ON THE INTERPRETATION OF THE HERTZSPRUNG-RUSSELL DIAGRAM

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Summary

A general discussion of laws of energy generation and opacity is given, together with a brief account of stellar evolution based on the view that stars consist almost entirely of hydrogen and helium. The Hertzsprung-Russell diagram is drawn, using mainly Kuiper's data on the nearest stars and also his estimates of surface temperatures and bolometric corrections. It is shown that this diagram strongly indicates that a switch of the law of energy generation occurs for stars of a luminosity about one-sixth that of the Sun. The brighter stars draw their energy from a highly temperature-sensitive process which is probably the Bethe cycle. The fainter stars draw their energy from a much less temperature-dependent process the proton-proton reaction (presumably).

Consideration of the Sun's central temperature shows that the astrophysical data definitely contradict the earlier estimates for the critical temperature of the Bethe cycle and tend to show that this temperature may even be slightly lower than the more recent experiments indicate.

I. Comparisons between the results of the theory of stellar constitution and the observations have generally been carried out by considering the mass-luminosity and the mass-radius relations. This approach has some advantages, especially in that it makes use of all the observable properties of stars and in that the physically fundamental quantity, the mass, appears as independent variable. On the other hand, the smooth outlines of the Hertzsprung-Russell diagram, when compared with the scatter of the empirical mass-luminosity relation, indicate that the observational determinations of the stellar masses are perhaps not as accurate and certainly not as numerous as measurements of luminosity and type. Accordingly it may be advantageous to compare the theoretical luminosity-surface temperature relation with the observational one. This task is simplified and greatly aided by the fact that the theoretical interpretation of the red giants and of the white dwarfs is now fairly certain. It is the purpose of the present paper to carry out such a comparison and it is claimed that, in spite of the exclusion of the observational determinations of stellar masses, valuable conclusions can be drawn. In the last section of the paper one special stellar mass is also considered, namely the mass of the Sun, which is known to a high degree of accuracy, and some further deductions are made.

2. The problem of the source of stellar energy has long been one of the most difficult questions of astrophysics. The modern development of nuclear theory has shown beyond reasonable doubt that, at least in the vast majority of all stars, thermonuclear processes are responsible for the generation of energy, and that the ultimate effect of these processes is the conversion of hydrogen into helium. Although this conversion usually takes place in several stages, one of these will
normally be far less likely to occur than the others and will therefore limit the rate of generation of energy. It may perhaps be useful to discuss the general properties of these critical stages of the various processes in some detail.

Owing to the relatively large amount of energy released in the conversion of four protons into a helium nucleus the energy requirements of a star are covered if only a minute fraction of the inter-particle collisions near the centre of the star leads to the requisite nuclear reaction. The investigation concerns therefore highly improbable processes and this naturally adds to its difficulties (only about one collision in $10^{20}$ is required to produce energy).

The only collisions of interest are those in which the participating nuclei are of the requisite types. One of them must be either hydrogen or must be derived from hydrogen by a process more likely to occur than the critical stage now under discussion. Hence one of the nuclei must be fairly common. The other one may have to be a comparatively rare one like, say, carbon, but the abundance of no element is likely to be so low as to contribute a very substantial factor to the improbability of the reaction.

It follows then, that the collision is still very unlikely to lead to the reaction even if the two nuclei are of the required types. In the terms of nuclear physics, a very small cross-section for the energy-generating process will be sufficient to keep up the energy supply of the star.

Collision cross-sections are in general functions of the relative velocity of the particles, increasing rapidly with the velocity. While, however, for most reactions the dependence on relative velocity is extremely steep (30th to 40th power of the velocity), for a very few it is less critical and for at least one the effective dependence, in the important range, may be as low as the 6th power.

