar differential rotation. The coupling becomes strong at high degree $l$, when frequency spacings between interacting modes become small, calling for a more accurate theoretical prediction of the mode-coupling effects in helioseismic data analysis. In this paper, the currently available description based on the quasi-degenerate perturbation theory is developed to a higher order, and extended to include the possible contributing effects of solar meridional circulation. A semi-analytic asymptotic description is developed for effects of the centrifugal forces on the oscillation frequencies of high degree $l$. Numerical estimates of the corresponding effects in the observational data are discussed.

Key words: waves – Sun: helioseismology – Sun: oscillations – Sun: rotation.

1 INTRODUCTION

Sounding solar subsurface layers with $p$ modes of high degree $l$ is the most challenging area in global solar seismology. The diagnostic potential of the high-degree modes is exceptionally important. When the horizontally averaged component of the solar stratification is addressed by helioseismic inversions (measurement of the solar He abundance, calibration of the equation of state), a very high accuracy can potentially be achieved by properly averaging the huge amount of data ($2l + 1$ modes of different azimuthal order $m$ at each $l$ and radial order $n$). In helioseismic inversions with modes of lower degree $l$, targeted at the diagnostic of the deep solar interior, the properties of the outer layers restored with high-degree modes provide invaluable outer boundary conditions. When horizontal inhomogeneities are addressed (active regions, fluid flows on a supergranular scale or below), modes of high degree $l$ are the only modes that allow proper spatial resolution in the horizontal dimensions. Since modes of the high-degree domain can be interpreted in terms of surface waves (e.g. Vorontsov 2006), it is the domain where we have an important link between methods of global and local helioseismology, a link which still needs a deeper theoretical investigation.

A large volume of high-resolution Doppler velocity measurements has been accumulated with the Solar and Heliospheric Observatory (SOHO) MDI instrument in observations covering more than a solar cycle, and new data of better spatial resolution are now coming from the Solar Dynamics Observatory (SDO) HMI project.

The major difficulties are the accurate measurements of the oscillation frequencies (Korzennik, Rabello-Soares & Shou 2008; Larson & Schou 2008; Rabello-Soares, Korzennik & Shou 2008). Principally, the difficulties come from spatial leakage in the spherical-harmonic decomposition of the Doppler-velocity images, which arises because spherical harmonics are not orthogonal over a visible hemisphere. At high degree, modes of the same radial order become closely separated in frequency, resonant line profiles become wider and individual spatial leaks blend into a continuous ridge in the observational power spectra. To recover the underlying mode parameters, we need an accurate response function (leakage matrix; see e.g. Vorontsov & Jefferies 2005), an accurate model of the acoustic line profiles (e.g. Jefferies, Vorontsov & Giebink 2006) and an accurate physical description of the mode-coupling effects induced by internal velocity fields, effects which become strong at high degree $l$.

This study is focused on the effects of mode coupling. The mode coupling induced by advection effects caused by solar differential rotation was first addressed by Woodard (1989); a more accurate description, convenient for helioseismic data analysis, was developed by Vorontsov (2007). When taken into account in the observational data analysis, this effect allows us to eliminate the apparent systematic errors in frequency measurements in the degree range up to somewhere between $l = 150$ and 200 (Vorontsov et al. 2009). Further (unpublished) attempts to extend the degree range to higher $l$ reveal that the systematic errors reappear again, accompanied with rapid degradation in the quality of the theoretical fits to the observational power spectra. Since mode coupling becomes strong, and its theoretical description is based on a perturbation analysis, we have to address the accuracy of the theoretical description, as well as
smaller contributions from sources other than differential rotation, first of all the effects of solar meridional circulation.

In Section 2, we extend the perturbational analysis of the effects of differential rotation to higher order by lifting the simplifying assumptions adopted in the previous work (Vorontsov 2007). The effects of the meridional circulation are addressed in parallel, within a single analysis. Together with higher order effects in the mode-coupling coefficients, we address the corresponding effects induced in the oscillation frequencies. In Section 3, we develop a semi-analytical description of frequency corrections produced by centrifugal effects. Our numerical estimates are described in Section 4, and Section 5 contains a short summary.

2 DIFFERENTIAL ROTATION AND MERIDIONAL CIRCULATION

Throughout this paper, we use a tilde to designate the eigenfrequencies \( \tilde{\omega} \) and eigenfunctions \( \tilde{u} \) of linear adiabatic oscillations of a non-rotating Sun with no meridional flows. In the operator form

\[
\rho_0 \tilde{\omega}^2 \tilde{u}_l = \tilde{H} \tilde{u}_l, \tag{1}
\]

with the equilibrium density \( \rho_0 = \rho_0(r) \), self-adjoint operator \( \tilde{H} \) and displacement field

\[
\tilde{u}_l = \tilde{R}U(r)Y_{lm}(\theta, \phi) + V(r) \nabla Y_{lm}(\theta, \phi), \tag{2}
\]

where \( \nabla = \hat{\theta} \frac{\partial}{\partial \theta} + \hat{\phi} \sin^{-1} \theta \frac{\partial}{\partial \phi} \) is the angular part of the gradient operator and the time dependence is separated as \( \exp(-i \tilde{\omega} t) \).

As in this paper we only address the axisymmetric components of the fluid flows, mode interaction is limited to modes of the same azimuthal order \( m \). At high degree \( l \), the interaction is also limited by modes of the same radial order \( n \) (we have small frequency separations along the \( p \)- and \( f \)-mode ridges). Therefore, we drop \( n \) and \( m \) from indexing the solutions for brevity, when this does not lead to confusion. In the perturbational analysis which follows, we assume the unperturbed eigenfunctions to be normalized as

\[
(\tilde{u}_l, \rho_0 \tilde{u}_l) = \delta_{l\ell}, \tag{3}
\]

with the scalar product defined as

\[
(\tilde{u}_l, \tilde{u}_s) = \int_V u_l^\dagger \cdot u_s \, dv, \tag{4}
\]

where the integration is performed over the spherical volume \( V \) occupied by the Sun.

For a configuration with a slow differential rotation and meridional circulation, equation (1) is replaced with

\[
\rho_0 \tilde{\omega}^2 u_l = (\tilde{H} + 2\tilde{o}_l \delta H)u_l, \tag{5}
\]

where

\[
\delta H = -i \tilde{\omega}_0 (v \cdot \nabla), \tag{6}
\]

and \( v \) is the sum of two axisymmetric stationary velocity fields

\[
v = v_{\text{tot}} + v_{\text{mer}}, \tag{7}
\]

which are represented by their decomposition in terms of vector spherical harmonics as

\[
v_{\text{tot}} = - \sum_{s=1,2,\ldots} w_s(r) \hat{r} \times \nabla Y_{s,0}(\theta, \phi), \tag{8}
\]

where

\[
w_s(r) = \left( \frac{4\pi}{2s + 1} \right)^{1/2} r_{\Omega_s}(r), \quad s = 1, 2, \ldots \tag{9}
\]

and \( \Omega_s \) are the coefficients in the expansion of the angular velocity of rotation \( \Omega = \Omega(r, \theta) \):

\[
\Omega(r, \theta) = \sum_{s=1,2,\ldots} \Omega_s(r) \frac{dP_s(\cos \theta)}{d \cos \theta}, \tag{10}
\]

and

\[
v_{\text{mer}} = \sum_{s=1,2,\ldots} [u_s(r) \tilde{Y}_{s,0}(\theta, \phi) + v_s(r) \nabla Y_{s,0}(\theta, \phi)]. \tag{11}
\]

The fluid flow \( v \) is assumed to be stationary, and hence \( v_{\text{mer}} \) satisfies the mass-conservation equation

\[
\nabla \cdot (\rho_0 v_{\text{mer}}) = 0, \tag{12}
\]

which is equivalent to

\[
\frac{d}{dr}(\rho_0 r^2 u_s) = \rho_0 r s(s + 1) v_s, \quad s = 1, 2, \ldots, \tag{13}
\]

with the surface boundary conditions \( u_s(R) = 0 \).

