A STELLAR MODEL WITH A MIXED OPACITY LAW

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Summary

Stellar models are integrated in which the opacity is due both to photoelectric effect and to electron scattering. The integrations are performed by applying a method outlined in an earlier paper.*

i. In an earlier paper a method has been described, by the use of which the equations of stellar structure may be integrated by means of variables which are homology-invariant (h-variables). This method singles out the hydrostatic equation as the most important condition and one which is expressible immediately in terms of h-variables, while it places the energy transport equation in a secondary position (as it is physically). In the most important simple cases, the energy transport equation is expressible in terms of h-variables and the methods of Paper I are applicable. There is, however, a slightly more complicated case of considerable interest, mentioned briefly in Paper I, which will be discussed here. The energy transport equation is no longer expressible in terms of h-variables, but involves one variable which, while not homology-invariant, is very slowly varying. This case can be brought within the scope of the earlier method by slight modifications and approximations.

The model used in this paper is the generalized Cowling model, which consists of a core in convective equilibrium with an atmosphere in radiative equilibrium, radiation pressure being neglected throughout and the molecular weight being assumed to be constant. Energy generation is assumed to take place in the core only. The particular case mentioned above and to be discussed in this paper is that in which, in the region in radiative equilibrium, both the free-free transitions of electron scattering and the photoelectric effect make contributions to the opacity. The guillotine factor in the photoelectric effect is neglected. This is a case of considerable physical interest, and does not appear to have been discussed elsewhere. It should be noticed that the relative contribution of the two sources to the opacity varies from point to point, and the contribution from electron scattering always increases relatively from the surface towards the centre. It can be shown from the equations that at low temperatures, that is, very near the star's surface, the contribution of photoelectric effect to the opacity (Kramers' opacity) always completely outweighs that due to electron scattering. Thus a radiative atmosphere with opacity due purely to electron scattering does not in fact exist, although such an atmosphere may exist with opacity due purely to photoelectric effect. It is thus of considerable physical interest to investigate a whole range of possible configurations from the case of pure Kramers' opacity to those in which, at some point interior to the surface but close to it, electron scattering opacity is predominant. Since, at the surface, Kramers' opacity must always predominate, it is models of this last type which are in fact of interest when

* C. M. Bondi and H. Bondi, M.N., 109, 62, 1949. This will be referred to as Paper I.
investigating stars in which, through a large proportion of the volume, electron scattering opacity predominates.

2. The contribution of photoelectric effect to the opacity is given by the expression \( \alpha (1 - X^2) \rho /T^3 \), where \( \alpha \) is a constant, \( X \) is the proportion of hydrogen by mass, \( \rho \) is the density and \( T \) the temperature. The corresponding contribution of electron scattering is \( \kappa (1 + X) \), where \( \kappa \) is a constant.

The energy transport equation in a radiative region can be expressed conveniently in the form

\[
\frac{dT}{dr} = \text{opacity} \times \frac{3L_\odot}{16\pi ac \ r^2 T^3},
\]

where \( L_\odot \) is the total luminosity and the usual notation is used. It is convenient to introduce constants \( B \) and \( C \), defined by

\[
B = \frac{3\alpha (1 - X^2)}{16\pi ac} L_\odot
\]

and

\[
C = \frac{3\kappa (1 + X)}{16\pi ac} L_\odot.
\]

\( B \) and \( C \) are constant in any one configuration but vary between configurations.

It has been pointed out above that the relative contribution of the two sources of opacity varies throughout the radiative region, so that no constant can express their ratio at all points of a configuration. However, it is clear that some parameter involving \( B \) and \( C \) can be defined so that all possible configurations form an ordered sequence, a member of the sequence being defined by a particular value of the parameter. A convenient form for this parameter was found to be

\[
E = \frac{\mathcal{R} r_*^4 \rho^6}{4\pi G C \Omega \mu},
\]

where \( \mathcal{R} \) is the gas constant, \( G \) the constant of gravitation, \( \mu \) the molecular weight and \( r_* \) the radius of the star. The particular significance of \( E \) will be seen later.

