ON THE MASSES OF SATURN'S SATELLITES

Sir Harold Jeffreys, F.R.S.

(Received 1933 January 12)

Summary

The principal determinations of the masses of Saturn's satellites are rediscussed, with special attention to the standard errors. A new determination of the mass of Titan is made from the motion of the orbital plane of Iapetus, and an attempt is made to estimate the errors remaining in Woltjer's solution from Hyperion. The values resulting are as follows, expressed as fractions of the mass of Saturn:

- Mimas \((6.69 \pm 0.20) \times 10^{-8}\);
- Enceladus \((1.27 \pm 0.53) \times 10^{-7}\);
- Tethys \((1.141 \pm 0.030) \times 10^{-6}\);
- Dione \((1.825 \pm 0.061) \times 10^{-6}\);
- Rhea \((0.4 \pm 3.8) \times 10^{-6}\);
- Titan \((2.412 \pm 0.018) \times 10^{-4}\);

and Struve's constants for Saturn are

\[ k/a_0^2 = 0.024305 \pm 0.000048, \]
\[ l/a_0^4 = (1.0 \pm 2.0) \times 10^{-4}. \]

1. The mass of Titan was determined almost simultaneously by S. Newcomb from the motion of the apse of Hyperion and by H. Struve from a comparison of the motions of the orbital plane of Iapetus and the apse of Titan. The results differed by a factor of nearly 3, but successive corrections to the theory of Hyperion have produced a rough agreement. There seems, however, to have been no new solution from Iapetus. Struve's discussion* is statistically very unsatisfactory. It is repeated with little alteration by Tisserand.† Using the same data, I applied the method of least squares, supposing that standard errors in \(i\) and \(\theta\) (longitude of the node) were in the ratio \(\sin i\) to 1, and giving equal weight to all data. The result was (the mass of Saturn being taken as 1)

\[ m_{\text{T}} = (2.324 \pm 0.075) \times 10^{-4}, \]
\[ 1/m_{\text{T}} = 4.308 \pm 138 \text{ (s.e.)}. \]

Struve's value was given as \(4.678 \pm 350\). The standard error suggests that with modern data the method might yield an uncertainty under 1 per cent.

J. Woltjer‡ gives seven determinations from Hyperion, ranging from 3.767 to 4.172, and adopts the value 4.033.

H. Struve§ discussed the observations of all the satellites and made important contributions to the theory. G. Struve‖ rediscussed the observations, with others up to 1928, and derived new masses. Estimates of uncertainty are given

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† F. Tisserand, Théorie de Mécanique Céleste, 4, 98–102, 1896.
for the equations of condition, but the final masses and their uncertainties are based on judgment and not on least squares solutions. Accordingly it cannot be said that their uncertainties are known, which is unfortunate in view of their importance for an understanding of the constitutions of these bodies. In the course of the present work it was found that G. Struve’s uncertainties ranged from half to twice the standard errors.

2. The following analysis is mainly based on G. Struve’s data. Some of the equations of condition were rediscussed but no important discrepancy was found. Uncertainties have been converted to standard errors.

The motions of the nodes and apses derived from the secular terms in the disturbing functions due to the ellipticity of Saturn and the attractions of the satellites are denoted by $\pm \beta n$; they are theoretically equal and opposite, to the accuracy available. Approximate commensurability of the mean motions produces systematic changes of the secular motions for the pairs Enceladus–Dione and Titan–Hyperion. The solar effect has been subtracted from all observed values. The theoretical values as functions of the masses are given by G. Struve (Heft 4, p. 53) for the first six satellites. A discussion of the observations of Titan and Iapetus is given in the posthumous Heft 5.

For the pairs Mimas–Tethys and Enceladus–Dione, though the masses are small, good determinations are possible because fairly large perturbations are produced by approximate commensurability of the mean motions.

The mean motions in degrees per day given by G. Struve will be adopted (Table I). The value for Hyperion is Woltjer’s.

<table>
<thead>
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<th>Table I</th>
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<tbody>
<tr>
<td>Mimas</td>
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<td>Iapetus</td>
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The observed values of $\beta n$, in degrees per year, are in Table II.

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<tr>
<td>Mimas</td>
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<td>Titan</td>
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The mean motion of the apse of Titan is given (Heft 5, pp. 12–13) as $31^1.41 \pm 0.21$ per year; from this must be subtracted the solar part $1^1.34$ (H. Struve, p. 172), leaving the value given. The value for the node is a rough determination from the data in G. Struve’s table (Heft 5, p. 11). The values for Iapetus need further discussion.

I consider first the periodic disturbances.
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2.1. Mimas and Tethys.—The theory is outlined by Tisserand (pp. 129–138). G. Struve (Heft 4, p. 10) writes the perturbations of mean longitude of Mimas in the form

\[ \Delta l = A \sin \mu (\tau - \tau_0) + B \sin 3\mu (\tau - \tau_0) \]

and determines

\[ A = -44°.390 \pm 0°.122, \]
\[ \mu = (5°.0864 \pm 0°.0085)/\text{year}, \]
\[ \tau_0 = 1866.27 \pm 0.06. \]

For Tethys \( A \) is replaced by \( A_1 \) and the same values of \( \mu, \tau_0 \) are adopted; then

\[ A_1 = 2°.065 \pm 0°.028. \]

The ratio \( m_{\text{M}}/m_{\text{T}} \) then follows from

\[ \frac{a_1 m_{\text{M}}}{a m_{\text{T}}} = 2 \frac{A_1}{A} = 0.09304. \]

The value of \( m_{\text{T}} \) depends on the use of elliptic functions, but substantially

\[ m_{\text{T}} \propto \frac{1}{\mu^2 \gamma_1}, \]

where \( \gamma, \gamma_1 \) are the inclinations of the two orbital planes to Saturn's equator. Allowing for the uncertainties of these and of \( \mu \) gives a proportional standard error of \( \pm 2 \times 0.0017 \pm 0.025 \pm 0.009 = \pm 0.026 \). Then

\[ m_{\text{T}} = \frac{1}{876400} (1 \pm 0.026) = 1.141 \times 10^{-8} (1 \pm 0.026). \]

The relative uncertainty of \( m_{\text{M}} \) is \( 0.14 \) from that of \( A_1 \); that of \( A \) is unimportant, but those of \( \mu, \gamma, \gamma_1 \) enter again; then

\[ m_{\text{M}} = \frac{1}{14960000} (1 \pm 0.026 \pm 0.014) \]

\[ = 6.60 \times 10^{-8} (1 \pm 0.030). \]

G. Struve gives 837,000 and 915,000 as extreme values of \( 1/m_{\text{T}} \), corresponding to about 1.5 times the standard error. On the other hand, he gives the uncertainty of the mass of Mimas as 5–10 per cent, the latter being too high on a similar scale.