We represent the self-adjoint operator \( \delta H \) by the sum of two operators induced by the toroidal (differential rotation) and poloidal (meridional circulation) flows,

\[
\delta H = \delta H_{\text{rot}} + \delta H_{\text{mer}}. \tag{14}
\]

The general expressions for the matrix elements of \( \delta H \) can be found in Lavely & Ritzwoller (1992). Simplification, which is appropriate at high degree \( l \), and our resulting expressions for the matrix elements of \( \delta H_{\text{rot}} \) and \( \delta H_{\text{mer}} \) are described in Appendix A.

We now implement the quasi-degenerate analysis to solve equation (5). We are looking for solutions as

\[
\delta u_l = \delta \tilde{u}_l + \delta o_l, \tag{15}
\]

with imposition of orthogonality constraints

\[
(\delta \tilde{u}_l, \rho_0 \delta \tilde{u}_{l+p}) = 0, \quad p = 0, \pm 1, \ldots. \tag{17}
\]

In equations (16) and (17), \( \delta \tilde{u}_l \) designates the remaining part of the perturbation to the eigenfunction, which is not covered by the linear combination of \( \tilde{u}_{l+p} \): spheroidal vector fields coming from the interaction with modes of different radial order \( n \), and toroidal vector fields. The magnitude of \( \delta \tilde{u}_l \) is assumed to be of the order of the perturbation.

We substitute expressions (15) and (16) into equation (5), and take the scalar product of both sides with \( \tilde{u}_{l+p} \). \( p = 0, \pm 1, \ldots \). Using the property that both \( \tilde{H} \) and \( \delta H \) are self-adjoint, and dropping terms which are of second order in the perturbation, we arrive at the system of linear algebraic equations (of infinite size)

\[
\left( \delta^2_{l+p} - \delta^2_{l} \right) \delta o_l + \sum_{p'=0,\pm 1,\ldots} (\delta \tilde{u}_{l+p}, \delta H \delta \tilde{u}_{l+p'})c_{l,p'} = 0, \tag{18}
\]

which represents an algebraic eigenvalue problem

\[
Zc_l = \delta o_l e_l, \tag{19}
\]

with eigenvectors \( e_l = (\ldots, c_{l,0}, c_{l,1}, \ldots)^T \) of a self-adjoint matrix \( Z \) with elements

\[
Z_{pp'} = \frac{\delta^2_{l+p} - \delta^2_{l}}{2\delta o_l} \delta o_{p'} + (\delta \tilde{u}_{l+p}, \delta H \delta \tilde{u}_{l+p'}). \tag{20}
\]

The eigenvectors \( e_l \) describe the expansion coefficients in the composite eigenstate, represented (to leading order) by a linear
combination of the unperturbed eigenfunctions of the neighbouring modes in the interaction chain (equation 16). Equation (19) can be solved analytically, if two approximations are implemented simultaneously: (a) the matrix elements \((\hat{u}_{l,p}, \delta H \hat{u}_{l,p'})\) depend on the difference \(p - p'\) only, i.e. matrix \(Z\) has a Toeplitz structure everywhere except for the main diagonal; and (b) the unperturbed eigenvalues \(\tilde{\omega}_{l,p}^2\) are equidistant in \(p\), i.e. the diagonal elements of \(Z\) are equidistant. These are the approximations on which the currently available description of the mode coupling at high degree \(l\) (by differential rotation) is based (Vorontsov 2007). These approximations are valid asymptotically at high \(l\) and are expected to produce accurate results at least when the interaction is not too strong, so that the composite eigenstates do not spread over modes covering a wide range of degree (i.e., the coupling coefficients \(c_{l,p}\) drop fast with \(|p|\) when \(|p|\) is small). However, no estimate of the numerical accuracy of the resulting coupling coefficients is still available. To address this issue, we lift the above approximations by introducing correction terms and evaluate the resulting corrections induced in the coupling coefficients (and, in parallel, in the oscillation frequencies).

We denote the matrix elements as

\[
M_{l+p,l+p} = (\hat{u}_{l+p}, \delta H \hat{u}_{l+p})
\]

and allow the interpolation of \(M_{l+p,l+p}\) to non-integer values of \(p\) and \(p'\). We introduce the two-term expansions

\[
M_{l+p,l+p} = M_{l} \, \tilde{\omega}_{l,p} + \frac{p + p'}{2} d_{p,p'},
\]

where \(d_{p,p'}\) is defined as

\[
d_{p,p'} = \frac{\partial}{\partial l} M_{l+p,l+p}.
\]

and

\[
\tilde{\omega}_{l,p} - \tilde{\omega}_{l,p'} = p \left( \frac{\partial \tilde{\omega}_{l}}{\partial l} \right)_n + \frac{p^2}{2} \left( \frac{\partial^2 \tilde{\omega}_{l}}{\partial l^2} \right)_n
\]

\[
= 2 \tilde{\omega}_{l,p} \left( \frac{\partial \tilde{\omega}_{l}}{\partial l} \right)_n + \frac{p^2}{2} \left( \frac{\partial^2 \tilde{\omega}_{l}}{\partial l^2} \right)_n.
\]

In these new approximations, matrix \(Z\) is a sum of two self-adjoint matrices,

\[
Z = Z^{(0)} + Z^{(1)},
\]

dominant part \(Z^{(0)}\) with elements

\[
z_{pp'}^{(0)} = p \left( \frac{\partial \tilde{\omega}_{l}}{\partial l} \right)_n \delta_{p,p'} + M_{l} \, \tilde{\omega}_{l,p} + \frac{p + p'}{2} d_{p,p'},
\]

and a small correction

\[
z_{pp'}^{(1)} = \frac{p^2}{4 \tilde{\omega}_{l}} \left( \frac{\partial^2 \tilde{\omega}_{l}}{\partial l^2} \right)_n \delta_{p,p'} + \frac{p + p'}{2} d_{p,p'}.
\]

Since matrix \(Z\) has an infinite size, we note that these representations only need to be accurate in a limited range of \(|p|\) and \(|p'|\), where the solution \(c_{l,p}\) differs significantly from zero.