If the cross-section depends extremely sensitively on the relative velocity, then the rate of energy generation in a star will depend very critically on the central temperature. For only the very fastest particles will make an effective contribution to the rate of energy generation, and their number depends, by the Maxwell distribution law, very critically on the temperature. Accordingly the energy-producing region will be closely confined to the innermost parts of the star. It is also well known that in this case the core of the star must be in convective equilibrium (13). A final consequence of this extreme sensitivity of the cross-section is that the process may be investigated in the laboratory. For it is possible to produce much faster particles than occur in stars and hence the cross-section may become large enough to be measurable.

If the cross-section does not depend so sensitively on the relative velocity, then the rate of energy generation in the star will vary only like a low power of the temperature. Accordingly the energy-producing region will occupy a substantial fraction of the star and the core need not be convective, owing to the low-power dependence on velocity it is not possible to increase the cross-section sufficiently in laboratory conditions to make it measurable. Hence such processes can only be investigated theoretically. Although only one process of this type is known, the possibility of the existence of others cannot yet be absolutely excluded.

A further sub-division of thermonuclear processes can be carried out by examining whether (a) they concern only hydrogen nuclei and those others which are produced in the process, or whether (b) other elements take part. In case (b), owing to the very small abundance of elements other than hydrogen, it is probably only necessary to consider processes in which the other elements act purely as
catalysts and are therefore not used up in the reaction. It is possible, though
unlikely, that stars of very low luminosity use non-catalytic processes of type (b).

All processes of type (a) must start with the formation of deuterons from two
protons, although various subsequent developments are possible. However,
the first stage imposes the most severe limitation. This process is well known (1),
but owing to its low-power velocity dependence can only be studied theoretically.
In stellar conditions the rate of energy generation due to it is given approximately by

\[ \epsilon_{bb} = \epsilon_0 X^2 \rho^2 T^{3.5}, \]  

(1)

where \( \epsilon_0 \) is a constant, \( X \) the hydrogen abundance and \( \rho \) is the density.

Highly velocity-sensitive processes are amenable to laboratory study and
the only one of importance is of class (b) and is the famous carbon-nitrogen
cycle discussed first by Bethe (2) and recently re-examined experimentally (3).
It appears that in stellar conditions

\[ \epsilon_{CN} = \epsilon_1 X Z_{12} \rho^2 T^n, \]  

(2)

where \( \epsilon_1 \) is a constant and \( Z_{12} \) is the abundance of carbon.

Low-power chain processes can only be studied theoretically, and although the
discussions given tend to show that no such process is of importance, the existence
of an important process of this type (temperature-insensitive catalytic chain)
cannot yet be completely excluded.

Finally it may be mentioned that processes involving triple and higher
collisions between atomic nuclei have been shown to be of no importance.

3. A knowledge of the coefficient of opacity of stellar material at all relevant
 temperatures and densities is required for a discussion of stellar structure. Since
these are far beyond the range of laboratory conditions the coefficient of opacity
must be determined theoretically. This is unfortunately not an easy matter,
since extensive computations are required, and hence our knowledge of the
coefficient of opacity is a little vague.

The two main contributions to the opacity are the scattering of light by free
electrons and the photoelectric effect. The first leads to a coefficient which is
proportional to the number of free electrons but otherwise constant. In a state
of complete ionization

\[ \kappa_{el} = 0.19(1 + X) \text{ cm}^2/\text{g}. \]  

(3)

The photoelectric effect was first investigated by Kramers (4) who showed
that hydrogen and helium atoms were far less effective agents of opacity than the
heavier elements. At that time it was believed that these heavier elements
were a considerable fraction of the stellar material and unfortunately all his
and subsequent investigations and computations (5) were carried out on that
basis, which is now known to be quite incorrect.