It should be observed that when \( E \) and the chemical composition have been specified, the integrations lead to only a single infinity of possible stars, that is, given the mass, the radius and luminosity are determined. Thus, the assumption of a particular mode of generation of energy, such as Bethe's law, defines a single star of a particular mass, luminosity and radius.

3. We now consider the form of the equations. The hydrostatic equation is, in the usual notation,

\[
\frac{dP}{dr} = -\frac{G \rho M}{r^2}, \tag{1}
\]

while conservation of mass is given by

\[
\frac{dM}{dr} = 4\pi r^2 \rho. \tag{2}
\]

The energy transport equation is

\[
\frac{dT}{dr} = -\frac{B \rho^2}{r^2 T^6} - \frac{C \rho}{r^2 T^3}. \tag{3}
\]

Since radiation pressure is neglected,

\[
P = \frac{\mathcal{R} \rho T}{\mu}. \tag{4}
\]

* The suffix \( s \) refers to surface values.
The variables $S$, $Q$, $N$ introduced in Paper I are defined by

$$S = \frac{4\pi Pr^4}{GM^2}, \quad Q = \frac{Pr}{GM\rho}, \quad 1 - N = \frac{d(\log \rho)}{d(\log P)}.$$

As shown there, the hydrostatic and mass conservation conditions together yield the equation:

$$\frac{S \, dQ}{Q \, dS} = \frac{S + N - Q}{1 + 2S - 4Q}.$$  \hfill (5)

Since

$$\frac{r \, dT}{T \, dr} = N \frac{r \, dP}{P \, dr},$$

it follows that

$$\frac{N}{Q} = \frac{C\rho}{rT^4} + \frac{Bp^2}{rT^{7.5}}$$

or

$$N = \frac{C}{G} \left( \frac{A}{\mu} \right)^4 \frac{\rho^4}{MP^8} (1 + \sigma).$$  \hfill (6)

where

$$\sigma = \frac{B}{C} \left( \frac{A}{\mu} \right)^{3.5} \frac{\rho^{4.5}}{P^{8.5}}.$$  \hfill (7)

$\sigma$ is therefore the ratio of the contributions of Kramers’ and electron opacities at any point. It is not an h-variable.

The energy transport equation now becomes

$$\frac{S \, dN}{N \, dS} = \frac{S + (1 - 4N) - \frac{\sigma}{1 + \sigma} (4.5N - 1)}{1 + 2S - 4Q}.$$  \hfill (8)

or, putting $\sigma/(1 + \sigma) = \nu$,

$$\frac{S \, dN}{N \, dS} = \frac{S + (1 - 4N) - \nu (4.5N - 1)}{1 + 2S - 4Q}.$$  \hfill (8')

From the definition of $E$ and of $\sigma$, it follows that

$$\frac{\nu^6}{(1 - \nu)^4} = E \frac{SN^5}{Q^7} \left( \frac{r}{r_0} \right)^3.$$  \hfill (9)

This brings out the significance of $E$ and also emphasizes the fact that $\nu$ is not an h-variable, for on the right-hand side of equation (9) $S$, $N$, $Q$, but not $r/r_0$ are h-variables.

From its definition, it is necessary that $0 \leq \nu \leq 1$. $\nu = 0$ corresponds to opacity due entirely to electron scattering, and $\nu = 1$ corresponds to opacity due entirely to photoelectric effect. On the other hand, $\nu^6/(1 - \nu)^4$ can vary between 0 and infinity, and hence the right-hand side of equation (9), which can be treated as a function of $S$, can range from 0 to infinity. Hence $\nu$ is a slowly varying function of $S$. Or, to express it otherwise, if the right-hand side of (9) is known approximately, $\nu$ can be determined accurately. For example, if $E[SN^5(r/r_0)^3/Q^7]$ is equal to $1.09 \times 10^3$, then $\nu$ is equal to 0.60, whereas if $E[SN^5(r/r_0)^3/Q^7]$ is equal to $1.66 \times 10^3$, then $\nu$ has only changed to 0.61.

We now discuss briefly the point which has been mentioned above, that at the surface the opacity is due entirely to photoelectric effect.