We will treat matrix \(Z^{(1)}\) as a small perturbation in the algebraic eigenvalue problem (equation 19), looking for its solutions as

\[
\delta \omega_{l} = \delta^{(0)} \omega_{l} + \delta^{(1)} \omega_{l},
\]

\[
c_{l} = c_{l}^{(0)} + c_{l}^{(1)}.
\]

\subsection{The zero-order problem}

When \(Z^{(1)}\) is discarded, we have the zero-order problem

\[
Z^{(0)} c^{(0)}_l = \delta^{(0)} \omega_l c^{(0)}_l,
\]

which allows an exact analytical solution. The derivation is described in Appendix B. The result is

\[
\delta^{(0)} \omega_l = M_{l} = (\hat{u}_{l}, \delta H \hat{u}_{l}),
\]

\[
c_{l}^{(0)} = \frac{1}{\pi} \int_0^{\pi} \cos \left( pt - \sum_{k=1,2, \ldots}^{p} \frac{2}{k} \Re(b_k) \sin(kt) \right)
\times \exp \left( i \sum_{k=1,2, \ldots}^{p} \frac{2}{k} \Im(b_k) \cos(kt) \right) \, dt, \quad p = 0, \pm 1, \ldots.
\]

Here, \(b_k\) specify the dimensionless non-diagonal matrix elements,

\[
b_k = - \left( \frac{\partial \tilde{\omega}_{l}}{\partial l} \right)_n \sum_{s=1,2, \ldots \text{odd}} \frac{(-1)^{s+k} (s-k)!! (s+k)!!}{(s+k)!} \times P_s \left( \frac{m}{l} \right) (\Omega_k)_{l_k}, \quad k = 1, 2, \ldots
\]

\[
= \left( \frac{2s+1}{4\pi} \right)^{1/2} \times \frac{(s-k-1)!! (s+k-1)!!}{(s+k)!} \left( \frac{m}{l} \right) \left( \frac{v_s}{r} \right)_{l_k}, \quad k = 1, 2, \ldots
\]

where \((\cdot)\) designates the average value of a corresponding quantity \([\Omega_k \langle r \rangle \text{ or } v_s \langle r \rangle \langle l \rangle]\) in the propagation domain of modes with degree \(l\), the average being performed with the kinetic energy density of the oscillations as a weighting function (cf. equations A6 and A7).

The frequency shifts \(\delta^{(0)} \omega_{l} \) are determined by the diagonal matrix elements (equation 31), which are governed by the effects of differential rotation only:

\[
\delta^{(0)} \omega_l = \sum_{s=1,3, \ldots} (-1)^{s+k} \frac{s!! s!!}{s!} P_s \left( \frac{m}{l} \right) (\Omega_k)_{l}.\]

The frequency shifts are only contributed by \(\Omega_k\) with odd \(s\), i.e. by the components of the differential rotation that are symmetric around the equatorial plane. Using the standard representation of the frequency shifts in terms of frequency-splitting coefficients,

\[
\delta \omega = \sum_{s=0,1,2, \ldots} a_s P_s^{(1)}(m) \simeq \sum_{s=0,1,2, \ldots} a_s l P_s \left( \frac{m}{l} \right),
\]

where the approximate equality sign refers to the asymptotic limit of high degree \(l\) (for more details, see Vorontsov 2007), we have

\[
a_s \simeq (-1)^{l-k} \frac{s!! s!!}{s!} (\Omega_k)_{l}, \quad s = 1, 3, \ldots
\]

For even \(k\), the real parts of \(b_k\) are coming from terms with odd \(s\) (equation 34), and when the odd splitting coefficients are available from the observations, the real parts of \(b_k\) are also available as

\[
\Re(b_k) = \left( \frac{\partial \tilde{\omega}_{l}}{\partial l} \right)_n \sum_{s=1,3, \ldots} (-1)^{s+k} \frac{s!! s!!}{s!} P_s \left( \frac{m}{l} \right) \left( \Omega_k \right)_{l_k}, \quad k = 2, 4, \ldots
\]
Therefore, when we limit the analysis by addressing the effects of the equatorially symmetric components of the differential rotation only, the mode-coupling coefficients can be calculated from the observable rotational-splitting coefficients. This result, which is convenient for data analysis, has been established by Vorontsov (2007). Note that the present study is extended to allow the treatment of the antisymmetric components (Ω, with even s). Although we do not have any observational evidence of the asymmetry in the solar differential rotation, even a small asymmetry can have a notable impact on the mode coupling.

As can be seen from equations (34) and (35), the dependence of \( b_m \) on the azimuthal order \( m \) is such that its real part is an odd function of \( m \), and the imaginary part is an even function of \( m \). Therefore, we have a symmetry

\[
b_m(-m) = -b^*_m(m).
\]

According to equation (32), this symmetry in the matrix elements brings the corresponding symmetry in coupling coefficients:

\[
c^0_{l,p}(-m) = c^{(0)}_{l,-p}(m).
\]

The solution to the zero-order problem (equation 30), which was described above (equations 31 and 32), is only one of the infinite set of solutions to the eigenvalue problem. This particular solution satisfies identification \( u_l \rightarrow \bar{u}_l, \alpha_l \rightarrow \bar{\alpha}_l \) when the perturbation goes to zero. The entire set of solutions will be needed to get the first-order corrections. Due to the symmetry properties of matrix \( Z^{(0)} \) (equation 26), the other solutions only differ from that described above by incorporating the eigenvalue \( \delta^{(0)} \alpha_l \) with spacing \( (\partial \alpha_l / \partial t)_l \), between the diagonal elements, multiplied by an arbitrary integer, and the corresponding shift in indexing the components of the eigenvector \( \epsilon^{(0)}_l \). We replace the upper index \( (0) \) with \( (0, j) \) for designating these solutions:

\[
\delta^{(0,j)} \alpha_l = \delta^{(0)} \alpha_l + j \frac{\partial \alpha_l}{\partial t}_l, \quad j = \pm 1, \pm 2, \ldots.
\]

\[
\epsilon^{(0,j)}_{l,p} = \epsilon^{(0)}_{l,p} + j \alpha_l, \quad j = \pm 1, \pm 2, \ldots.
\]

### 2.2 First-order corrections

We now address the solutions (equations 28 and 29) to equation (19) with \( Z = Z^{(0)} + Z^{(1)} \) (equation 25), considering matrix \( Z^{(1)} \) with elements specified by equation (27) as a small perturbation:

\[
(Z^{(0)} + Z^{(1)}) (\epsilon^{(0)}_l + \epsilon^{(1)}_l) = (\delta^{(0)} \alpha_l + \delta^{(1)} \alpha_l) \left( \epsilon^{(0)}_l + \epsilon^{(1)}_l \right).
\]

In terms of linear perturbation, we have

\[
Z^{(1)} \epsilon^{(0)}_l + Z^{(1)} \epsilon^{(1)}_l = \delta^{(1)} \alpha_l \epsilon^{(0)}_l + \delta^{(0)} \alpha_l \epsilon^{(1)}_l.
\]

We impose an orthogonality constraint on the correction to the eigenvector

\[
\epsilon^{(0),w}_l \cdot \epsilon^{(1)}_l = \sum_{p=0,1,\ldots} \epsilon^{(0),w}_{l,p} \epsilon^{(1)}_{l,p} = 0.
\]

Taking the dot products of both sides of equation (45) with \( \epsilon^{(0),w}_{l,p} \) and using the properties that vector \( \epsilon^{(0),w}_l \) has unit norm (equation B8) and matrix \( Z^{(0)} \) is self-adjoint, we get

\[
\delta^{(1)} \alpha_l = \epsilon^{(0),w}_l \cdot Z^{(1)} \epsilon^{(1)}_l.
\]

Using the explicit form of \( Z^{(1)} \) (equation 27) and implementing equations (B17) and (B19) of Appendix B, we obtain

\[
\delta^{(1)} \alpha_l = \frac{1}{4 \epsilon_0} \left( \frac{\partial \tilde{\omega}^2}{\partial t} \right)_l \sum_{k=-\infty}^{\infty} b^*_k b_k + \sum_{k=-\infty}^{\infty} b^*_k d_k.
\]

Mode interaction has a repulsive effect on the eigenvalues, which vanishes when the interaction is uniform along the interaction chain (p- or f-mode ridge). Equation (48) describes the corrections to the eigenvalues induced by a slightly non-equidistant separation of the undistorted eigenvalues \( \tilde{\omega}^2 \) (first term) and by a slow variation of the matrix elements (second term) along the interaction chain. The magnitude of \( \delta^{(1)} \alpha_l \) is quadratic in the magnitude of fluid-flow velocity, i.e. this frequency correction is of second order in the perturbation. In their dependence on the azimuthal order \( m, d_l \) have the same symmetry as that of \( b_l \) (equation 40):

\[
d_l(-m) = -d^*_l(m).
\]

Therefore, \( \delta^{(1)} \alpha_l \) is an even function of \( m \), which means that the second-order frequency corrections come to the even splitting coefficients.