Kramers found that

\[ \kappa_{pe} = \kappa_{00} \frac{Z(1 + X)\rho}{gT^{3.5}}, \]  

(4)

where \( Z \) is the abundance of elements other than \( H \) and \( He \), \( \kappa_{00} \) is a constant
equal to about \( 10^{25} \) in c.g.s. units, whereas \( g \) is the so-called guillotine factor
which depends on the density, temperature and composition of the material,
but generally only varies slowly. Extensive tabulations of \( g \) have been made by
Strömgren and by Morse (5) but unfortunately on the erroneous assumption that
Z is not too small (say \(Z > 0.05\)). New tabulations of \(g\) for very small \(Z\) are sorely needed.

Various formulae of approximations to the somewhat irregular variations of \(g\) have been tried, notably \(g = \text{const.}, \, g \sim \rho^{1/4}, \, g \sim \rho^{1/2}T^{-3/4}, \, g \sim T^{3/2}, \) etc.

At low densities (such as are more common in more massive and luminous stars) electron scattering predominates, but in average and small stars the photoelectric effect is probably more important.

The present situation is far from satisfactory, but it seems justified to put

\[
\kappa = \kappa_0(1 + X)\rho^{-1}/T^{v-3},
\]

where \(Z\) has been included in the constant \(\kappa_0\), and

- in big stars: \(\sigma = 1, \, v = 3\),
- in other stars: \(1.5 \leq \sigma \leq 2, \, 6 \leq v \leq 8\).

An additional complication may arise in very massive stars through the action of radiation pressure, but this is probably quite unimportant in all other stars and its effects will be neglected.

4. Even without detailed integrations of the equations of stellar structure, deductions of considerable interest can be made from homology relations.

We consider general models of stars. Universal constants will be left out in the proportionality relations. Similarly the abundance \(Z\) of elements other than \(H\) and \(He\) will be taken to be constant and will hence be omitted.

The notation used will be:

- \(L\) = luminosity of star,
- \(M\) = mass of star,
- \(r\) = radius of star,
- \(\rho\) = central density,
- \(\mu\) = central molecular weight,
- \(X\) = central hydrogen abundance,
- \(T\) = surface hydrogen abundance,
- \(T_c\) = central temperature,
- \(GM\) = an invariant characteristic of the whole structure of the star (6, 7),
- \(F = \frac{\rho_c}{\rho_{\text{mean}}} = \frac{4\pi r^2 \rho_c}{3M}\) = an invariant characteristic of the whole structure of the star (6),
- \(A\) = numerical constant characteristic of the structure of the surface layers (6).

If \(\epsilon \sim X^\psi \rho^\alpha T^n\) (\(\psi = 1\) for catalytic reactions, \(\psi = 2\) for the \(p-p\) reaction, \(\alpha = 2\) usually), then by the definitions of \(D\) and \(E\)

\[
L \sim r^3 X_c \rho_c^2 T_c^n \sim D^n \mu_c E^n X_c^{\psi} M^{2+\psi/2\alpha-3-\psi}. \]

Also if

\[
\kappa \sim (1 + X)\rho^{-1}/T^{v-3}
\]

then the mass-luminosity-radius relation is

\[
L \sim A^{\psi+\sigma} \frac{\mu_c^{\psi+1}}{1+X} \frac{M^{\psi+1-\sigma}}{r^{v-3\sigma}}.
\]

Finally,

\[
L \sim r^2 T_c^4.
\]
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We shall first derive the relation between $L$ and $T_e$ if only $M$ and $r$ are allowed to vary, that is we are considering a homologous sequence of stars. If $M$ and $r$ are eliminated between (6), (7) and (8), then

$$L \sim T_e^q,$$

where

$$q = \frac{\eta(2\sigma + 1) + \alpha(2\nu + 3) - 3(\nu + 1 - \sigma)}{\eta(2\sigma - 1) + \alpha(2\nu + 1) - (\nu + 1 - \sigma)}.$$  

(10)

In the most important case $\alpha = 2$ and then

$$q = 4 \frac{\eta(2\sigma + 1) + \nu + 3 + 3\sigma}{\eta(2\sigma - 1) + 3\nu + 1 + \sigma}.$$  

(11)

Values of $q$ for a number of representative values of $\alpha$, $\eta$, $\sigma$ and $\nu$ are tabulated below.