2.2. Enceladus and Dione.—The largeness of the perturbations in this case also depends on the fact that the mean motions are nearly as 2 to 1, but the theory is much more difficult. In the case of Mimas and Tethys the argument of the libration is \( 4l - 2l - \theta - \theta_1 \), and thus depends on a product of the inclinations and not on the eccentricities. For Enceladus and Dione it does depend on the eccentricities. H. Struve used a linearized theory, which is used in part by G. Struve. Woltjer* pointed out that higher terms need attention, and G. Struve quotes his solution also. It allows, however, only for perturbations of Enceladus by Dione, and a somewhat fuller theory is needed. The following method is that of H. Struve extended to allow for certain second-order terms. Letters with suffix \( \chi \) refer to Dione, without them to Enceladus.


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The rates of change of the relevant orbital elements are given by (H. Struve, 1898, p. 178)

\[
\frac{dn}{dt} = 3m_1n^2\{Ae \sin (2l_1 - l - \omega) - Be_1 \sin (2l_1 - l - \omega_1)\},
\]
(1)

\[
\frac{de}{dt} = m_1nA \sin (2l_1 - l - \omega),
\]
(2)

\[
\frac{d\omega}{dt} = \beta n - m_n A \frac{e}{e_1} \cos (2l_1 - l - \omega),
\]
(3)

\[
\frac{dn_1}{dt} = -6mn_1^2\{A_1e \sin (2l_1 - l - \omega) - B_1e_1 \sin (2l_1 - l - \omega_1)\},
\]
(4)

\[
\frac{de_1}{dt} = -mn_1B_1 \sin (2l_1 - l - \omega_1),
\]
(5)

\[
\frac{d\omega_1}{dt} = \beta_1n_1 + mn_1 B_1 \frac{B_1}{e_1} \cos (2l_1 - l - \omega_1),
\]
(6)

\[
A = 0.7532, \quad B = 0.2714, \quad A_1 = 1.1943, \quad B_1 = 0.4348, \quad AB_1 - A_1B = 0.0034.
\]
(7)

Put

\[
V_0 = (2n_1 - n)t, \quad V = 2l_1 - l = V_0 + V',
\]
(8)

\[
h = e \sin (\omega - V_0), \quad k = e_1 \cos (\omega - V_0), \quad h = e_1 \sin (\omega - V_0),
\]
(9)

\[
v = \beta n - (2n_1 - n), \quad v_1 = 2n_1 - n - \beta_1n_1.
\]
(10)

Then

\[
\frac{dh}{dt} = nk - m_nA \cos V',
\]
(11)

\[
\frac{dV}{dt} = -nV + m_nA \sin V',
\]
(12)

\[
\frac{dh_1}{dt} = -v_1h_1 + mn_1B_1 \cos V',
\]
(13)

\[
\frac{dV_1}{dt} = v_1h_1 - mn_1B_1 \sin V',
\]
(14)

\[
\frac{d^2l}{dt^2} = \frac{dn}{dt} = 3mn^2\{-Ah \cos V' + Ak \sin V' + Bh_1 \cos V' - Bk_1 \sin V'\},
\]
(15)

\[
\frac{d^2l_1}{dt^2} = \frac{dn_1}{dt} = -6mn_1^2\{-A_1h \cos V' + A_1k \sin V' + B_1h_1 \cos V' - B_1k_1 \sin V'\},
\]
(16)

\[
\frac{d^2V'}{dt^2} = P(h \cos V' - k \sin V') - Q(h_1 \cos V' - k_1 \sin V'),
\]
(17)

where

\[
P = 12mn_1^2A_1 + 3mn^2A,
\]
(18)

\[
Q = 12mn_1^2B_1 + 3mn^2B \div BP / A.
\]
(19)

For any constant \(a, \gamma\)

\[
V' = \alpha + \gamma t
\]
leads to an exact solution of (11) to (14) and (17). This corresponds to the addition of constants to \(n, n_1\) and the epochs. But it also indicates that the period
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equation contains $\gamma^2$ as a factor, and that we can choose the constants so that $V'$ has no constant or secular part. This identifies $n$, $n_1$ with the observed mean motions.

For $V'$ small, if we neglect $V''$, there is a constant solution

$$h = h_1 = 0, \quad V' = 0,$$

$$k = k_0 = \frac{m_1 A}{v}, \quad k_1 = k_{10} = -\frac{mn_1 B_1}{v_1}.$$  \hfill (20)

Accents denoting departures from this constant solution, we put

$$h = c \sin \gamma t, \quad k' = \frac{\gamma}{v} c \cos \gamma t, \quad h_1 = c_1 \sin \gamma t, \quad k_1' = -\frac{\gamma}{v_1} c_1 \cos \gamma t, \quad V' = \lambda \sin \gamma t$$

and retain the first powers of $c$, $c_1$, $\lambda$. Then, cancelling a factor $\gamma^2 \lambda$, we have the period equation

$$\frac{Pk_0}{\gamma^2 - v^2} - \frac{Qk_{10}}{\gamma^2 - v_1^2} = I$$

with

$$c = -\frac{m_1 A v \lambda}{\gamma^2 - v^2}, \quad c_1 = \frac{mn_1 B v_1 \lambda}{\gamma^2 - v_1^2}.$$  \hfill (21)