We now look for the first-order correction to the coupling coefficients in the form of a linear combination of eigenvectors of the zero-order problem:

\[
\epsilon^{(1)}_l = \sum_{j=1,2,\ldots} \alpha_j \epsilon^{(0,j)}_l.
\]

Taking the dot products of the both sides of equation (45) with \( \epsilon^{(0,j)*}_l \), \( j = \pm 1, \pm 2, \ldots \), we have

\[
\alpha_j = \frac{\epsilon^{(0,j)*}_l \cdot Z^{(1)} \epsilon^{(1)}_l}{\delta^{(0)} \alpha_l - \delta^{(0,j)} \alpha_l}, \quad j = \pm 1, \pm 2, \ldots.
\]

Using equations (42) and (43), which specify the entire set of zero-order solutions, and expanding the result in the same way as with the frequency correction, we arrive at

\[
-j \left( \frac{\partial \alpha_l}{\partial t} \right)_n \alpha_j = \frac{1}{4 \epsilon_0} \left( \frac{\partial^2 \tilde{\omega}^2}{\partial l^2} \right)_l \left( \sum_{k=-\infty}^{\infty} b_k b^*_k + j b_j \right) + \sum_{k=-\infty}^{\infty} d_k b^*_k + \frac{1}{2} j d_j.
\]

One could argue that terms with \( (\partial^2 \tilde{\omega}^2 / \partial l^2)_l \) in our final expressions for the corrections to the oscillation frequencies and to the coupling coefficients (equations 48 and 52) are asymptotically small at high degree \( l \) and can be neglected. Indeed, at high degree \( l \), the frequencies of \( f \) modes satisfy, with a high accuracy, the simplest dispersion relation \( \omega^2 = kg \), where \( g \) is the surface gravitational acceleration and \( k \) is the wavenumber, \( k \approx l/R \). The linear variation of \( \tilde{\omega}^2 \) with \( l \) is also predicted by the dispersion relation of high-degree \( p \) modes in a polytropic envelope:

\[
\tilde{\omega}^2 \approx \frac{2 (n + \frac{1}{2} \mu) \Gamma_1}{\mu + 1} \Gamma_1^2 g,
\]

where \( \mu \) is the polytropic index and \( \Gamma_1 \) is the adiabatic exponent (e.g. Deubner & Gough 1984). But the stratification of the solar convective envelope in the propagation domain of high-degree \( p \) modes deviates substantially from that of a polytrope, principally because of the rapid variation of the adiabatic exponent in the region of partial ionization of H and He. Therefore, we retain these terms for further numerical evaluation.

### 3 EFFECTS OF CENTRIFUGAL FORCES

The analysis of the previous section was based on the oscillation equations where only terms that are linear in the fluid-flow velocity (advection terms) were retained in the momentum equation. Effects of the centrifugal forces, together with effects of the centrifugal distortion of the equilibrium configuration, have been discarded.
and need to be addressed separately. In solar seismology, these effects are very small, so that only the leading-order correction to the oscillation frequencies needs to be addressed. A straightforward perturbational analysis can be well implemented; for the relevant work in solar seismology, the reader is referred to Gough & Thompson (1990), Dziembowski & Goode (1992) and Antia et al. (2000). In this section, we develop simple analytical estimates, applicable to modes of high degree \( l \).

We implement a local analysis, considering the high-degree \( p \) and \( l \) modes as surface waves, with the dispersion relation

\[
k = k(\omega, g),
\]

where \( k \) is the horizontal wavenumber, \( k \approx lR \), and \( g \) is the effective gravity. When there is no rotation, the waves propagate along a big circle \( C_0 \) inclined at an angle \( \alpha \) to the rotation axis, with

\[
\sin^2 \alpha = \frac{m^2}{l^2}
\]

(with \( \alpha = 0 \) for the meridional propagation). The rotational distortion of the equilibrium configuration changes the wavepath from \( C_0 \) to \( C \). The frequency of the standing wave is governed by the quantization condition

\[
\oint_C k \, ds = 2\pi l.
\]

Considering a small variation induced by the distortion, we have

\[
\oint_{C_0} \delta \ln k \, ds + \oint_C \delta \ln g \, ds = 0,
\]

or

\[
\oint_{C_0} \delta \ln k \, ds + \oint_{C_0} \delta \ln \alpha \, ds + \oint_C \delta \ln g \, ds + \oint_C \delta \ln s \, ds = 0.
\]

The frequency correction will be evaluated from this expression.

We describe the geometry of the distorted solar surface and the gravitational potential near the surface by three-term expansions

\[
R(\theta) \simeq R_\odot [1 + \epsilon_2 P_2(\cos \theta) + \epsilon_4 P_4(\cos \theta)],
\]

\[
\psi \simeq -\frac{GM_\odot}{r} \left[ 1 - \left( \frac{R_\odot}{r} \right)^3 J_2 P_2(\cos \theta) - \left( \frac{R_\odot}{r} \right)^5 J_4 P_4(\cos \theta) \right],
\]

where \( J_2 \) and \( J_4 \) are the gravitational moments. Accordingly, the surface angular velocity is described by two dominant terms (cf. equation 10):

\[
\Omega(\theta) \simeq \Omega_\odot + \epsilon_\theta P_\theta(\cos \theta) + \epsilon_\phi P_\phi(\cos \theta),
\]

which gives

\[
\Omega^2(\theta) \simeq \Omega_\odot^2 + 3 \Omega_\odot \delta_\odot (5 \cos^2 \theta - 1),
\]

when we neglect \( \Omega_\odot^2 \) compared with \( \Omega_\odot \delta_\odot \).

Using equations (59) and (60), at the distorted surface, we have

\[
\nabla \psi \big|_{R = R(\theta)} \simeq \hat{F}_0 \left[ 1 - 2(\epsilon_2 + 2J_2) P_2(\cos \theta) - 2(\epsilon_4 + 3J_4) P_4(\cos \theta) \right] + \hat{g}_0 \left[ J_2 \frac{dP_2(\cos \theta)}{d\theta} + J_4 \frac{dP_4(\cos \theta)}{d\theta} \right],
\]

discarding terms that are quadratic in the oblateness coefficients \( \epsilon_2, \epsilon_4 \), where \( g_0 = GM_\odot/R_\odot^2 \) is the gravitational acceleration at the undistorted solar surface.