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This table indicates the variation of $L$ with $T_e$ for stars of different masses but the same constitution. This variation will be of considerable assistance in interpreting the Hertzsprung-Russell diagram, but it will also be necessary to consider what different types of composition are likely to arise and how they affect the distribution of stars in the Hertzsprung-Russell diagram. For this purpose stellar evolution has to be considered to some extent.

5. In this and subsequent sections the term "normal star" will be used to describe a homogeneous star of arbitrary mass in equilibrium, the material of the star being mainly (say 95 per cent to 99 per cent) hydrogen, the rest consisting of a little helium and other elements. This is supposed to be the state of a star which has condensed from interstellar material and in which thermonuclear energy generation fully covers the loss of energy by radiation but has not yet converted any appreciable amounts of hydrogen into helium. The sequence of normal stars will form a curve in the luminosity-type diagram. The slope of the tangent to this curve at any point will be described by the value of $q$ discussed in Section 4. The curvature of the curve is due to the fact that
for different sizes of star different laws of energy generation and opacity predominate and hence the value of \( q \) varies along the curve. The switch from one law to another will always lead to a convex shape of the curve (i.e. \( q \) increasing with \( T_e \)). Concave regions can only be due to a suitable variation of the exponents in any one law such as an increase of \( \eta \) with decreasing \( T_e \).

In the state preceding the normal one the star is contracting under gravity without nuclear energy generation. The radius of such a "pre-normal" star will clearly be greater than the radius of a normal star of the same mass. Since for any opacity law proposed (\( \nu \rightarrow \infty \)) is small (usually positive), it follows from the mass-luminosity-radius relation that the luminosity differs little from the normal luminosity (it is probably a little smaller, especially since \( A \) is probably a little less than its normal value), but \( T_e \) will be smaller owing to the bigger radius. Accordingly the representative points of the pre-normal stars will lie to the right of the "normal" curve. Owing to the brevity of the Helmholtz-Kelvin time scale very few stars may be expected to be in this state.

Of far greater importance are the developments due to the conversion of hydrogen into helium, and those due to the accretion of interstellar matter.

Considering first the conversion of hydrogen into helium, we note that it will take place mainly close to the centre of the star. In the course of time substantial amounts of helium will be formed there, but it is an open question whether they will stay there or whether they will diffuse throughout the star. The problem is still obscure in spite of the amount of work that has been done on it, and it seems quite possible that in fact the speed of diffusion depends on the rotation rate of the star and therefore differs from star to star. In view of this uncertainty both alternatives will be considered here. Since we first leave out the question of accretion, the mass \( M \) of the star will be taken to be constant during the evolution.

(i) Diffusion much less rapid than production of helium.—In this case the helium will be chiefly confined to the inner regions of the star. If the core is convective the helium will be distributed throughout the core, but will not penetrate far into the radiative envelope. If the core is radiative, then this implies that \( \eta \) is low (high \( \eta \) leads to instability and convection). Accordingly the energy generation is fairly widely spread and helium will be produced in a region of similar dimension to the convective cores usually considered (radius varies between \( r/10 \) and \( r/3:5 \)).

In either case it seems (7) that the invariants \( D, E \) and \( A \) differ little from their normal values. By hypothesis \( \mu \) and \( X \) are unaltered and hence by equation (7) \( L\nu^{3:3} \) is virtually unchanged. But, by (6), \( L\nu^{3:3-3:4} \) is changed owing to the alteration of \( \mu \) and \( X \).