In the neighbourhood of the surface, that is, as $T$ tends to zero, it is necessary that $\rho$ should vary like $T^{3.25}$. Hence, the second term on the right-hand side of
equation (3) which arises from electron scattering, is proportional to \( T^{0.25} \), while the first term, which arises from photoelectric effect, is constant. Therefore, as \( T \) tends to zero, the photoelectric effect term becomes relatively large. This is equivalent to the statement that the surface condition on \( \sigma \) is that \( \sigma \) tends to infinity as \( r \) tends to \( r_s \), or that \( \nu \) tends to \( I \) as \( r \) tends to \( r_s \).

4. The problem therefore becomes that of integrating the equations (5) and (8') with \( S \) as independent variable, where \( \nu \) is given by (9). The problem could be solved in exactly the same manner as outlined for various other problems in Paper I but for the fact that \( r/r_s \) enters into (9).

It might be suggested that a differential equation might be used to determine \( \nu \), since in fact \( \nu \) satisfies the equation

\[
\frac{S}{\nu(I-\nu)} \frac{d\nu}{dS} = -\frac{4.5N-I}{I+2S-4Q},
\]

which, apart from \( \nu \), contains only \( h \)-variables. However, the simplicity of this method is somewhat illusory, since a solution for \( \nu \) must involve a constant of integration which is not homology-invariant.

In fact it was found that since complete integrations were available for the cases of opacity due purely to photoelectric effect and due purely to electron scattering, \( r/r_s \) could be reasonably well estimated for intermediate cases. Since, as has been pointed out, \( \nu^5/(I-\nu)^4 \) is a very quickly varying function of \( \nu \), this was sufficient to give an accurate estimate of \( \nu \).

Since the neighbourhood of the star's surface is of greatest interest in the integration, the changeover from the mixed opacity law to the Kramer's opacity law occurring there, the integration was performed in two parts; for convenience the dividing line being at \( S=0.1 \). First, a value of \( E \) was chosen. To each value of \( E \) there corresponds a single infinity of solutions running to the origin, and any such solution can be characterized by the value of \( Q \) at some specified value of \( S \), say \( S=0.1 \). For any such solution, \( N \) is then completely determined as a function of \( S \). The required solution is that which gives the correct conditions of fit on to the convective core, the solution for which is known, being an incomplete Emden polytrope for \( n = 5/3 \). Some value of \( Q \) at \( S=0.1 \) was therefore assumed, and from this assumption the value of \( N \) at \( S=0.1 \) for this solution was evaluated. In fact, an estimate of \( N \) at \( S=0.1 \) must be made, but this can be made very accurately, since a good approximation can be made by interpolation between the two extreme cases (integrations in these two cases already existing). This approximation can then be tested by the smoothness of the resulting integration of \( N \) as a function of \( S \), which is highly stable. The value of \( \nu \) at \( S=0.1 \) can be obtained from the values of \( Q \) and \( N \) there and from an estimate of \( r/r_s \).

The integration of the equations for increasing \( S \) (i.e. inwards) then proceeded exactly as in Paper I, using the graphs described there, except that \( \nu \) had to be evaluated at each stage from an assumed value of \( r/r_s \). If no fit on to the core was obtained, another value of \( Q \) was taken and the integration repeated. The procedure was continued until the correct value of \( Q \) at \( S=0.1 \) was found. In practice, the correct value of \( Q \) at \( S=0.1 \) was not difficult to find, as it can be estimated by interpolation between the two extreme cases. It was generally found that not more than two integrations had to be performed.

Having determined, for a given value of \( E \), the value of \( Q \) at \( S=0.1 \) which gives a point of fit on to the Emden core, we consider the integration outwards.
from $S = 0.1$ to the surface. Integration in this direction is highly unstable for
$N$ and the most satisfactory procedure is as follows. At $S = 0.0$ (the surface)
$N = \frac{4}{17}$, while at $S = 0.1$ a good estimate of it has been obtained. Also at the
surface $dN/dS$ is infinite. The run of $N$ must lie between the two extreme cases
and therefore some idea of the correct run of $N$ can be obtained. A point which
assists this construction is that the value of $E$ gives some indication of whether the
correct solution is close to the Kramers’ opacity solution or the electron opacity
solution. In the former case, the construction of the run of $N$ is straightforward,
since the surface boundary condition for $N$ is the same, and the first approximation
is frequently sufficient. In the latter case, more care is required and higher
approximations may be necessary. That is, when a complete solution has been
obtained from the assumed run of $N$, the resulting $dN/dS$ must be compared with
the assumed $dN/dS$. If there is any considerable discrepancy, a new run of $N$
must be assumed and the integration repeated.