We now introduce vector \( \mathbf{R}(\theta) = \hat{F} R(\theta) \), with

\[
\frac{d\mathbf{R}}{d\theta} = \hat{F} \frac{d\mathbf{R}(\theta)}{d\theta} + \hat{\theta} \mathbf{R}(\theta).
\]

On the surface, the effective gravity is orthogonal to the level surface:

\[
[\nabla \psi - \Omega^2 r \sin \theta (\hat{F} \sin \theta + \hat{\theta} \cos \theta)]_{R = R(\theta)} \cdot \frac{d\mathbf{R}}{d\theta} = 0.
\]

Equation (65) assumes that the pressure gradient is orthogonal to the level surface – in other words, the level surface \( r = R(\theta) \) is defined by equation (65) as a surface of constant pressure. Because the solar rotation is slow and nearly uniform (\( \Omega_\odot \ll \Omega_\odot \)), we discard a small possible mismatch between surfaces of constant pressure and constant density, which may be induced by differential rotation. From this equation, we obtain two relations between the oblateness coefficients, gravitational moments and rotation rates:

\[
\epsilon_2 + J_2 \simeq -\frac{R_\odot^3}{GM_\odot} \left( \frac{1}{3} \Omega_\odot^2 + 8 \Omega_\odot \delta_\odot \right),
\]

\[
\epsilon_4 + J_4 \simeq -\frac{6}{7} \frac{R_\odot^3}{GM_\odot} \delta_\odot \Omega_\odot.
\]

The effective gravity \( g \) can be evaluated as

\[
g = \frac{|\nabla \psi - \Omega^2 r \sin \theta (\hat{F} \sin \theta + \hat{\theta} \cos \theta)|_{R = R(\theta)} \times \frac{\partial \hat{F}}{\partial \theta}|_{R = R(\theta)} | \theta |}{\partial \theta}.
\]

Using this expression, we obtain

\[
\delta \ln g \simeq 2(\epsilon_2 + J_2) - \frac{8}{3} (\epsilon_4 + J_4) \left[ 4 \epsilon_2 + 6 J_2 + \frac{20}{3} (\epsilon_4 + J_4) \right] P_2(\cos \theta) - (6 \epsilon_4 + 10 J_4) P_4(\cos \theta),
\]

which is the integrand in the second term of equation (58). The last term in equation (58) is

\[
\oint_C \delta \ln \epsilon_2 + \oint_{C_0} \delta \ln \epsilon_4.
\]

We are thus led to evaluating the integrals \( \oint_C P_2(\cos \theta) d\theta \) and \( \oint_{C_0} P_4(\cos \theta) d\theta \). We rotate the spherical coordinate system \( (r, \theta, \phi) \) by an angle \( \beta = \pi/2 - \alpha \) to the rotation axis, such that the wavepath \( C_0 \) is in the equatorial plane of the new coordinate system \( (r', \theta', \phi') \), and implement the transformation property of spherical harmonics under finite rotation (e.g. Edmonds 1960):

\[
Y_{l0}(\theta, \phi) = \sum_{m=-l}^{l} d_{m}^{(l)}(\beta) Y_{lm}(\theta', \phi').
\]

We then have

\[
\oint_{C_0} P_l(\cos \theta) d\theta = 2\pi R_\odot d_{l0}^{(l)} \left( \frac{\pi}{2} - \alpha \right) P_l(0),
\]

since terms with \( m \neq 0 \) in equation (71) have zero contribution to the integral. With

\[
P_l(0) = (-1)^{l/2} (l - 1)!!(l - 1)!! \frac{1}{l!}, \quad l = 0, 2, 4, \ldots
\]

\((P_l(0) = 0 \text{ when } l \text{ is odd})\), and with (Varshalovich, Moskalëv & Khersonskii 1988)

\[
d_{l0}^{(l)}(\beta) = P_l(\cos \beta),
\]
we get, using \( \cos \beta = \sin \alpha = \pm ml \),
\[
\oint_{c_0} P_2(\cos \theta) ds = -\pi R_\odot P_2 \left( \frac{m}{l} \right),
\]
and
\[
\oint_{c_0} P_4(\cos \theta) ds = \frac{3\pi}{4} R_\odot P_4 \left( \frac{m}{l} \right).
\]
Using the simplest high-degree asymptotic approximation
\[
\frac{\partial \ln k}{\partial \ln g} = -1
\]
(equation 53) and
\[
\omega \frac{\partial \ln \omega}{\partial \ln k} = l \left( \frac{\partial \omega}{\partial l} \right)_n,
\]
with \( \delta \omega \) described by its contribution to the even splitting coefficients
\[
\delta \omega = \delta^{(c)} a_0 l + \delta^{(c)} a_2 l P_2 \left( \frac{m}{l} \right) + \delta^{(c)} a_4 l P_4 \left( \frac{m}{l} \right),
\]
the effects of the centrifugal forces give
\[
\delta^{(c)} a_0 \simeq 2 \left[ \frac{\epsilon_2 + J_2}{3} (\epsilon_4 + J_4) \right] \left( \frac{\partial \omega}{\partial l} \right)_n,
\]
\[
\delta^{(c)} a_2 \simeq \frac{5}{3} \frac{\epsilon_2 + 3J_2}{3} \left( \frac{\partial \omega}{\partial l} \right)_n
\]
and
\[
\delta^{(c)} a_4 \simeq \frac{3}{8} (7\epsilon_4 + 10J_4) \left( \frac{\partial \omega}{\partial l} \right)_n.
\]
The solar gravitational moments can be evaluated most accurately, using a model of the interior structure, from the internal differential rotation inferred from helioseismic measurements at lower degree \( l \). The result obtained by Roxburgh (2001) is \( J_2 = 2.21 \times 10^{-7} \) and \( J_4 = -4.46 \times 10^{-9} \). With the surface values of \( \Omega_1 = 2\pi \times 435 \) nHz and \( \Omega_2 = -2\pi \times 15 \) nHz (e.g. Vorontsov et al. 2002), equations (66) and (67) give \( \epsilon_2 = -5.80 \times 10^{-6} \) and \( \epsilon_4 = 5.65 \times 10^{-7} \). We note that the oblateness coefficient \( \epsilon_4 \) is governed almost entirely by the centrifugal forces, with only 1 per cent contribution coming from the distortion in the gravitational field \( \mathcal{J}_4 \). To address possible uncertainties, these figures can be compared with the results of Antia et al. (2000) (who apparently used another sign convention for gravitational moments): \( J_2 = -2.18 \times 10^{-7}, J_4 = 4.64 \times 10^{-9}, \epsilon_2 = -5.84 \times 10^{-6} \) and \( \epsilon_4 = 6.2 \times 10^{-7} \). Adopting the gravitational moments from Roxburgh (2001) and the oblateness coefficients evaluated as described above, we obtain
\[
\delta^{(c)} a_0 \simeq -1.26 \times 10^{-5} \left( \frac{\partial \omega}{\partial l} \right)_n,
\]
\[
\delta^{(c)} a_2 \simeq -1.42 \times 10^{-5} \left( \frac{\partial \omega}{\partial l} \right)_n,
\]
and
\[
\delta^{(c)} a_4 \simeq 1.47 \times 10^{-6} \left( \frac{\partial \omega}{\partial l} \right)_n.
\]

### 4 NUMERICAL ESTIMATES

We start with addressing the most challenging issue – the accuracy of the theoretical description of the mode coupling by differential rotation, which is currently implemented in the helioseismic data analysis (Vorontsov 2007).