Whether the product \( X\nu^{2} \mu \) increases or decreases depends on the values of \( \psi \) and \( \eta \), and on the amount of helium produced. For \( \mu \rightarrow \infty \) as \( X \rightarrow 0 \), and so, in the final stages of evolution, the product tends always to zero. However, in the early stages, which are more important for our purposes, the product increases or decreases according as to whether \( 5\eta/8 \) is greater or less than \( \psi \) (since \( \mu \sim 1/\nu (1-5Y/8) \), \( X=1-Y \). Since \( \eta \) always exceeds 2 the product will increase in the early stages of evolution for any catalytic process (\( \psi=1 \)) but will be almost unaltered at first for the p-p reaction (\( \eta=3:5 \), \( \psi=2 \)) and will soon start to decrease. Furthermore, for all likely laws of opacity and energy generation (\( 3\nu-3:4+\eta \))\( >\nu^{3:3} \). Therefore \( \nu \) will increase in the early stages of evolution (unless the p-p reaction is responsible to energy generation, in which
case $r$ hardly changes). The increase will be larger the greater $\eta$ is. For very large values of $\eta$ the radius $r$ will be proportional to $\mu_e$. Since $(v - 3\sigma)$ is generally a small positive number, the luminosity $L$ will, by virtue of (7), decrease slightly. In the early stages of evolution the representative point in the type-luminosity diagram will therefore not move if the p–p reaction predominates and will otherwise move to the right and a little down, but the displacement will not be very large (say up to at most a factor of 1.4 in $T_e$ if both $\eta$ and $Y_e$ are large).

In the later stages of evolution there may be a movement to the left (higher $T_e$) and slightly upwards, particularly so in the case of the p–p reaction.

(ii) **Diffusion much more rapid than production of helium.**—In this case the star will be homogeneous. Hence $D$, $E$ and $A$ will have their normal values, but $\mu_e = \mu_a$ will be increased and $X_e = X_a$ will be decreased. It follows that

$$Lr^{3x-3+\eta} \sim \mu^x X^y, \quad Lr^{x-3\sigma} \sim \mu^{x+1} (1 + X).$$

The detailed calculations are a little tedious and the result is that in every case both $L$ and $T_e$ increase as $\mu$ increases, that is as hydrogen is converted into helium. The luminosity $L$ increases for Kramers’ opacity approximately like the seventh power of $\mu$ and for electron opacity more like $\mu^4$. The increase in $T_e$ is also fairly rapid and, in the luminosity-type diagram, the representative point moves upwards but not quite as steeply as the normal curve, so that the new position of the representative point is a little to the left of the normal curve. For each magnitude (factor of $10^{0.4}$) increase in $L$ due to this development, the increase in $T_e$ is about 5 per cent more than would correspond to motion along the normal curve (this excess is about the same for all likely laws of opacity and energy generation).

When allowance is made for the fact that mixing is not likely to lead to a perfectly uniform star but that $\mu_e$ will probably be slightly greater than $\mu_a$, then there will be a slight reduction in $T_e$, so that it is probably sufficient to say that, with good mixing and without accretion, evolution leads to upward motion more or less along the normal curve.

(iii) **Effects of accretion.**—We now consider the possibility of the accretion of interstellar matter (chiefly hydrogen). It is clear that if a normal star accretes matter substantially like its own it will merely be converted into another normal star of rather greater mass. Similarly if a star considered under (i) (which has an appreciable abundance of helium near the centre but not near the surface) accretes substantial amounts of interstellar matter it is merely turned into a similar star of greater mass.

However, if a star of the type considered under (ii) accretes interstellar hydrogen then very remarkable results occur (7, 8). The luminosity is little altered, but the newly acquired hydrogen envelope is very tenuous and extends to large distances. Very considerable increases in $r$ and therefore decreases in $T_e$ may result, leading to the formation of a red giant star. The degree of extension depends critically on the molecular weight of the original star and also on the amount of hydrogen accreted. For maximum extension this should exceed 10 per cent but not 80 per cent of the mass of the original star. It is essential for the formation of such red giants that the hydrogen of the envelope should not be mixed with the helium of the interior. Presumably the difference in molecular weight would, at least for sufficiently rapid accretion, permit the formation of such a distinct envelope. A red giant will therefore be formed if the
speed of accretion exceeds the speed of mixing, and if this in turn exceeds the rate of helium production.