Once a run of $N$ has been assumed for $0 \leq S \leq 0.1$ only equation (5) remains
to be integrated. In this region the use of the graphs of Paper I is not appropriate,
and the equation is best solved in the form

$$
\frac{d \log Q}{d \log S} = \frac{S + N - Q}{1 + 2S - 4Q}.
$$

(5')

As $S$ decreases, $\log Q$ and hence $Q$ can be obtained at each stage. $r/r_0$ can then
be obtained from the quadrature

$$
\log r/r_0 = -\int_0^Q \frac{dQ}{S + N - Q}.
$$

(11)

Hence $\nu$ can be calculated and $dN/dS$. This is compared with the assumed
d$N/dS$ and if necessary the integration is repeated with the new run of $N$ assumed.
This second run of $N$ has in most cases been found to be unnecessary, but has
been required and been found sufficient in the cases nearest to pure electron
scattering opacity.

5. In order to make full use of the completed integration, we must obtain the
relations between the mass, luminosity and radius of the star, given its com-
position. These involve knowing the ways in which $Q$ and $\nu$ vary with $S$ at the
surface. Since $N = \frac{4}{17}$ at $S = 0$, it is necessary that in the neighbourhood of the surface,

$$
Q = AS^{4.17},
$$

where $A$ is a constant which can be determined from the integration. Also, from
(8') and (10),

$$
N = \frac{4}{17} \left[ I + \frac{5}{35} S^{4.17} \right]
$$

and

$$
\nu = I - \frac{5}{35} S^{4.17}
$$
in the neighbourhood of the surface, where $\delta$ is given by

$$
\delta^{4.17} = \left(\frac{17}{4} \right)^5 \frac{A^2}{E}.
$$

(12)

From (6) and the definitions of $S$, $Q$, $N$, it follows that

$$
\frac{C}{\pi M^3 (\mu G)^4} = \frac{4 N (1 - \nu) Q^4}{S},
$$

(13)
while, multiplying (6) by (7), we have

\[ \frac{B\nu^{0.5}}{\pi^2 M^{0.5}} \left( \frac{R}{\mu G} \right)^{7.5} = \frac{16\nu Q^{0.3} N}{S^2}. \]  

(14)

Consider these equations at the surface.

Then

\[ \frac{C}{\pi M^3} \left( \frac{R}{\mu G} \right)^4 = \frac{18}{11} \delta A^4 = K_1 \quad \text{say}, \]  

(15)

and

\[ \frac{B\nu^{0.5}}{\pi^2 M^{0.5}} \left( \frac{R}{\mu G} \right)^{7.5} = \frac{64}{17} A^{8.5} = K_2 \quad \text{say}. \]  

(16)

Recalling the definitions of B and C, these are the required relations between \( M_s, r_s \) and the luminosity \( L_s \), which enters through \( B \) and \( C \). It is therefore essential to evaluate \( A \) and hence \( \delta \).

One method of evaluating \( A \) is directly from the integration, that is by continuing the outward integration for decreasing \( S \) until \( \log Q - \frac{4}{11} \log S \) becomes constant. This is extremely tedious and inconclusive and another method presents itself, which is much more satisfactory.

Consider again equation (15) and divide it by (13). Then

\[ \left( \frac{M}{M_s} \right)^3 = \frac{48 A^4}{17} \frac{S}{N(1-v)Q^4} \frac{S}{N(1-v)Q^{4E_{11}}} A^{611} \text{ from (12)}. \]  

(17)

But \( M/M_s \) can also be obtained by a quadrature, which can very easily be performed. For

\[ \log \frac{M}{M_s} = -\int_s \frac{dS}{1 + 2S - 4Q}. \]  

(18)

We can therefore evaluate \( M/M_s \) at some arbitrary point \( S_1 \) from (18) and so find \( A \) from (17).