4.1 Mode coupling by the differential rotation

We first address the correction to the mode-coupling coefficients, specified by the expansion coefficients \( \alpha_j \) (equations 50 and 52). To simplify the estimates, we approximate the differential rotation by accounting for its dominant component \( \Omega_1 \) only. This component is measured by the rotational-splitting coefficient \( \alpha_1 \); for modes of high degree \( l \), \( \alpha_1 \) is about \( 2\pi \cdot 21.5 \) nHz (e.g. Vorontsov et al. 2009). The dimensionless matrix elements \( b_j \) (equation 33) are all real and only differ from zero when \( k = \pm 2 \); the values of \( d_j \) (equation 23) are all real and differ from zero when \( k = 0 \) and \( k = \pm 2 \). Explicitly, we have
\[
b_2 = \frac{5}{4} l \left( \frac{\partial \omega}{\partial l} \right)_n^{-1} a_1 m \left( 1 - \frac{m^2}{l^2} \right),
\]
\[
d_2 = \frac{5}{2} a_1 m^3 l^5,
\]
and \( d_2 = 2d_2, b_{-2} = b_2 \) and \( d_{-2} = d_2 \). The sum \( \sum_{n=0}^{\infty} b_j b_{-j} \) is 2\( \delta_l \) when \( j = 0 \), \( b_2 \) when \( j = \pm 4 \) and zero otherwise. The sum \( \sum_{n=0}^{\infty} d_j b_{-j} \) is 2\( \delta_l \) when \( j = 0 \) or \( j = \pm 2 \), \( d_2 b_2 \) when \( j = \pm 4 \) and zero otherwise. We thus have
\[
\mp 2 \left( \frac{\partial \omega}{\partial l} \right)_n^{\pm} \approx \pm \frac{1}{2} \left( \frac{\partial^2 \omega}{\partial l^2} \right)_n b_2 + d_2 (2b_2 \pm 1),
\]
and
\[
\mp 4 \left( \frac{\partial^2 \omega}{\partial l^2} \right)_n \approx \frac{1}{4} \left( \frac{\partial^2 \omega}{\partial l^2} \right)_n b_2 + d_2 b_2.
\]
Among modes with different azimuthal order \( m \), \( b_2 \) reaches its maximum when \( m^2 = \Omega_l^2 \). Among multiplets with different radial order \( n \), \( b_2 \) is biggest for \( f \) modes, where
\[
\left( \frac{\partial \omega}{\partial l} \right)_n \approx \frac{1}{2 \Omega_l^2} \left( \frac{GM_\odot}{R_\odot^3} \right)^{1/2}.
\]
At given \( l \), the maximum value of \( |b_2| \) is then evaluated as
\[
\max |b_2| \approx 2.1 \times 10^{-4} l^{3/2}.
\]
Note that the coupling is strong, with neighboring components having nearly the same amplitudes in the composite eigenstates, when \( |b_2| \) is about 1. For \( f \) modes, equation (87) shows that we have a strong coupling starting with \( l \) of 300.

In the same way, we get an estimate
\[
\max \left| \left( \frac{\partial \omega}{\partial l} \right)_n d_2 b_2 \right| \approx 8.5 \times 10^{-8} l^2.
\]
For a strong coupling (\( |b_2| \approx 1 \) or bigger), we thus have
\[
\max \left| \alpha_j \right| \approx 4.2 \times 10^{-8} l^2 + 8.5 \times 10^{-8} l^2,
\]
where
\[
\gamma = \frac{1}{2} \left( \frac{\partial \omega}{\partial l} \right)_n^{\pm} / \left( \frac{\partial^2 \omega}{\partial l^2} \right)_n.
\]
To evaluate \( \gamma \), we are addressing the theoretical eigenfrequencies of a solar model. The result is that in the observable \((l, \omega)\) domain of solar oscillations, \( \gamma \) does not exceed 0.22. This result confirms the expectation that the effect of the non-uniform eigenvalue spacing along the interaction chain is small. The final estimate is
\[
\max |\alpha_j| \approx 9 \times 10^{-9} l^2.
\]
With \( \alpha_j \) less than \( 8 \times 10^{-3} \) at \( l = 300 \) and \( 9 \times 10^{-2} \) at \( l = 1000 \), we conclude that the correction to the coupling coefficients can be safely ignored in data analysis, at least up to degree \( l \) of about 1000.
We now address the corrections that are induced in the oscillation frequencies (equation 48). In the same way as with \( \alpha_{l} \), we obtain

\[
\max \left| \frac{1}{\partial \omega}{\partial l} \right|^{-1} \delta^{(1)} \omega_{l} < 1.8 \times 10^{-2} R^{2}.
\] (92)

The right-hand side is \( 1.6 \times 10^{-2} \) at \( l = 300 \) and \( 1.8 \times 10^{-4} \) at \( l = 1000 \). With \( \partial \omega / \partial l \), measured at several \( \mu \text{Hz} \), the effect in the oscillation frequencies may be well observable. Therefore, we resort to a more detailed analysis, which involves numerical computation.

The computations were done directly for the contribution of the mode-coupling effects to the even frequency-splitting coefficients. The angular integrals of the triple products of the Legendre polynomials were evaluated using Wigner’s 3-j symbols. The integrals containing the derivatives of the Legendre polynomials (coming from \( d_{l} \)) were evaluated using integration by parts and the explicit expressions

\[
x \frac{d}{dx} P_{l}(x) = 2 P_{l}(x) + P_{0}(x),
\]

\[
x \frac{d}{dx} P_{l}(x) = 4 P_{l}(x) + 5 P_{l}(x) + P_{0}(x),
\]

\[
x \frac{d}{dx} P_{l}(x) = 6 P_{l}(x) + 9 P_{l}(x) + 5 P_{l}(x) + P_{0}(x),
\] (93)

which can be obtained from recurrence relations for the Legendre polynomials. The results obtained for solar \( f \) modes in the degree range from 100 to 300 using two dominant components of the differential rotation measured by the rotational-splitting coefficients \( \alpha_{2} = 2 \pi \times 21.5 \text{ nHz} \) and \( \alpha_{3} = -2 \pi \times 4.5 \text{ nHz} \) (Vorontsov et al 2009) are shown in Fig. 1. Also shown is the contribution of the centrifugal effects (evaluated using equations 81) and the measured solar values of \( \delta^{(2)} \omega_{l} \), \( \alpha_{l} \), and \( \delta^{(4)} \omega_{l} \). The magnitude of random errors in the measurements can be judged from the scatter in the plots: individual data points, corresponding to the separate frequency multiplets, shall fall on smooth curves.

In the \( \alpha_{2} \) coefficient (Fig. 1a), the contribution of the mode coupling is comparable with that of centrifugal forces and starts to dominate from degree \( l \) of about 200. Taken together, the two effects appear to provide an adequate fit to the observational data. The \( \alpha_{3} \) coefficient (Fig. 1b) is overwhelmingly dominated by the mode-coupling effects. Interestingly, we again have a good fit to the observations. If the mode coupling were discarded, we would have to attribute the significant observational values of \( \alpha_{3} \) to asphericities induced by the solar magnetic activity; however, it would be difficult to reconcile such an interpretation with the fact that the data were collected when the solar activity was near its minimum (the first year of \textit{SOHO} MDI measurements). The comparison with observations suggests that when the solar activity is near its minimum, the \( \alpha_{2} \) coefficient of solar \( f \) modes is governed by the joint effect of rotational distortion and mode coupling, the \( \alpha_{3} \) coefficient is dominated by the mode-coupling effects, and the effects of magnetic activity in both \( \alpha_{2} \) and \( \alpha_{3} \) are hardly detectable. Since the effects of magnetic activity in even splitting coefficients are known to correlate well with surface magnetic indices, this suggestion can be verified in future by accurate measurement of the temporal variation of \( \alpha_{2} \) and \( \alpha_{3} \). In the \( \omega_{l} \) coefficient (Fig. 1c), the effects of mode coupling are below the observational uncertainties.