The representative points of stars of this type lie to the right of the normal curve, the displacement depending on the circumstances.

6. When hydrogen conversion has progressed very considerably it becomes impossible for the star to generate enough energy. Gravitational collapse sets in and presumably continues until either disruption (novae, supernovae) due to centrifugal force occurs, or until the white-dwarf stage is reached. This final stage is known to be very long lived and has been studied in considerable detail. The intermediate stages of collapse are not so well understood but probably a star passes through them with considerable speed, so that at any given time very few stars are in this intermediate state. Accordingly it is of little importance to the interpretation of the luminosity-type diagram and will not be further considered here.

7. A fairly complete picture of the evolutionary tracks of stars in the luminosity-type diagram has now been given. Before this can be applied to the diagram representing stars of our galaxy, account has to be taken of the fact that there is an upper limit to the age of such stars. The age of our galaxy and hence the age of the oldest stars has been estimated as between $5 \times 10^9$ years and $2 \times 10^{10}$ years. Of course we must expect to find numerous stars much younger than that, but none can be older than this limit.

The implications of this age limit are clear. It is readily worked out that the Sun can have converted at most 4 per cent (15 per cent for the higher age limit) of its hydrogen into helium. Since the luminosity is a very steep function of the mass, it follows that, say, stars of bolometric magnitude 7 cannot have converted more than 3 per cent of their hydrogen into helium, but stars of $M_{bol}=2$ may have converted as much as 16 per cent (or 60 per cent if the upper limit of age of the galaxy is taken). For stars of $M_{bol}>7$ there can be hardly any deviation from the normal curve if mixing is thorough (as it is likely to be in such small stars). If however the 3 per cent of helium is confined to a small region near the centre, producing there an appreciable abundance of $He$, a slight extension may result.

For stars of $M_{bol}=2$ a moderate degree of red giant formation (as discussed in 5 (iii)) becomes quite possible, but considerable red giants are not to be expected until $M_{bol}<0$.

We must therefore expect that the stars of low luminosity lie close to the normal curve and possibly a little to the right of it, but that for the more luminous stars a greater spread to the right sets in. Unless the deviation discussed in 5 (ii) becomes very important (and this is certainly unlikely for $M_{bol}>2$) the normal curve will be the lower left boundary of the band representing the stars of the main sequence. This boundary therefore has to be used for comparison with the values of q computed at the end of Section 4.

8. For our purposes it is evidently desirable to compare the theory with a highly homogeneous set of observations in which the lower part of the main sequence is well represented. For this reason the data given in Kuiper's article on the nearest stars (9) were taken as the primary material. These data are particularly valuable because the stars are selected by distance rather than by apparent brightness, and also because particular care was taken to determine spectral classes by consistent criteria. Even in this homogeneous collection it
was considered desirable to omit all those stars where Kuiper considers some of the data to be dubious (italized in his list). In order to have some more material, especially for the upper ranges of the main sequence, visual and spectroscopic binaries from Kuiper's article on the empirical mass-luminosity relation (10) were also included. Again all data involving binaries of uncertain mass ratio were neglected.

These stars from Kuiper's article form the principal material worked on, but in order to illustrate the position of the sub-dwarfs better a few selected sub-dwarfs from Joy's list (11) were also included. Joy's colour classes are not in perfect agreement with Kuiper's. Therefore his general data were not included and the few sub-dwarfs selected are there solely for the purpose of illustrating the position of a class of which Kuiper gives few examples.

![Graph](https://example.com/graph.png)

**Fig. 1.**—Ordinates: Absolute bolometric magnitude. Abscissae: $\log T_e$ (spectral type).

- ● Stars from ref. (9) (nearest stars).
- + Stars from ref. (10) (binaries).
- × Stars from ref. (11) (sub-dwarfs).