\( P/P_e^*, \rho/\rho_e \) and \( T/T_e \) were then calculated as in Paper I. The results of these integrations are given in Table I.

<table>
<thead>
<tr>
<th>Table Ia</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E ) infinite, i.e. opacity entirely due to photoelectric effect. ( v=1 ) throughout. ( At ) the surface ( Q=0.2351S^{11/17} )</td>
</tr>
</tbody>
</table>

<table>
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<th>( S )</th>
<th>( Q )</th>
<th>( N )</th>
<th>( r/r_s )</th>
<th>( M/M_s )</th>
<th>( P/P_e )</th>
<th>( \rho/\rho_e )</th>
<th>( T/T_e )</th>
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<td>0.0553</td>
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The convective core fits on at \( S=2.12 \), the radius of the core in Emden units being 1.19.

* The suffix \( e \) refers to central values.
TABLE IB

<table>
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<th>S</th>
<th>Q</th>
<th>N</th>
<th>ν</th>
<th>r/r₈</th>
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The convective core fits on at S=1.92, the radius of the core in Emden units being 1.245.

TABLE IC

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<th>N</th>
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<td>0.654</td>
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<td>0.230</td>
<td>0.426</td>
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</table>

The convective core fits on at S=1.38, the radius of the core in Emden units being 1.415.

TABLE ID

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<th>ν</th>
<th>r/r₈</th>
<th>M/M₈</th>
<th>P/Pₑ</th>
<th>ρ/ρₑ</th>
<th>T/Tₑ</th>
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<td>0:0125</td>
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<td>0:0052</td>
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<td>0:0100</td>
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<td>0:764</td>
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<td>0:652</td>
</tr>
</tbody>
</table>

The convective core fits on at S=1.03, the radius of the core in Emden units being 1.58.

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TABLE I

\[ E = 0, \text{ i.e. mathematically limiting case of opacity due entirely to electron scattering} \]
\[ \nu = 0 \text{ throughout. At the surface, } Q = 0.218S^4 \]

<table>
<thead>
<tr>
<th>S</th>
<th>Q</th>
<th>N</th>
<th>r/rs</th>
<th>M/Ms</th>
<th>P/Pc</th>
<th>( \rho/\rho_c )</th>
<th>T/Tc</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0.2500</td>
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<td>1</td>
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<td>0.2559</td>
<td>0.615</td>
<td>0.947</td>
<td>0.005</td>
<td>0.027</td>
<td>0.196</td>
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<tr>
<td>0.1</td>
<td>0.132</td>
<td>0.2661</td>
<td>0.532</td>
<td>0.888</td>
<td>0.020</td>
<td>0.073</td>
<td>0.274</td>
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<td>0.170</td>
<td>0.2888</td>
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<td>0.357</td>
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<td>0.204</td>
<td>0.2958</td>
<td>0.419</td>
<td>0.691</td>
<td>0.095</td>
<td>0.226</td>
<td>0.420</td>
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<td>0.5</td>
<td>0.273</td>
<td>0.3254</td>
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<td>0.177</td>
<td>0.343</td>
<td>0.516</td>
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<td>0.340</td>
<td>0.3541</td>
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<td>0.643</td>
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<tr>
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<td>0.453</td>
<td>0.4000</td>
<td>0.285</td>
<td>0.326</td>
<td>0.343</td>
<td>0.527</td>
<td>0.652</td>
</tr>
</tbody>
</table>

The convective core fits on at \( S = 1.03 \), the radius of the core in Emden units being 1.58.

In order to investigate the changes in the mass-luminosity-radius relation, it is of interest to tabulate \( K_1 \) and \( K_2 \) against \( E \). The expressions \( K_1 \) and \( K_2 \) are non-dimensional, and it is also of interest to determine corresponding dimensional quantities defined by

\[
\frac{\mu^4 M_a^3}{(1 + X) L_a} \quad \text{and} \quad \frac{\mu^{7.5} M_a^5}{(1 - X^2) r_8^5 L_a}.
\]