4.2 Mode coupling by meridional circulation

The effects of meridional circulation are small compared with those of differential rotation, and we only address their contribution to the mode-coupling coefficients to leading order (equation 32). When there is no meridional circulation, the coupling coefficients \( c_{2,p}^{(0)} \) are all real. A slow meridional flow brings corrections of the order of \( v_{\text{mer}}^{2} \) to the real part of \( c_{2,p}^{(0)} \) and induces an imaginary part with magnitude of the order of \( v_{\text{mer}}^{2} \). Therefore, the effects of meridional flows in the observational power spectra of solar oscillations are of the order of \( v_{\text{mer}}^{2} \), if small spatial leaks with imaginary amplitudes (e.g., those coming from a small CCD tilt; Vorontsov & Jefferies 2005) are neglected.

For a numerical estimate, consider a simple model of two-cell circulation (one cell in each hemisphere) with flow velocity near the surface (where high-degree modes are trapped) described as

\[
v_{\text{mer}}(R, \theta) = \frac{1}{2} \left( \frac{5}{4 \pi} \right) ^{1/2} R \sin 2 \theta.
\] (94)

The matrix elements of this perturbation (equation 35) differ from zero when \( k = \pm 2 \) only; explicitly, we have

\[
\text{Im}(b_{2}) = -\frac{3}{4} \left( \frac{5}{4 \pi} \right) ^{1/2} \left( \frac{\partial \omega}{\partial l} \right) ^{-1} \frac{v_{2}(R)}{R} l \left( 1 - \frac{m^{2}}{l(l+1)} \right).
\] (95)
Consider zonal \((m = 0)\) modes only, which are most influenced by the effects of meridional circulation (since these are waves propagating along the meridian). Another convenience in choosing \(m = 0\) modes is that they are not affected by differential rotation (with fluid flow orthogonal to wave propagation). Defining the maximum surface velocity as \(v_{\text{max}}\), i.e.

\[
v_{\text{mer}}(R, \theta) = -\hat{\theta} v_{\text{max}} \sin 2\theta, \quad v_{\text{max}} = \frac{3}{2} \left( \frac{5}{4\pi} \right)^{1/2} v_2(R),
\]

we have

\[
b_2 = -\frac{i}{2} l \left( \frac{\partial \hat{\omega}}{\partial l} \right)^{-1} \frac{v_{\text{max}}}{R}.
\]

Consider \(f\) modes with \(l = 300\) as an example, which have \(\delta \hat{\omega}/\delta l \approx 2 \pi \times 2.86 \mu\text{Hz}.\) With a representative value of about 15 m s\(^{-1}\) for \(v_{\text{max}}\) (e.g. Schou 2003; Zhao & Kosovichev 2004), we have \(b_2 \approx -0.17i.\) For the coupling coefficients, we obtain \(c_{l, \pm 2} \approx -0.08i\) (only modes with degree of the same parity interact). We thus expect that mode coupling induced by the meridional circulation is hardly detectable in the observational power spectra, at least when \(l < 300\). The effect can be well measurable, however, when addressed in the cross-correlations between complex Fourier-amplitude spectra, as was first suggested by Woodard (2000).

Since the diagonal matrix elements of the meridional circulation are identically zero, the effect in the oscillation frequencies is of second order in fluid velocity (the symmetry is such that changing the flow to the opposite leaves frequencies unchanged). Evaluating \(\delta^{(l)}_{\text{PHD}}\) using equation (48) for the same model of meridional circulation and for modes with \(m = 0\), we have

\[
d_{k=2} = \pm \frac{3}{4} l \left( \frac{5}{4\pi} \right)^{1/2} \frac{v_2(R)}{R}.
\]

Discarding a smaller term with \(\hat{\delta}_k \partial \hat{\omega}/\partial \delta^2\) in equation (48), we have

\[
\delta^{(l)}_{\text{PHD}} = -\frac{45}{32\pi l} \int \left( \frac{\partial \hat{\omega}}{\partial l} \right)^{-1} \left[ \frac{v_2(R)}{R} \right]^2.
\]

For \(f\) modes with a degree of 300 and \(v_{\text{max}} \approx 15\) m s\(^{-1}\) as before, we get \(\delta^{(l)}_{\text{PHD}} \approx -2\pi \times 0.53 \text{ nHz}.\) This is a very small quantity; an inherent frequency resolution of one-year measurements is 32 nHz. We conclude that the effects of meridional circulation in the oscillation frequencies are far below the detectability level of helioseismic measurements.

The influence of meridional circulation on solar p-mode frequencies has been addressed earlier by Roth & Stix (2008) by means of direct numerical treatment of the perturbation equations. The frequency shifts that were obtained at high degree \(l\) are far bigger than ours (our numerical estimate refers to \(f\) modes, but at the same degree \(l\) this only makes the discrepancy worse). To resolve the controversy, we address an alternative way of evaluating the frequency shift.

Consider high-degree modes of frequency \(\omega\) with \(m = 0\) as surface waves with dispersion relation \(k = k(\omega),\) propagating along the meridian. Advection by a slow meridional flow changes the local wavenumber from \(k\) to \(k + \delta k\). In a local frame of reference moving together with the fluid flow, the frequency is \(\omega + \delta \omega,\) with \(\delta \omega = \delta \omega_0\) such that \(\delta k = (dk/d\omega)\delta \omega_0\). Let the fluid velocity be \(v\), and the wave propagates in the direction of the flow. The phase speeds of the same wave in the moving frame of reference and in the frame that is at rest differ by:

\[
\frac{\omega}{k + \delta k} - v = \frac{\omega + \delta \omega_0}{k + \delta k}.
\]

From this equation, we obtain the variation induced in the wave-number:

\[
\delta k = -k \frac{\partial k}{\partial \omega} v + k \left( \frac{\partial k}{\partial \omega} \right)^2 v^2 + O(v^3).
\]

The wave propagates along a big circle \(C_0\), with the phase integral

\[
\oint_{C_0} k \, ds = 2\pi l.
\]

Taking a small variation induced by the fluid flow, we have

\[
2\pi R \frac{\partial k}{\partial \omega} \delta \omega + \oint_{C_0} \delta k \, ds = 0.
\]

With \(\delta \omega\) evaluated in equation (101), we have

\[
\delta \omega \approx -\frac{1}{2\pi R} \left( \frac{\partial \hat{\omega}}{\partial l} \right)^{-1} \int_0^\pi v^2(\theta) \, d\theta.
\]

With flow velocity specified by equation (94), we have

\[
\int_0^\pi v^2(\theta) \, d\theta = \frac{45}{32} v_2(R),
\]

arriving at exactly the same result for the frequency shift as that provided by the mode-coupling analysis (equation 99).

5 SUMMARY

In this paper, the perturbational analysis of the solar oscillation-mode coupling at high degree \(l\) is developed to the approximation of higher order by lifting the simplifying assumptions which were used in the previous study (Vorontsov 2007). The analysis is extended to incorporate, into a single theoretical description, the effects of meridional circulation. A simplified semi-analytical description is developed for the effects of centrifugal distortion in oscillation frequencies of modes with high degree \(l\).

The most important conclusion for current work in global solar seismology is that the earlier description of the mode coupling by differential rotation (Vorontsov 2007) provides an adequate accuracy of the coupling coefficients in the degree range of up to about 1000.

The effects of mode coupling on the oscillation frequencies of modes with high degree \(l\) are comparable to or bigger than the effects of centrifugal distortion and need to be taken into account in the interpretation of the even component of the oscillation frequency splittings.

The contribution of the effects of meridional circulation to the mode coupling is definitely measurable by the cross-correlation analysis suggested by Woodard (2000), but it is hardly detectable in the oscillation power spectra, at least in the degree range of up to 300. The effects of meridional circulation on the oscillation frequencies at high degree \(l\) are far below the detectability level of helioseismic measurements.