Finally the Sun was included in the collection, the data used being those given by Kuiper (12). The bolometric corrections and the relations between spectral type and surface temperatures were also taken from this paper. All these stars are represented in Fig. 1 in which $\log_{10} T_e$ is the abscissa and $M_{bol}$ the ordinate.
Although in this way a very homogeneous and satisfactory set of data was obtained, it was found that data on stars of spectral type later than M2 were not usable. As Kuiper points out, the bolometric correction depends so sensitively on the spectral class and is so large that it is a very uncertain guide to the bolometric magnitudes. There are also no direct determinations of the surface temperatures of these very red dwarfs. Kuiper's doubts are strengthened by the appearance of the luminosity-surface temperature diagram in this region, which shows a sharp downward turn. As was pointed out before, such a concave shape is very unlikely on theoretical grounds and is presumably due to an overestimate of the effective temperatures and a consequent under-estimate of the bolometric correction.

Information about these stars would be of the greatest value, since they must all be in a very early stage of evolution and should hence form a one-parameter series. Possibly the best method would be to measure red magnitudes of the stars between K6 and M8. The bolometric correction would then be small and slowly varying. The problem of determining and defining the surface temperatures of these stars would still be formidable. A complete luminosity-surface temperature diagram for this region would however give clear answers concerning the homogeneity and similarity of stellar material in our neighbourhood.

Returning to the sample taken and represented in Fig. 1, there still remains the question whether it is a fair sample. It can clearly only be representative of our galaxy, but whether dwarfs in our neighbourhood are like dwarfs elsewhere in the galaxy remains to be seen. The most important variable condition is probably the density of interstellar material, but there is little reason to believe that ours is an exceptional region in this respect.

9. The main features of the appearance of Fig. 1 are fairly clear. The band of the main sequence stands out very clearly. Near the top of the diagram the red-giant sequence starts on the right. The sub-dwarfs appear below the main sequence, the white dwarfs would be in and beyond the extreme left-hand bottom corner, but have not in fact been drawn in.

The left-hand edge of the main sequence which should be the normal curve is well represented by the straight lines drawn in on the diagram. In the upper part the $q$ value is about 6.5, in the lower region it is only about 3.5. The transition from one line to the other seems to be quite sharp. As will be seen from Table I, the upper $q$ value requires $\eta$ to have a high value (such as the carbon cycle would give) and the opacity law seems to be intermediate between Kramers' and electron opacity. This is very much what is to be expected.

The value of $q$ for the lower ranges points very strongly towards a low value of $\eta$ such as corresponds to the p-p reaction. The opacity law must clearly be of the Kramers form, possibly with a rather high value of $\nu$ ($\nu=7$ ?).

The $L-T_\epsilon$ diagram lends very strong support to the view that the mechanism of energy generation has a high value of $\eta$ for bright stars and a low value for small stars. The transition appears to take place fairly suddenly at $M_{\text{bol}}=6.5 \pm 0.5$, i.e. for stars one-sixth as luminous as the Sun.

The width of the main sequence in the lower ranges is probably largely due to observational errors, but for brighter stars is chiefly due to evolutionary developments.

10. Whereas the features of the diagram discussed so far are in good agreement with current theories, there are two features which cannot be so readily explained.
The first one of these is the curious "hole" in the main sequence near $M_{\text{bol}} = 7$. Stars of spectral type K5–8 seem to be either of $M_{\text{bol}} = 6.5$ or $M_{\text{bol}} = 7.5$. The observations are too scanty for definite impressions but such a curious formation not far from the switch-over from low to high $\eta$ may be significant.

The second odd feature is the sub-dwarfs. It is clear from the discussion of stellar evolution here given that they cannot easily be fitted into the early or middle stages of any evolutionary sequence. It is not impossible that they form part of the late stages, the final collapse to the white-dwarf stage, but their grouping, their occurrence for late spectral types and their numbers make this hypothesis unattractive. It may be that an exceptional composition, by its effect on the mode of energy generation, is responsible for the curious position of the sub-dwarfs in the diagram.