To the second order in $k_0$, $k_{10}$ the roots are given by

$$\gamma^2 - v^2 = Pk_0 \left( I - \frac{Qk_{10}}{v^2 - v_1^2} \right),$$

$$\gamma_1^2 - v_1^2 = -Qk_{10} \left( I - \frac{Pk_0}{v^2 - v_1^2} \right),$$

and

$$\Delta l = -3m_1 n^2 \left\{ \left( \frac{Ak_0}{\gamma^2 - v^2} - \frac{Bk_{10}}{\gamma^2 - v_1^2} \right) \lambda \sin \gamma t + \left( \frac{Ak_0}{\gamma_1^2 - v^2} - \frac{Bk_{10}}{\gamma_1^2 - v_1^2} \right) \lambda_1 \sin \gamma_1 t \right\},$$

$$\Delta l_1 = 6mn_1^2 \left\{ \left( \frac{A_1 k_0}{\gamma_1^2 - v^2} - \frac{B_1 k_{10}}{\gamma_1^2 - v_1^2} \right) \lambda \sin \gamma t + \left( \frac{A_1 k_0}{\gamma_1^2 - v^2} - \frac{B_1 k_{10}}{\gamma_1^2 - v_1^2} \right) \lambda_1 \sin \gamma_1 t \right\},$$

$$\Delta l = -\frac{2mn_1^2 A_1}{m_1 n^2} \frac{A}{A} \Delta l = -\frac{1}{2} m \frac{A_1}{A} \Delta l.$$

$k_0$ is small and $k_{10}$ still smaller, and if we neglect $m$ the solution reduces to that of Woltjer. Note that $\Delta l$, $\Delta l_1$ are not small compared with $\lambda$, $\lambda_1$, since $\gamma^2 - v^2$ and $\gamma_1^2 - v_1^2$ contain the small factors $m$, $m_1$.

G. Struve gives $l$, $e$ and $\varpi$ for mean dates, each summarizing observations over about a year. With $e$ and $\varpi$ he gives also $2l_1 - l$, which never differs from $\varpi$ by more than a few degrees. Thus $k$ from (g) is practically the recorded $e$, and the mean value of the latter is an estimate of $k_0$. Thus $k_0$ is the forced eccentricity of Enceladus. The variations of $e$ and of $e \sin (2l_1 - l - \varpi)$ are however too small to give useful determinations of the periodic parts of $h$ and $k$. The values of $e$ give

$$k_0 = 0.004.48 \pm 0.000.21.$$  \hfill (27)

G. Struve's value (p. 33) is $0.004.44 \pm 0.000.09$ (p.e.) but the uncertainty seems to be a mistake.

For Dione the mean eccentricity given by Struve's table (p. 34) is

$$e_1 = 0.002.11 \pm 0.000.09.$$  \hfill (28)
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Struve gives $0.00221$ without an uncertainty. (Some of these small changes are due to differences in the allocation of weights where two sets of observations correspond to one year. I have checked most of the summary values by means of $\chi^2$, and took the larger uncertainty in cases of disagreement, which seldom occurred.) As for Enceladus the periodic variations $h'_1, k'_1$ are too small to be detected from the variations of $e$ and $\omega$.

The periodic parts of $l$ are found to be (G. Struve, p. 23)

$$\Delta l = (14'39 \pm 1'20) \sin \{53^\circ.75 \pm 5^\circ.28 + (32^\circ.51 \pm 0^\circ.30) t\}
\quad + (14'06 \pm 1'19) \sin \{117^\circ.28 \pm 5^\circ.47 + (93^\circ.14 \pm 0^\circ.40) t\},$$

and direct comparison of this with $l_1$ gives

$$\frac{\Delta l_1}{\Delta l} = x = +0.065 \pm 0.025.$$  

(29)

The speeds of the harmonic terms are estimates of $\gamma, \gamma_1$, but for their interpretation we need also preliminary estimates of $\beta n, \beta_1 n_1$. The free eccentricity of Dione ($\delta_1$) leads observationally to

$$d\omega_1/dt = 30^\circ.75 \pm 0^\circ.21;$$  

(31)

there is no direct evidence at all for Enceladus. H. and G. Struve proceeded by interpolation from the values for Mimas and Tethys. I extend their method by fitting a form $Cn^{78} + Dn^{-1}$ to all the observed values out to Rhea, giving equal weights to each. This amounts roughly to regarding the secular motions as mainly due to the ellipticity of Saturn and perturbations by Titan, with a small part due to roughness of the Titan term and to the effects of the other satellites, which is taken as random. The object is only to get preliminary estimates of $\beta n, \beta_1 n_1$, which can be improved later. The result is, with $\beta n$ in degrees per year, $n$ in degrees per day,

$$C = 3.45008 \times 10^{-4}, \quad D = +53,$$

with the following results.

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<tbody>
<tr>
<td>Mimas</td>
<td>$365.26 + 0.14 = 365.40$</td>
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<tr>
<td>Enceladus</td>
<td>$152.57 + 0.20 = 152.77$</td>
</tr>
<tr>
<td>Tethys</td>
<td>$72.23 + 0.28 = 72.51$</td>
</tr>
<tr>
<td>Dione</td>
<td>$30.35 + 0.40 = 30.75$</td>
</tr>
<tr>
<td>Rhea</td>
<td>$9.43 + 0.67 = 10.10$</td>
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The estimated standard error of one value is $0.23$. This is more than the apparent uncertainties of the observed values, and it will be worth while to attempt greater detail. It is clear, however, that the two observed values for Mimas differ by 2.5 times the standard error of their difference, and that the interpolation gives $\beta n$ for Enceladus with as much accuracy as the speeds of the two harmonic terms already possess. For Dione the interpolate happens to agree exactly with the observed value. I take

$$\beta n = 152^\circ.77/1^\circ + \delta(\beta n),$$

(32)

$$\beta_1 n_1 = 30^\circ.75/1^\circ + \delta(\beta_1 n_1),$$

(33)

where the corrections are unlikely to exceed $0.3/1^\circ$. Then

$$2n_1 - n = 123^\circ.44/1^\circ, \quad \nu = 29^\circ.33 - \delta(\beta n), \quad \nu_1 = 92^\circ.69 + \delta(\beta_1 n_1).$$