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APPENDIX A: MATRIX ELEMENTS

We start with the exact expressions for the required matrix elements, which have been derived by Lavelly & Ritzwoller (1992) [note that an alternative convention exp(i\omega t) for time dependence was used in their paper, which is equivalent to changing the sign of \omega]. With unperturbed eigenfunctions normalized according to equation (3), the exact expressions are

\[
\langle \hat{u}_r, \delta H_{\text{lin}} \hat{u}_l \rangle = (-1)^{s+1} \times \left[ (l+1)!/(l'+1)!(2l+1)(2l'+1) \right]^{1/2} \times \sum_{s=1,2} \left( \begin{array}{ccc} s & l' & l \\ 0 & 0 & -m \end{array} \right) \times \int_0^R \rho_0 r^2 \Omega_s(r) \left[ U'U - U'V - V'U \right] \left[ l(l+1) + l'(l'+1) - s(s+1) \right] V'V dr,
\]

and

\[
\langle \hat{u}_r, \delta H_{\text{non-lin}} \hat{u}_l \rangle = i(-1)^{s+1} \left[ (2l+1)(2l'+1) \right]^{1/2} \times \sum_{s=1,2} \left( \begin{array}{ccc} s & l' & l \\ 0 & 0 & -m \end{array} \right) \times \left( 2s + 1 \right) \left( 4r \right) \int_0^R \rho_0 r^2 \left( \frac{U'\dot{U} - U'U}{2} \right) l(l+1) + l'(l'+1) - s(s+1) dr.
\]

where the dot denotes differentiation with respect to \( r \).

For modes of high degree \( l \), these expressions can be significantly simplified using the semiclassical approximation for Wigner's 3-j symbols (Brussaard & Tolhoek 1957; Dahlen & Tromp 1998). When \( s \ll l \) and \( s \ll l' \), we have

\[
\left( s \ l' \ l \ 0 \ m \ -m \right) \simeq \left( -1 \right)^{l'+m} \left( \frac{l + l' - 1}{2l'!} \right)^{1/2} \left( \frac{l + l' - 1}{2l!} \right)^{1/2} P_{l'}^{-l}(m_T),
\]

where \( P_{l'}^{-l}(m_T) \) is an associated Legendre polynomial. For the product of two 3-j symbols, which enter equations (A1) and (A2), we get

\[
\left( s \ l' \ l \ 0 \ 1 \ -1 \right) \left( s \ l' \ l \ 0 \ m \ -m \right) \simeq (-1)^{l + l' - 1} \left( \frac{l + l' - 1}{2l'!} \right)^{1/2} \left( \frac{l + l' - 1}{2l!} \right)^{1/2} P_{l'}^{-l}(m_T),
\]

when \( s + l' + l \) is odd (and zero otherwise), and

\[
\left( s \ l' \ l \ 0 \ 0 \ 0 \right) \left( s \ l' \ l \ 0 \ m \ -m \right) \simeq (-1)^{l + l' - 1} \left( \frac{l + l' - 1}{2l'!} \right)^{1/2} \left( \frac{l + l' - 1}{2l!} \right)^{1/2} P_{l'}^{-l}(m_T),
\]

when \( s + l' + l \) is even (and zero otherwise). The first of the 3-j symbols in equation (A4) has been evaluated using equation (A3) with \( m = 1 \), and the leading-order behaviour of the corresponding Legendre polynomial at small argument. Since radial eigenfunctions of interacting modes (with \( |l| \ll l \) and the same radial order \( n \) are nearly the same, the matrix elements are reduced to

\[
\langle \hat{u}_r, \delta H_{\text{lin}} \hat{u}_l \rangle \simeq l \sum_{s=1,2} \left( -1 \right)^{s+1} \int_0^R \rho_0 r^2 \Omega_s(r) \left[ U^2 + l(l+1) \right] V'V dr,
\]

\[
\langle \hat{u}_r, \delta H_{\text{non-lin}} \hat{u}_l \rangle \simeq l \sum_{s=1,2} \left( -1 \right)^{s+1} \int_0^R \rho_0 r^2 \Omega_s(r) \left[ U^2 + l(l+1) \right] V'V dr.
\]
and
\[
(\tilde{u}_l, \delta H_{\text{sec}} \tilde{u}_l) \simeq i(l' - l) \sum_{c=0,1} \left( -1 \right)^{c+l'} \left( \frac{2c+1}{4\pi} \right)^{1/2} 
\times \frac{ (s-l'+l-1)!(s+l'-l-1)! }{ (s+l'-l)! } 
\times \int_0^R \rho_0 \rho v_l(r) \left[ U^2 + (l+l')V^2 \right] \, dr.
\]
(A7)

Note that all the matrix elements of \( \delta H_{\text{sec}} \) are real, and those of \( \delta H_{\text{int}} \) are purely imaginary. Their Hermitian symmetry (both the operators are self-adjoint) is ensured by the relation
\[
P_p^{-1} = \left( -1 \right)^p \frac{(s-l)!}{(s+l)!} P_p^1(c).
\]
(A8)

**APPENDIX B: THE SOLUTION OF THE ZERO-ORDER PROBLEM OF MODE COUPLING**

In component notation, the algebraic eigenvalue problem (equation 30) is
\[
\left[ p \left( \frac{\partial \tilde{u}_l}{\partial t} \right) - \delta^{(0)}_{ll} \right] c_p + \sum_{p' = 0, \pm 1 \ldots} M_{pp'} c_{p'} = 0,
\]
where \( c_p = c_p^{(0)} \) is for brevity. The eigenvalue which goes to zero when matrix elements go to zero (so that \( \omega_l \to \omega_l^0 \)) is the central diagonal element \( M_{ll} \) (equation 31). The proof that this is indeed an eigenvalue comes from the existence of the non-trivial (non-zero) solution to the homogeneous system of algebraic equations (B1), which is derived below. With \( \delta^{(0)}_{ll} = M_{ll}, \) the algebraic system (B1) is
\[
p c_p = \sum_{p' = 0, \pm 1 \ldots} b_{p-p'} c_{p'} = 0,
\]
where coefficients \( b_{p-p'} \) are defined by equation (33). Replacing the summation index with \( k = p - p', \) we have
\[
p c_p = \sum_{k = \pm 1, \pm 2 \ldots} b_k c_{p-k} = 0,
\]
the equation which can be considered as a single recurrence relation between \( c_p \), with \( p = 0, \pm 1, \ldots \). Introducing a comparison equation
\[
dc(t) + ib(t)c(t) = 0
\]
with
\[
b(t) = \sum_{p = -\infty}^{\infty} b_p e^{-ipt},
\]
it is easy to see that equation (B3) is nothing else but the relation between the Fourier coefficients \( c_p \) of \( c(t), \)
\[
c_p = \frac{1}{2\pi} \int_{-\pi}^{\pi} c(t) e^{ipt} \, dt.
\]
(Solving equation (B4) with \( b_0 = 0 \) and \( b_{-p} = b_p^* \), we have
\[
c(t) = B \exp \sum_{p = 1, 2 \ldots} \frac{1}{p} \left( b_p e^{-ipt} - b_p e^{ipt} \right).
\]
Since the argument of the exponent in this expression is purely imaginary, we have \( |c(t)| = |B| \). Setting \( B = 1 \) to ensure the normalization of the expansion coefficients
\[
\sum_{p = 0, \pm 1 \ldots} c_p \overline{c}_p^* = \frac{1}{2\pi} \int_{-\pi}^{\pi} |c(t)c(t)| \, dt = 1.
\]
(B8)