TABLE II

<table>
<thead>
<tr>
<th>( E )</th>
<th>( \bar{\rho}/\rho_c )</th>
<th>( \bar{R}T_{r_8} r_8^5 L_a / \mu GM_a )</th>
<th>( \mu^4 M_a^3 / \mu GM_a (1 + X) )</th>
<th>( K_1 )</th>
<th>( \mu^{7.5} M_a^5 / r_8^5 (1 - X^2) L_a )</th>
<th>( K_2 )</th>
<th>( K_1 + 10K_2^{11} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.049</td>
<td>0.80</td>
<td>[64.2168]</td>
<td>[3.3538]</td>
<td>( \infty )</td>
<td>0</td>
<td>[3.3538]</td>
</tr>
<tr>
<td>0.637</td>
<td>0.0496</td>
<td>0.782</td>
<td>[64.2543]</td>
<td>[3.3163]</td>
<td>[147.1514]</td>
<td>[8.9462]</td>
<td>[3.5431]</td>
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<tr>
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<td>0.0310</td>
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<td>[3.0827]</td>
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<td>[6.8372]</td>
<td>[2.2172]</td>
</tr>
<tr>
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<td>0.0273</td>
<td>0.913</td>
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<tr>
<td>( \infty )</td>
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<td>0.901</td>
<td>( \infty )</td>
<td>0</td>
<td>( \infty )</td>
<td>[144.8661]</td>
<td>[5.2314]</td>
</tr>
</tbody>
</table>

These are given in Table II. It can be seen that these can be compared respectively with the corresponding quantities for opacity due entirely to electron scattering, and opacity due entirely to photoelectric effect.

The non-dimensional quantities

\[
\frac{\rho_{\text{mean}}}{\rho_c} \quad \text{and} \quad \frac{\bar{R}T_{r_8}}{\mu GM_a}
\]

are also given and can be compared with the extreme cases. For the sake of comparison, the extreme cases are also given in the tables, being the solutions given in Paper I.

Since \( K_1 \) is zero for infinite \( E \) and \( K_2 \) is zero for \( E = 0 \), while \( 4EK_1^{11} = K_2^{10} \), for purposes of interpolation for various values of \( E \) it is convenient to tabulate \( K_1 + 10K_2^{11} \) against \( E \). Various combinations of this type have been examined but this was found to be the most satisfactory. Thus, for any \( E \), it is possible by interpolation to evaluate \( K_1 \) and \( K_2 \).
It will be seen from these tables that the internal arrangements of stars change
over smoothly from one extreme case, through the intermediate cases, to the other
extreme case. The ratio of core radius to the radius of the star and the central
condensation seem to vary monotonically from one extreme to the other, though
there appears to be a slight waviness in the expression for $T_c$ in terms of $M_s$ and $r_s$.
Columns 4 and 6 of Table II show how the addition of another source of opacity
diminishes the luminosity. The infinity symbol occurring in these columns
arises since, with fixed coefficients of opacity, pure electron opacity corresponds
formally to a star of infinite mass, and pure Kramers' opacity to a star of zero mass.
Now since for electron opacity $L_e \sim M_e^3$, $M_e^{5.5} L_e^{-1}$ tends to infinity as $M_e$ tends
to infinity. The other infinity arises in a similar way and also has only formal
significance.

Table III gives the detailed solutions in $h$-variables for the lower values of
$E$ for $S \leq 0.1$. These are of interest, since the changeover from Kramers' opacity
to mixed opacity takes place there for small $E$.

<table>
<thead>
<tr>
<th>$S$</th>
<th>$Q$</th>
<th>$N$</th>
<th>$\nu$</th>
</tr>
</thead>
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<tr>
<td>0.0001</td>
<td>0.02560</td>
<td>0.23555</td>
<td>0.676</td>
</tr>
<tr>
<td>0.001</td>
<td>0.04385</td>
<td>0.23611</td>
<td>0.634</td>
</tr>
<tr>
<td>0.01</td>
<td>0.07545</td>
<td>0.2380</td>
<td>0.575</td>
</tr>
<tr>
<td>0.1</td>
<td>0.140</td>
<td>0.2500</td>
<td>0.500</td>
</tr>
</tbody>
</table>

* Owing to the instability of $N$ when integrated outwards, no great accuracy is claimed for this
figure, but $Q$ and $\nu$ are accurately determined.

Newnham College,
Cambridge: