INTEGRATION OF THE COWLING STELLAR MODEL

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Summary

A numerical integration of Cowling's stellar model is performed to give a solution of known accuracy. The method is described and the solution tabulated.

1. Introduction.—The stellar model introduced by Cowling (1) has become a basic model for stars of the main sequence. The integration of the equations for the model were performed by Cowling (1) and have been repeated by other workers (2, 3). However, these computers did not claim any considerable accuracy and made no estimate of the accuracy attained. There is now a demand for a definite integration of known accuracy. In response to requests reaching him, Mr D. H. Sadler, Superintendent of H.M. Nautical Almanac Office, suggested to the writer that she should attempt to provide this. Mr Sadler also suggested the particular choice of dependent variables which have actually been used.

The purpose of the rather high degree of accuracy which it is here sought to achieve is principally:

(a) to discover whether previous solutions have been appreciably affected by the mathematical "instability" of the equations of stellar structure;

(b) to estimate the degree of accuracy attained by the various methods of integration used by other workers, especially those of C. M. and H. Bondi (3). These methods are not intended to yield high accuracy but it is useful if the results obtained can be compared with those of a method of known accuracy so that for other similar cases the degree of approximation can be inferred.

It is not, of course, implied that the accuracy arrived at in the present work is of direct value for comparison with observational results. Neither the accuracy of the observations nor that of the assumptions underlying the equations would warrant this.

2. Problem.—The Cowling stellar model consists of two regions: a central convective core and an outer region in radiative equilibrium. All the sources are assumed to lie within the convective zone and the polytropic index \( n = (\gamma - 1)^{-1} \) has there its least value 1.5.

The usual physical variables are defined as follows:

\[ \begin{align*}
    r &= \text{distance from centre,} \\
    M &= \text{mass contained within a radius } r, \\
    \rho &= \text{density of matter at distance } r \\
    \text{from centre,} \\
    P &= \text{total pressure at distance } r \text{ from centre,} \\
    T &= \text{temperature,} \\
    L &= \text{luminosity.}
\end{align*} \]

\[ \begin{align*}
    \alpha &= \text{factor of proportionality in Kramers' law,} \\
    a &= \text{Stefan's constant,} \\
    c &= \text{speed of light,} \\
    G &= \text{constant of gravitation,} \\
    \mathcal{R} &= \text{gas constant,} \\
    \mu &= \text{molecular weight.}
\end{align*} \]
The equations of hydrostatic equilibrium and conservation of mass are
\[
\frac{dP}{dr} = -\frac{GM\rho}{r^2}, \\
\frac{dM}{dr} = 4\pi r^2 \rho,
\]
(1) (2)

In the convective zone
\[
\rho \propto T^{3/2},
\]
which can be written
\[
\rho = AT^{3/2},
\]
(3)

whereas in the outer region in radiative equilibrium, using Kramers' opacity law,
\[
\frac{d}{dr} \left(\frac{3}{2} a T^4\right) = \frac{-\alpha L\rho^2}{4\pi r^2 T_{\text{eff}}^4}.
\]
(4)

Changing \( r, T, \rho \) and \( M \) to the dimensionless quantities \( x, y, z \) and \( \omega \), where \( r = Xx, T = Yy, \rho = Zz, M = W\omega, \) equations (1) to (4) become
\[
\frac{d(yz)}{dx} = -\frac{5\omega z}{2z^2},
\]
(5)
\[
\frac{d\omega}{dx} = 2z^2,
\]
(6)
\[
z = y^{3/2}, \text{ in convective zone},
\]
(7)
\[
\frac{dy}{dx} = -\frac{kz^2}{x^2y^{3/2}}, \text{ in outer region},
\]
(8)

and the constants \( X, Y, Z \) and \( W \) are given by
\[
X = u^9V^{-1}, \ Y = u^{-2}V^4, \ Z = Au^{-3}V^6, \ W = 4\pi Au^{24}V^3,
\]
where \( u = (5\beta / 8\pi \mu GA)^{1/17}, \ V = (3aL\beta / 16\pi ack)^{1/17}, \) and \( k \) is a numerical constant (see below).

The problem is to find the solution of these equations such that: at the centre, \( x = 0, y \) and \( z \) are finite and \( \omega = 0; \) at the interface of the two regions, \( x = x_0, y, z, \omega \) and their first derivatives with respect to \( x \) are continuous; and at the outer boundary, \( x = R, y \) and \( z \) tend to zero together (i.e. \( T \) and \( \rho \) are zero together at the boundary of the star). For any \( x_0 \) the continuity conditions of \( dy/dx \) at \( x = x_0 \) give a unique numerical value of \( k \). For the solution satisfying all the required conditions there is a unique \( x_0 \) and hence a unique value of \( k \).

The solution of equations (5), (6) and (7) is the solution (4) of the Emden equation for \( n = 1.5 \). Hence a solution satisfying the required conditions is one that is the same as the solution of Emden's equation up to \( x = x_0 \), and which for \( x \geq x_0 \) satisfies equations (5), (6) and (8), and has \( y \) and \( z \) zero together for some \( x = R(\geq x_0) \).

Near the outer boundary \( x = R, \) a series solution of equations (5), (6) and (8) can be obtained. Outer boundary values of the variables are denoted by the suffix \( s \).

Put \( \xi = R/x - x \), so that \( \xi = 0 \) at the boundary. Then assuming that \( y \) can be expanded in a power series of the form \( y = y_s + y\xi + \cdots \) terms in higher powers of \( \xi \), expansions of \( z \) and \( \omega \) as power series in terms of \( \xi \) can be obtained by using equations (5), (6) and (8).
The purpose of deriving a series solution was to calculate boundary values of the variables (see next paragraph) and it was found that for the required accuracy it was sufficient to take the first two terms in \( y \), i.e. \( y = y_0 + Y_2 \xi \), and the terms dependent on this expression in the expansions for \( z \) and \( \omega \).

The resulting series are

\[
\begin{align*}
y &= Y \xi, \quad y_0 = 0, \\
z &= R Y^{17/4} k^{-1} \xi^{1 + 13/4}, \\
\omega &= \omega_0 - A_\omega \xi^{17/4} (1 + \xi)^{-8} (1 - \frac{5}{21} \xi + \frac{3}{35} \xi^2 - \frac{39}{1013} \xi^3 + \frac{291}{11165} \xi^4),
\end{align*}
\]

where

\[
\begin{align*}
\omega_0 &= 1.7 R Y, \\
A_\omega &= \frac{4}{17} R^{7/2} Y^{15/4} k^{-1/2}.
\end{align*}
\]

It may be noted that to this degree of approximation the function \( z y^{-13/4} \) is of constant value.

3. Method of integration.—The numerical integration was carried out using the variables \( x \), \( y \), \( \omega \) and \( \theta \), where \( \theta = z y^{-13/4} \). With these variables, equations (5), (6) and (8) become

\[
\begin{align*}
\frac{dy}{dx} &= - \frac{k_0^2}{x^2}, \\
\frac{d\omega}{dx} &= x^2 \theta y^{1/4}, \\
\frac{d\theta}{dx} &= - \frac{5\theta}{2x^2 y} (1.7 k \theta^2 - \omega).
\end{align*}
\]

The boundary condition for \( \theta \), to the same degree of approximation as in (9), is

\[
\theta = \theta_0 = (R Y / k)^{1/2},
\]

i.e. \( \theta \) tends to a constant non-zero value as \( x \to R \) and \( y \to 0 \).

This is a unique solution separating two sets of solutions for which either \( \theta \to 0 \) as \( y \to 0 \) or \( \theta \to \infty \) as \( y \to 0 \), according as \( x_0 \), the radius of the convective core, is smaller or larger than the required radius. (See Fig. 1.)

An approximate value of the radius of the convective core was found using the method of integration suggested by C. M. and H. Bondi (3). This gave a value of \( x_0 \) between 1.1920 and 1.1924.

The procedure used for the integration was as follows:

(a) for some \( x = x_0 \) calculate \( y_0 \), \( z_0 \), \( \omega_0 \) and their derivatives from Emden’s solutions for \( n = 1.5 \) (4);

(b) from equation (8) find \( k \) at \( x = x_0 \) using \( y_0 \), \( z_0 \), \( \omega_0 \), etc.;

(c) integrate equations (10), (11), (12) by the usual methods described in Interpolation and Allied Tables (5), i.e. assume values of \( y \) and \( 2h \frac{dy}{dx} \) are known for \( x_0 \), \( x_1 \), \( x_2 \) (where \( x_1 = x_0 + h \), \( x_2 = x_0 + 2h \)) and thus the differences \( \delta y \), \( \delta^2 y \), \( \delta^3 y \left(2h \frac{dy}{dx}\right) \), \( \delta^4 \left(2h \frac{dy}{dx}\right) \), etc., and then

(i) extrapolate a value of \( \delta^4 \left(2h \frac{dy}{dx}\right) \) at \( x_2 \),
(ii) calculate a value $y_E$ (say) for $y_3$ at $x_3$ by the equation

$$y_E = y_1 + \left( 2h \frac{dy}{dx} \right)_2 + \frac{1}{6} \delta^2 \left( 2h \frac{dy}{dx} \right)_2 + \text{higher terms},$$

where suffixes 1 and 2 denote values of functions at $x_1$, $x_2$,

(iii) use $y_E$ (also $\omega_E$ and $\theta_E$ which are similarly extrapolated) to calculate $\left( 2h \frac{dy}{dx} \right)_3$ from equations (10), (11) and (12),

(iv) difference $\left( 2h \frac{dy}{dx} \right)$ at $x_3$ to give a revised value of $\delta^2 \left( 2h \frac{dy}{dx} \right)_2$,

and, if necessary, to correct $y_E$ and $2h \frac{dy}{dx}$,

(v) calculate

$$\delta(y) = y_2 - y_1 = \frac{1}{6} \left\{ \left( 2h \frac{dy}{dx} \right)_1 + \left( 2h \frac{dy}{dx} \right)_2 + \frac{1}{6} \delta^2 \left( 2h \frac{dy}{dx} \right)_1 - \delta^2 \left( 2h \frac{dy}{dx} \right)_2 \right\}$$

which gives $y_2$ (also $\omega_2$ and $\theta_2$) ready for the next step in the integration.

Initial values of $y$, $\omega$ and $\theta$, etc. found from (a), together with two other sets of values calculated from series solutions of equations (10), (11) and (12) were sufficient to start the integration. The first few steps of the integration were performed using an interval of 0·01 in $x$; this was changed to 0·05 and later to 0·10. The integration was continued until it became clear from the behaviour of $\theta$ and $d\theta/dx$ that the solution was deviating from the required one. (See Fig. 1.)

![Diagram](https://example.com/diagram.png)

**Fig. 1.**—Diagram to show the extreme sensitivity of the solution for $\theta$ according to the choice of $x_0$.

A new integration was then started using different values of $x_0$ and $k$ and the procedure was repeated. From two integrations on either side of the required solution an intermediate integration could be started. It was not necessary to recalculate all the initial values of $y$, $\omega$, $\theta$, etc., since these could easily be interpolated from the initial values of the other integrations.
By this method it was possible to find the numerical solution as far as \( x = 6.5 \). To finish the integration, appeal was made to the series solution (9). Using the known values of \( y, \omega \) and \( \theta \) at \( x = 6.4 \) and 6.5, and assuming that \( \theta = \theta_s = \) constant, the values of \( R \) and \( Y \) were calculated from the expansion \( y = Y \xi \).

Then using the series for \( \omega \) and the values of \( R, Y \) and \( k \), the values of \( \omega_s \) and \( A_\omega \) were calculated. The value of \( \theta_s = (RY)^{1/2} k^{-1/2} \) was recalculated, but was found to differ from the assumed value of \( \theta_s \) by only a few units in the sixth decimal place. This discrepancy made no difference to the calculated solution however.

The integration was carried out using six decimal places, but owing to the cumulative effect of rounding-off errors, etc., the sixth decimal is unreliable and is not given in the table. In general, the fifth decimal should be reliable especially in \( \omega \) and \( \theta \) for which the solution is very stable.

4. Results.—The principal results obtained by this method of integration are as follows:

- Radius of convective core = 1.192296,
- \( k = 0.197409 \),
- Radius of star = 7.01168,
- Mass of star = 3.12610 Emden units,
- Mass of convective core = 0.45759 Emden units,
- Mean density/central density = 0.02749.

The number of decimals given corresponds to five decimals in the solutions and is therefore expected to be reasonably correct.

5. Comparison.—The results are compared with those obtained by other authors in the following table:

<table>
<thead>
<tr>
<th>Table I</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Comparison with previous solutions</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Present paper</td>
</tr>
<tr>
<td>T. G. Cowling (1)</td>
</tr>
<tr>
<td>F. Hoyle and R. A. Lyttleton (2)</td>
</tr>
<tr>
<td>C. M. and H. Bondi (3)</td>
</tr>
<tr>
<td>C. M. Bondi (6)</td>
</tr>
</tbody>
</table>

6. Boundary conditions.—While it is irrelevant to the performance of the computations, it is of interest to see the physical meaning of the so-called "instability of the solution". This instability of the solution is, in fact, a result of the stability of the model with regard to its boundary conditions. A slight departure from the required solution when integrating outwards leads to a solution of the equations differing greatly from the former near the boundary. Conversely, this means that a very large difference in the boundary conditions is needed to give an appreciable difference in the interior of the model. Tables II, III and IV illustrate the effect on the boundary conditions of a slight departure of \( \theta \) from its true value. Tables II and III give the solutions for two models A and B which start to deviate from the Cowling model at \( x = 6.60 \) and 6.75 respectively. Table IV gives a comparison of values of the temperature and radius for models A and B and
### Table II

**Boundary values for model A**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>θ</th>
<th>ω</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.60</td>
<td>0.0164</td>
<td>4.78 x 10^{-6}</td>
<td>3.052</td>
<td>3.1261</td>
</tr>
<tr>
<td>6.65</td>
<td>0.0143</td>
<td>3.06 x 10^{-6}</td>
<td>3.052</td>
<td>3.1261</td>
</tr>
<tr>
<td>6.70</td>
<td>0.0122</td>
<td>1.84 x 10^{-6}</td>
<td>3.050</td>
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<tr>
<td>6.75</td>
<td>0.0102</td>
<td>1.01 x 10^{-6}</td>
<td>3.043</td>
<td>3.1261</td>
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<tr>
<td>6.80</td>
<td>0.0082</td>
<td>4.97 x 10^{-7}</td>
<td>2.992</td>
<td>3.1261</td>
</tr>
<tr>
<td>6.85</td>
<td>0.0065</td>
<td>1.99 x 10^{-7}</td>
<td>2.574</td>
<td>3.1261</td>
</tr>
<tr>
<td>6.90</td>
<td>0.0057</td>
<td>5.72 x 10^{-8}</td>
<td>1.119</td>
<td>3.1261</td>
</tr>
<tr>
<td>6.95</td>
<td>0.0056</td>
<td>1.37 x 10^{-8}</td>
<td>0.283</td>
<td>3.1261</td>
</tr>
<tr>
<td>7.00</td>
<td>0.0056</td>
<td>3.29 x 10^{-9}</td>
<td>0.068</td>
<td>3.1261</td>
</tr>
<tr>
<td>7.05</td>
<td>0.0056</td>
<td>8.1 x 10^{-10}</td>
<td>0.017</td>
<td>3.1261</td>
</tr>
<tr>
<td>7.10</td>
<td>0.0056</td>
<td>2.1 x 10^{-10}</td>
<td>0.004</td>
<td>3.1261</td>
</tr>
<tr>
<td>7.15</td>
<td>0.0056</td>
<td>5.2 x 10^{-11}</td>
<td>0.001</td>
<td>3.1261</td>
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</table>

### Table III

**Boundary values for model B**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>θ</th>
<th>ω</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.80</td>
<td>0.0082</td>
<td>4.99 x 10^{-7}</td>
<td>3.052</td>
<td>3.1261</td>
</tr>
<tr>
<td>6.85</td>
<td>0.0062</td>
<td>2.03 x 10^{-7}</td>
<td>3.051</td>
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</tr>
<tr>
<td>6.90</td>
<td>0.0043</td>
<td>5.96 x 10^{-8}</td>
<td>3.017</td>
<td>3.1261</td>
</tr>
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<td>6.95</td>
<td>0.0029</td>
<td>8.43 x 10^{-9}</td>
<td>1.559</td>
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</tr>
<tr>
<td>7.00</td>
<td>0.0027</td>
<td>4.7 x 10^{-10}</td>
<td>0.098</td>
<td>3.1261</td>
</tr>
<tr>
<td>7.05</td>
<td>0.0027</td>
<td>2.6 x 10^{-11}</td>
<td>0.005</td>
<td>3.1261</td>
</tr>
</tbody>
</table>

### Table IV

**Comparison of y and x for a common density z = 5.0 x 10^{-11} for Cowling model and models A and B**

<table>
<thead>
<tr>
<th></th>
<th>Cowling model</th>
<th>Model A</th>
<th>Model B</th>
</tr>
</thead>
<tbody>
<tr>
<td>y (Temp.)</td>
<td>0.0004</td>
<td>0.0056</td>
<td>0.0027</td>
</tr>
<tr>
<td>x (Radius)</td>
<td>7.00</td>
<td>7.15</td>
<td>7.04</td>
</tr>
</tbody>
</table>

### Table V

**Numerical solutions of the Cowling stellar model**

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>z</th>
<th>ω</th>
<th>θ</th>
</tr>
</thead>
<tbody>
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<td>0</td>
<td>1.0</td>
<td>1.0</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.99833</td>
<td>0.99751</td>
<td>0.00033</td>
<td>These values not used in integration</td>
</tr>
<tr>
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<td>0.99335</td>
<td>0.9905</td>
<td>0.00265</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.98510</td>
<td>0.97773</td>
<td>0.00888</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.97365</td>
<td>0.96074</td>
<td>0.02083</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.95910</td>
<td>0.93929</td>
<td>0.04014</td>
<td></td>
</tr>
<tr>
<td>0.6</td>
<td>0.94159</td>
<td>0.91367</td>
<td>0.06823</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.92125</td>
<td>0.88424</td>
<td>0.10626</td>
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<tr>
<td>0.8</td>
<td>0.89828</td>
<td>0.85136</td>
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<tr>
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<td>0.87285</td>
<td>0.81547</td>
<td>0.21537</td>
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<tr>
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<td>0.84517</td>
<td>0.77099</td>
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</tr>
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<td>1.1</td>
<td>0.81547</td>
<td>0.73640</td>
<td>0.37069</td>
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<tr>
<td>1.2</td>
<td>0.78398</td>
<td>0.69415</td>
<td>0.46528</td>
<td>1.53092</td>
</tr>
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<td>1.3</td>
<td>0.75224</td>
<td>0.64965</td>
<td>0.57026</td>
<td>1.63877</td>
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<tr>
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<td>0.60300</td>
<td>0.68438</td>
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<td>1.5</td>
<td>0.69097</td>
<td>0.55526</td>
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<td>1.84609</td>
</tr>
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<td>1.6</td>
<td>0.66144</td>
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<td>0.93364</td>
<td>1.94434</td>
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<tr>
<td>1.7</td>
<td>0.63267</td>
<td>0.46036</td>
<td>1.06526</td>
<td>2.03831</td>
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<td>0.60469</td>
<td>0.41484</td>
<td>1.19913</td>
<td>2.12764</td>
</tr>
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<td>1.9</td>
<td>0.57752</td>
<td>0.37143</td>
<td>1.33351</td>
<td>2.21202</td>
</tr>
</tbody>
</table>

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Cowling’s model at a common density of \( z = 5 \times 10^{-11} \). For the two models A and B, the temperature remains approximately constant after \( x = 7 \times 10^{-6} \), while the density continues to approach zero.

It should be noted that these differences in boundary conditions are independent of other boundary changes such as a change of opacity coefficient.
Integration of the Cowling stellar model

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Girton College,
Cambridge:
1950 September 27.

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(2) F. Hoyle and R. A. Lyttleton, M.N., 102, 218, 1942.