II. It will be seen from Fig. 1 that the Sun ($M_{\text{bol}} = 4.62, \text{dG}2$) is well on the part of the main sequence drawing its energy from a high-$\eta$ process (carbon cycle). Information about the critical temperature of this process may be obtained from a study of the structure of the Sun.

The factor of chief interest in this connection is the invariant $D = R T^\sigma \mu G M$. The value of this quantity for a homogeneous star with a convective core (necessary for high $\eta$) depends slightly on the opacity law. For electron opacity ($\sigma = 1, \nu = 3$) $D = 0.80 (6)$, for ordinary Kramers' opacity ($\sigma = 2, \nu = 6.5$) $D = 0.90 (6)$, while for modified Kramers' opacity ($\sigma = 1.75, \nu = 6.5$) $D = 1.15 (7)$. It seems therefore that for a homogeneous star of the mass of the Sun $D$ cannot differ very much from unity.

The central temperature of a star of solar mass and radius is readily seen to be $2.3 \mu D \times 10^7$ deg. If the Sun were a pure hydrogen star its central temperature would probably be between $10^7$ deg. and $1.25 \times 10^7$ deg. This figure is almost certainly inapplicable for three reasons:

(i) If the Sun were a normal star its representative point would be very close to the left-hand boundary of the main sequence.

(ii) Geological evidence indicates that the Sun must be at least $4 \times 10^9$ years old and hence must contain at the very least 3 per cent $He$.

(iii) According to nuclear physics the temperature given is too low for the $C$–$N$ cycle.

How then does the Sun differ from a normal star? If it were about $2 \times 10^{10}$ years old there might be as much as 15 per cent $He$. If this were well mixed it would lead to an appreciable deviation from the normal curve but in the wrong sense. If it were well mixed and then a hydrogen envelope had accreted, the extension of radius might lead to a displacement from the normal curve in the right direction, but it would probably not be very appreciable owing to the small difference in atomic weight between envelope and interior. If however the mixing were poor there might be an appreciable concentration of helium in the core even for a moderate age. During a life of say $5 \times 10^8$ years the core would have acquired a helium abundance of some 30 per cent. The effect of this on $D$ would be small (cf. Section 5), but the central value of $\mu$ would have to be substituted in the formula for $T_n$, leading to a value between $1.25 \times 10^7$ deg. and $1.56 \times 10^7$ deg. The upper value agrees well with estimates based on the most recent experiments on the cross-sections of the $C$–$N$ cycle (3). The older figure of $1.9 \times 10^7$ deg. (2) is clearly almost incompatible with the astrophysical data. It might be mentioned that the author had, on the basis of these considerations,
deduced that the usual estimate for the threshold temperature of thermonuclear energy generation was a good deal too high, even before the more recent experiments had come to his notice.

In fact the correct temperature is probably still a little lower than current nuclear physics estimates, and $1.4 - 1.5 \times 10^7$ deg. is probably not far from correct. Using the formulae of Section 4 and Fig. 1, it may be ascertained that the central temperature of stars of $M_{bol} = 6.5$ is about 10 per cent lower than the Sun's central temperature. These are the stars which are close to the switch in the mode of energy generation, so that one can deduce that this switch occurs for a central temperature of about $1.3 \times 10^7$ deg.

The results here presented are at variance with the calculations of Epstein (14), who deduced from the nuclear physics data that the switch in the mode of energy generation occurred for stars rather brighter than the Sun. The complete absence of a change of slope of the main sequence in that region, together with the change of slope discussed here, strongly suggests that his conclusions are erroneous, and that hence the stellar model and the nuclear data used by him stand in need of revision.


References

(4) H. A. Kramers, Phil. Mag., 46, 836, 1923.