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I take also

\[ m = 1.5 \times 10^{-7}(1 + \mu), \quad m_1 = 1.8 \times 10^{-8}(1 + \mu_1). \] (35)

Then

\[ k_0 = 0.00446 (1 + \mu_1)(1 + \frac{1}{29} \delta(\beta n)), \] (36)

\[ k_{10} = -0.000625 \mu (1 + \mu)(1 - \frac{1}{90} \delta(\beta_1 n_1)), \] (37)

\[ \frac{P k_0}{n^2} = (2.056 + 0.238 \mu + 3.870 \mu_1) 10^{-8} (1 + \frac{1}{29} \delta(\beta n)), \] (38)

\[ \frac{Q k_{10}}{n^2} = -(4.32 + 4.82 \mu + 3.81 \mu_1) 10^{-11} (1 - \frac{1}{90} \delta(\beta_1 n_1)), \] (39)

whence, with sufficient accuracy,

\[ \gamma = (1.1057 + 0.0106 \mu + 0.1723 \mu_1)(1 - \frac{1}{290} \delta(\beta n)), \] (40)

\[ \gamma = 32.43 + 0.34 \mu + 5.59 \mu_1 - \delta(\beta n), \] (41)

and

\[ 0.34 \mu + 5.59 \mu_1 = +0.08 \pm 0.30 + \delta(\beta n). \] (42)

From \( k_0 \)

\[ 0.00446 (\mu_1 + \frac{1}{29} \delta(\beta n)) = +0.000 02 \pm 0.000 21. \] (43)

Also

\[ (\gamma_1^2 - v_1^2)/v_1^2 = 10^{-5} \] (44)

and is negligible. The ratio \( \Delta l_1/\Delta l \) leads to

\[ 0.066 \mu - 0.066 \mu_1 = -0.001 \mu \pm 0.025. \] (45)

The mean eccentricity of Dione is \( \epsilon_1 = \epsilon_1 \); the amplitude of the mode with speed \( \gamma_1 \) for Enceladus is Struve's \( q \)

\[ q = 3BM_1 c_1 \left( \frac{n}{\nu_1} \right)^2 = 5.439 (1 + \mu_1) = 14.06 + 1.21. \] (46)

In \( \mu_1 \) we take \( c_1 \) from (28); then

\[ \epsilon_1 + 0.002 \mu_1 = 0.002 58 \pm 0.000 22. \] (47)

\( \delta(\beta n) \) and \( \delta(\beta_1 n_1) \) have been neglected where their effects are negligible in comparison with the uncertainties already present. On account of (44) the "periodic" term in the disturbing function due to Enceladus produces a negligible effect in the motion of the apse of Dione.

Then the equations of condition for \( \mu, \mu_1, c_1 \) are (42), (43), (28), (45), (47). The solution is

\[ \mu = -0.027 + 0.37 \delta(\beta n) \pm 0.35, \] (48)

\[ \mu_1 = +0.030 + 0.047 \delta(\beta n) \pm 0.034, \] (49)

\[ c_1 = 0.002 19 - 0.000 02 \delta(\beta n) \pm 0.000 08. \] (50)

Comparing with the equations of condition, and ignoring \( \delta(\beta n) \), we get \( \chi^2 = 3.2 \) on 2 degrees of freedom. Then

\[ m = 1.46 \times 10^{-7} (1 + 0.38 \delta(\beta n) \pm 0.36), \] (51)

\[ m_1 = 1.854 \times 10^{-8} (1 + 0.046 \delta(\beta n) \pm 0.033), \] (52)

\[ \frac{I}{m} = 6.850 \times 10^{-5}, \quad \frac{1}{m_1} = 539.400. \] (53)

G. Struve gives 6622 000 and 547 300, with uncertainties of 20 and 5 per cent. That for Enceladus is too low; that for Dione is about 1.5 times the standard error.
The solution is not quite so satisfactory as it appears, for a reason given by G. Struve (p. 37). The argument of the $\gamma_1$ term is essentially $2l_1 - l - \sigma_1$, and the constant term in this is (with probable errors)

$$-173^\circ.4 + 3^\circ.4 + (507^\circ.44'0 + 0^\circ.8) - (199^\circ.25'8 + 0^\circ.8) = 134^\circ.90 + 3^\circ.4.$$

The observed value is $117^\circ.28 \pm 3^\circ.69$, and the difference is $17^\circ.62 \pm 5^\circ.15$ (p.e.) or $\pm 7^\circ.62$ (s.e.). This would have contributed $5.4$ to $\chi^2$ had it been included in the comparison, making $\chi^2 = 8.6$ on 3 degrees of freedom. It did not arise in the actual solution because the phase affects none of the equations of condition, but there are also certain persistences of sign in the residuals that suggest unconsidered long-period perturbations. If these have been inadequately eliminated in the method of solution the uncertainties may need to be increased by a factor of 1.5 or so.

2.3. *Rhea.*—This has a forced eccentricity due to Titan, and the pericentre oscillates about that of Titan. G. Struve (Heft 1, p. 11) adopts $m_{T1} = \frac{1}{4083}$ and derives the libration constants by least squares. The residuals $e$ after allowance for the libration give a mean $(-2 \pm 6.7) \times 10^{-5}$; the adopted mean was $0.000 698$. Taking the mean eccentricity as proportional to $m_{T1}$ we get

$$m_{T1} = \frac{96 \pm 6.7}{98} \times \frac{1}{4033} = 0.000 242 9 (1 \pm 0.070),$$

$$\frac{1}{m_{T1}} = 4117 \pm 288.$$

2.4. *Titan.*—For the mean motion of the node on the proper plane G. Struve adopted H. Struve’s value of $0^\circ.506$/year, and he apparently did not try to improve it. For the apse and node I take the values in Table II.

An estimate of the mass of Iapetus is derived from the inclination of the orbit to Saturn’s equator, namely

$$m_{Ja} = 2.46 (1 \pm 0.78) \times 10^{-8}.$$

The uncertainty has as usual been converted into a standard error. The result is only an indication that the mass is comparable with those of Tethys and Dione.

Examination of possible effects of Hyperion shows that its mass is less than $\frac{1}{3000}$ of that of Titan.

2.5. *Hyperion.*—The theory of this satellite is far the most complicated of all. Woltjer’s treatment* is the fullest. The essential points may be indicated by comparison with the perturbations of Enceladus by Dione. The chief differences are that the mean motions are nearly in the ratio 4 to 3 instead of 2 to 1; the reaction of Hyperion on Titan can be neglected; and the mass of Titan is about 100 times that of Dione, so that approximations have to be carried to a much higher order. As the counterpart of $k_{10}$ is negligible, that of $\gamma_1$ reduces to $\gamma_1$.

The resemblances are as follows. (1) The $k_0$ term in the motion of Enceladus has a much larger counterpart. This is a forced eccentricity of about 0.1 combined with a motion of pericentre $4n - 3n'$, accents referring to Titan. (2) The free oscillations about the resulting periodic orbit, in the case of Enceladus, are represented by the term in mean longitude of speed $\gamma$. When the departures from a circular orbit are described in terms of an eccentricity and a moving apse, and the free eccentricity is less than the forced one, the apse can be described as librating about a point with constant motion. Hyperion shows a similar feature. (3) The $\gamma_1$ term in the mean longitude of Enceladus is a forced term due to the eccentricity

of Dione. The eccentricity of Titan produces a corresponding disturbance of Hyperion.

The differences are quantitative matters and are shown chiefly by the slowness of convergence of series for Hyperion. The disturbances are not so far from linearity that a partly linearized theory similar to that for Enceladus would be seriously in error. Inspection of Wolter's series shows that higher terms, though not negligible, are much smaller than the main terms. In these conditions it appears that further corrections can be treated by adapting the theory of Enceladus, the chief features having already been treated fully by Wolter.

Wolter (p. 66) quotes the following estimates of \( \frac{1}{m'_{\mathrm{Ti}}} = \frac{1}{m'} \) found from Hyperion. (a), (b), (c) are his own.

- (a) Mean motion of libration argument: 3986
- (b) Mean motion of pericentre: 4080
- (c) Mean motion of node: 3767 ± 140
- (d) Hill's method (computed by D. Brouwer): 4143
- (e) Eichelberger (special perturbations): 4172 ± 58
- (f) Longitude of pericentre (Samter): 4125
- (g) Liberation (Samter): 3910

The uncertainty of (c) is large essentially because the inclination of the orbital planes is small. Eichelberger's result* is from the direct computation of perturbations of the elements, but rests on only about six months' observations. This is only about \( \frac{1}{6} \) of the libration period, but as it stands the uncertainty should be valid.

Samter's discussion† is interesting historically because it shares with Eichelberger's the result that a mass \( m' = \frac{1}{4700} \) is definitely too low, and also on account of incidental remarks.

Brouwer's method‡ is an up-to-date version of G. W. Hill's. A periodic solution is found by numerical integration over half a synodic period; thus the eccentricity of Titan and the libration are neglected. The motion of the apse and the libration period do not contain the first power of \( e' \), and the second power is neglected by Wolter. On the other hand, the libration, which is large, affects the motion of the apse to an important extent, as pointed out by Samter.

Wolter assigns weights \( \frac{1}{3} \), \( \frac{1}{3} \) to (c) and (e). For the others he says that incompleteness of the theoretical computations is the chief source of uncertainty. He assigns weights \( \frac{1}{2} \) to (a) and (g), \( \frac{1}{3} \) to (b), (d), (f). Adopting these weights gives standard error per unit weight 118 on 6 degrees of freedom, and

\[
\frac{1}{m'} = 4033 ± 74.
\]

(The standard error is my computation.) The obvious criticism of this analysis is that the errors of the various estimates are largely due to omissions in computation, and they are unlikely to be independent. It seems to me that Eichelberger's method is the only one not open to this objection, and that its uncertainty cannot be raised to 190 or so without other evidence. The data for (d), (e), (f), (g) are included in those for (a), (b), and the methods of (d), (f), (g) take less full account of higher terms than (a), (b) do. I think, therefore, that all the useful information contained in these solutions is already contained in (a), (b), (c), (e). If only we could be sure that the computations for (a), (b) were complete, they would also include (e).

* W. S. Eichelberger, Publ. U.S. Naval Obs. (2), 6, Appendix 1, B1-17, 1911.
‡ D. Brouwer, B.A.N., 2, 119-120, 1924.
Woltjer does not take $n$ and $n'$ as data. Instead, he takes $n'$ and the mean eccentricity of Hyperion. With these and an adopted value of $m'$ the mean motion of pericentre can be calculated, and $n$ is fixed by the consideration that the mean motion of $4l - 3l' - w$ must be zero. His expansions are in power of $\mu = m'^{1/2}$, and he starts with the numerical values $m'^{-1} = 4.033$, $e = 0.1043$, amplitude of libration $36^\circ$. $m'^2$ and $e^2$ are neglected. On these grounds, since $e^2$ is about $\frac{1}{1200}$, we might expect that this neglect would not produce errors of more than 6 or so in $1/m'$; but this seems to be far from the case.

Adapting the results for Enceladus, with necessary changes in notation, we should have
\begin{align*}
    v &= \beta n - (3n' - 4n), \\
    \nu &= \gamma_1 = -\beta n + (3n' - 4n), \\
    P &= 3m'n^2A, \\
    Q &= 3m'n^2B, \\
    e &= k_0 = \frac{m'nA}{\nu}, \\
    k_{10} &= 0, \\
    \gamma^2 - \nu^2 &= Pk_0.
\end{align*}

Woltjer's equation for the mean motion of the libration argument corresponding to (4) is
\begin{equation}
    \gamma = 1.57164 n' \sqrt{m'},
\end{equation}
with no term in $m'$. The secular terms $\beta n$ have not been introduced at the start; a correction for them is introduced only at the very end in dealing with the apse motion. In the calculation $\gamma$ has been supposed expanded in a series $\Sigma \nu, \mu'$, and the terms of order $\mu^2$ in the period equation are of the form
\begin{equation}
    \mu^2 \nu_1 \frac{d\theta}{d\tau} - \frac{\partial^2 R_0}{\partial \rho^2} \frac{\partial R_1}{\partial \theta},
\end{equation}
where $R_0 = M/2a$, $\theta$ is the argument $4l - 3l' - w$ and $R_1$ is the part of the disturbing function not containing $e'$. Hence $\partial R_1/\partial \theta$ contains $m'e$, and the right side of (5) needs a factor $(e/0.1043)^{1/2}$ and is analogous to the right of (4). The term $\nu^2$ has been omitted. Now $v$ is the difference between the actual mean motion of the apse and the part due to the secular terms in the disturbing function, including those due to Titan, and is therefore of order $m'$. Its neglect is therefore consistent with the neglect of $m'^2$, but it is not negligible. It can be assessed as follows. Tisserand (vol. 4, p. 108) gives for the effect of Titan
\begin{equation}
    \frac{d\omega}{d\tau} = m'n \left( 4.53 + \frac{3.26}{e} \cos V \right)
\end{equation}
with $V = \theta$, $\cos V$ near $-1$. The ratio of the terms is $-6.91$. The actual mean value is $-18^\circ.6562$ in a year, to which the Sun, the ellipticity of Saturn, and the attractions of the other satellites contribute $+0^\circ.278$. We have therefore to share the difference $-18^\circ.934$ in the ratio $1$ to $-6.91$, and the separate contributions are $3^\circ.20$ and $-22^\circ.14$. Thus $v = -22^\circ.14/1^\circ$. From observation
\begin{equation}
    \gamma = 0.562039 \pm 0.000023/1^d,
\end{equation}
whence
\begin{equation}
    1 - v^2/\gamma^2 = 0.98836.
\end{equation}

Then Woltjer's estimate $m' = \frac{1}{3986}$ needs to be revised to
\begin{equation}
    m' = \frac{1}{3986} \times 0.98836 (1 - \epsilon) = 2.4796 \times 10^{-4} (1 - \epsilon),
\end{equation}
On the masses of Saturn's satellites

where

$$e = 0.104.3(I + \epsilon).$$  \(11\)

The uncertainty of the observational values is negligible. That of the computations is not, but this correction has probably included the most important effect depending on \(m'^2\).

To adapt the equation for \(k_0\) it might appear that the definition of \(\nu\) should again take into account the secular term due to Titan, in which case \(m'\) should be multiplied by

$$\frac{22^5.14}{18^9.93}(I + \epsilon).$$

I think however that this would be erroneous. If other disturbances were absent Weltjer's calculation of the apse motion would be correct; hence the part of \(\beta n\) due to Titan has already been taken into account, and his equation should be right except possibly that the term, already small, due to the other bodies may need a slight change. I therefore write his equation from the apse motion as

$$m' = \frac{I}{4080}(I + \epsilon) = 2.451 \times 10^{-4}(I + \epsilon).$$  \(12\)

The eccentricity is given as \(0.104.19 \pm 0.000.25\) (pp. 59, 64), whence

$$\epsilon = -0.001.1 \pm 0.002.4.$$  \(13\)

Direct comparison of the estimates of \(m'\) gives \(\epsilon = +0.005.8\), which contradicts the observed value. If we substitute \((13)\) into \((10), (12)\) we get respectively

$$m' = 2.482.3 \times 10^{-4}, 2.448.3 \times 10^{-4}.$$  

The discordance is about halved, and the mean would be

$$m' = 2.465.3 \pm 0.017.0, \quad I/m' = 4.056 \pm 28,$$

on \(1\) degree of freedom.

The difference is still larger than we should normally expect from the terms neglected. It should be noticed, however, that as the most important part of the perturbations arises near conjunction, the effect of \(e'^2\) is essentially compared with \((a - d')/a\), and may easily be of order \(1\) per cent. It is likely to be more important in the libration speed than in the apse motion, since the libration requires a further differentiation of the disturbing function. On an average \(e'^2\) will tend to increase negative powers of the distance between the satellites and therefore, probably, to increase the perturbations for a given mass of Titan. On this ground I think that it is incorrect to take the mean of the two estimates, and that allowance for \(e'^2\) might easily change both to about \(\frac{1}{4120}\). The uncertainty of this extrapolated value can hardly be put under \(1\) per cent.

3. Secular motions.—Calculated expressions for these are given (logarithmically) by G. Struve (p. 53) in degrees per year. \(k/a_0^2\) and \(l/a_0^4\) are constants depending on the figure of Saturn and the ring. In ordinary form the coefficients of the constants in \(\beta n\) for the various satellites are given in Table III.

<table>
<thead>
<tr>
<th>Mimas</th>
<th>Enceladus</th>
<th>Tethys</th>
<th>Dione</th>
<th>Rhea</th>
<th>Titan</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k/a_0^2)</td>
<td>(1.48440 \times 10^4)</td>
<td>(6.20243 \times 10^3)</td>
<td>(2.93777 \times 10^4)</td>
<td>(1.23530 \times 10^5)</td>
<td>(3.837 \times 10^6)</td>
</tr>
<tr>
<td>(l/a_0^4)</td>
<td>(3.949 \times 10^3)</td>
<td>(1.002 \times 10^4)</td>
<td>(3.097 \times 10^4)</td>
<td>(7.94 \times 10^4)</td>
<td>(1.265 \times 10^5)</td>
</tr>
<tr>
<td>(m_{Ml})</td>
<td>(2.601 \times 10^4)</td>
<td>(5.384 \times 10^4)</td>
<td>(1.467 \times 10^5)</td>
<td>(1.487 \times 10^5)</td>
<td>(1.333 \times 10^5)</td>
</tr>
<tr>
<td>(m_{En})</td>
<td>(2.947 \times 10^4)</td>
<td>(2.592 \times 10^4)</td>
<td>(3.474 \times 10^4)</td>
<td>(3.474 \times 10^4)</td>
<td>(3.632 \times 10^4)</td>
</tr>
<tr>
<td>(m_{En})</td>
<td>(6.792 \times 10^4)</td>
<td>(2.885 \times 10^5)</td>
<td>(1.599 \times 10^5)</td>
<td>(1.333 \times 10^5)</td>
<td>(1.392 \times 10^5)</td>
</tr>
<tr>
<td>(m_{E})</td>
<td>(2.093 \times 10^4)</td>
<td>(4.717 \times 10^4)</td>
<td>(1.599 \times 10^5)</td>
<td>(1.333 \times 10^5)</td>
<td>(1.392 \times 10^5)</td>
</tr>
<tr>
<td>(m_{R})</td>
<td>(5.849 \times 10^5)</td>
<td>(1.019 \times 10^6)</td>
<td>(1.862 \times 10^6)</td>
<td>(5.185 \times 10^6)</td>
<td>(1.287 \times 10^7)</td>
</tr>
<tr>
<td>(m_{T})</td>
<td>(3.831 \times 10^5)</td>
<td>(5.732 \times 10^5)</td>
<td>(8.268 \times 10^5)</td>
<td>(5.185 \times 10^6)</td>
<td>(1.287 \times 10^7)</td>
</tr>
</tbody>
</table>
The observed values are given in Table II. For the constants I take the following values.

\[ \frac{k}{a_0^3} = 2.5 \times 10^{-7} (1 + 10^{-8} \mu_0) ; \quad \frac{l}{a_0^3} = 7.0 \times 10^{-4} \mu_1 ; \]

\[ m_{\text{Mi}} = 6.69 \times 10^{-8} (1 \pm 0.03) ; \]

\[ m_{\text{Te}} = 1.44 \times 10^{-6} (1 \pm 0.026) ; \]

\[ m_{\text{Rh}} = 2.5 \times 10^{-8} \mu_2 ; \]

\[ m_{\text{Jm}} = 2.5 \times 10^{-6} \mu_4 . \]

In addition we have the equation from the eccentricity of Rhea

\[ m_{\text{Ti}} = 2.5 \times 10^{-4} (0.971 \pm 0.068) ; \quad \mu_3 = -2.8 \pm 6.8 . \]

In the first place I made a least squares solution from the above data alone. The effects of the first four satellites were calculated from the start; the uncertainties of their masses contribute ±0.001 to the calculated \( \beta n \) for Mimas and Enceladus, ±0.017 for Tethys, and ±0.00 for Dione. Thus these uncertainties can be ignored except for Tethys, and for Tethys they can be treated by raising the standard error of \( \beta n \) from ±0.03 to ±0.06. The contribution of Iapetus for Titan is \( 8 \times 10^{-4} \mu_4 \). According to the preliminary estimate \( \mu_4 \) is of order 1, and it would have to reach 4 for its contribution to be as large as the standard error for Titan's apse. At present, therefore, we ignore \( \mu_4 \).

The first solution gave a rather large \( \chi^2 \) and made me suspect that some of the smaller uncertainties, especially the very small one for Tethys, might be underestimates. This was checked as follows. The maximum displacements due to equal changes of \( \omega \) and \( \theta \) would be nearly proportional to \( ae \) and \( a \sin \gamma \), where \( a \) is the radius of the orbit and \( \gamma \) the inclination of the orbit to Saturn's equator; hence \( ae \sigma_\omega \) and \( a \sin \gamma \sigma_\gamma \) should have comparable values in all cases for equally good observing over equal periods. The same should apply to \( ae \sigma_\theta \) and \( a \sigma_\gamma \), with extensions to the various librations. With \( a \) in seconds of arc, the results were mostly between \( 0.02 \) and \( 0.03 \), except for the two for Mimas, which were \( 0.05 \) and \( 0.07 \), and for the node of Titan, which gave \( 0.055 \). The two estimates from Mimas already differ by \( 2.6 \) times the standard error of their difference, strongly suggesting that these uncertainties need to be doubled; but none of the standard errors, when so reduced to a common basis of comparison, is outstandingly smaller than the rest. The high accuracy for Tethys is due mainly to the rather large inclination. The large \( \chi^2 \) was therefore treated by doubling all the smaller observational uncertainties. It was not thought necessary for Tethys to double the contribution of the uncertainties of the masses of other satellites. The equations of condition, so modified, are as follows. I take:

\[ \mu_0 = -28 + \mu_0' , \quad \mu_1 = 1 + \mu_1' . \]

Mimas apse

- \( 0.37 \mu_0' + 2.76 \mu_1' + 0.01 \mu_2 + 0.005 \mu_3 = + 1.87 \pm 0.20 \)
- \( 0.6 \)

node

- \( 0.37 \mu_0' + 2.76 \mu_1' + 0.01 \mu_2 + 0.005 \mu_3 = + 1.50 \pm 0.20 \)
- \( 1.2 \)

Tethys node

- \( 0.073 \mu_0' + 0.22 \mu_1' + 0.05 \mu_2 + 0.002 \mu_3 = + 0.087 \pm 0.065 \)
- \( 0.7 \)

Dione apse

- \( 0.03 \mu_0' + 0.06 \mu_1' + 0.13 \mu_2 + 0.003 \mu_3 = + 0.20 \pm 0.42 \)
- \( 0.1 \)

Rhea node

- \( 0.01 \mu_0' + 0.01 \mu_1' + 0.006 \mu_3 = + 0.12 \pm 0.08 \)
- \( 1.5 \)

Titan apse

- \( 0.0005 \mu_0' + 0.0001 \mu_1' + 0.0043 \mu_2 = + 0.0078 \pm 0.0078 \)
- \( 1.0 \)

node

- \( 0.0005 \mu_0' + 0.0001 \mu_1' + 0.0043 \mu_2 = - 0.0014 \pm 0.0029 \)
- \( 0.0 \)

Rhea eccentricity

- \( \mu_3 = -2.8 \pm 6.8 \)
- \( 0.3 \)

- \( 5.4 \)

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The normal equations, reduced to unit weight, are as follows.

\[8.72 \mu_0 + 55.26 \mu_1 + 1.32 \mu_2 + 0.070 \mu_3 = + 33.5,\]
\[55.26 \mu_0 + 392.4 \mu_1 + 4.05 \mu_2 + 0.25 \mu_3 = + 236.5,\]
\[1.32 \mu_0 + 4.05 \mu_1 + 1.087 \mu_2 + 0.206 \mu_3 = + 267,\]
\[0.070 \mu_0 + 0.25 \mu_1 + 0.206 \mu_2 + 0.029 \mu_3 = + 0.180.\]

The solution is

\[\mu_0' = +0.0 \pm 0.9; \quad \mu_1' = +0.60 \pm 0.28; \quad \mu_2 = +0.2 \pm 1.5; \quad \mu_3 = +0.9 \pm 0.6.\]

\(\chi^2\) on 4 degrees of freedom is satisfactory, and the data for these satellites are consistent among themselves. The uncertainty of 6 per cent in the mass of Titan, however, calls for improvement, which may be sought in the data for Hyperion and Iapetus.

3.1. Secular movements of Iapetus.—The relevant formulae are in H. Struve’s paper of 1888. G. Struve gave data up to 1927, but his paper ends abruptly after linear forms have been fitted; presumably he would have brought H. Struve’s work up to date had he lived longer. The equations can be written as follows. \(\theta\) is again written for longitudes on the ecliptic.

\[
\sin i \frac{d\theta}{dt} = -\Sigma K_r \{\sin i \cos i \cos (\theta - \theta_r) \}
\times \{\cos i \cos i + \sin i \sin i \cos (\theta - \theta_r)\},
\]

\[
\frac{di}{dt} = \Sigma K_r \sin i \sin (\theta - \theta_r) \{\cos i \cos i + \sin i \sin i \cos (\theta - \theta_r)\}.
\]

These were integrated numerically; the values taken for the Sun and for Saturn’s equator, referred to the ecliptic and equinox of 1890.0, were

\[\theta_0 = 112^\circ 42', \quad i_0 = 2^\circ 29.5',\]
\[\theta_1 = 167^\circ 58.79 \pm 0.46, \quad i_1 = 28^\circ 04.44 \pm 0.25.\]

The last is based on G. Struve’s determinations from the orbital planes of the five inner satellites (Heft 4, p. 48) for 1889-25. These means, weighted according to the stated uncertainties, gave \(\chi^2 = 3.2\) on 8 degrees of freedom; there was therefore no occasion for modifying the weights, as Struve did. \(\theta_1\) was adapted to 1890.0 by allowing for precession. These values could be taken as constant. For Titan the motion of the orbital plane is appreciable. G. Struve’s solutions (Heft 5, p. 11) were used. Precession had already been removed from his values of \(\theta_{Titan}\). The satellites out to Rhea could be combined with the ellipticity of Saturn, but Titan needed separate treatment. (H. Struve had taken its orbital plane as identical with Saturn’s equator, but as these planes can be inclined at about 38’ and the orbit of Iapetus is inclined at about 10° to both, this might have produced an error of order 5 per cent in the mass of Titan.) We have also

\[K_0 = 4.073/I^x, \quad K_1 = 0.694/I^x, \quad K_{Titan} = 2.779(1 + 0.01 \mu_3)/I^x.\]

\(K_1\) has a standard error of about 1 per cent, mainly arising from the mass of Rhea, but this can lead to one of only 0.3 per cent in the mass of Titan. The initial values adopted for 1890.0 were \(\theta = 142^\circ 05', \ i = 18^\circ 24',\) and numerical integration was carried out to cover the interval 1770 to 1930. The secular motion is slow and it was enough to regard corrections to the initial values as carried through the entire period. The result was compared with the observed values (adapted to the fixed equinox used). These were much less accurate up to 1880.2 than later.
I took the standard error of $\theta$ to be cosec $i$ times that of $i$ in each interval, with the following results.

<table>
<thead>
<tr>
<th></th>
<th>Crude residuals</th>
<th>Revised residuals</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_\theta$</td>
<td>$\sigma_i$</td>
</tr>
<tr>
<td>Up to $i880.2$</td>
<td>$11^1.1$</td>
<td>$3^1.6$</td>
</tr>
<tr>
<td>1885-1927</td>
<td>$2^1.0$</td>
<td>$0^2.62$</td>
</tr>
</tbody>
</table>

The unknowns were estimated with these uncertainties and the standard errors redetermined from the new residuals. No substantial further correction was needed, and the results were

$$\delta \theta(1890.0) = +0^1.56 \pm 0^1.89,$$

$$\delta i(1890.0) = +1^1.59 \pm 0^1.35.$$  

$$\mu_3 = -5^1.5 \pm 2^1.1.$$  

The separate estimates of $\mu_3$ from $\theta$ and $i$ were $-4^2.8 \pm 4^0.0$ and $-5^7.7 \pm 2^4.4$. Effectively the uncertainties are on 16 degrees of freedom in comparison with usual standards, since two standard errors have been estimated from 20 equations. The residuals are as follows.

<table>
<thead>
<tr>
<th>Observer</th>
<th>O–C</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1787.7</td>
<td>Herschel and Bernard</td>
<td>$-20^1.7$</td>
<td>$-2^1.7$</td>
</tr>
<tr>
<td>1832.5</td>
<td>Bessel</td>
<td>$-5^7.4$</td>
<td>$-4^7$</td>
</tr>
<tr>
<td>1857.5</td>
<td>Jacob</td>
<td>$+9^5.5$</td>
<td>$-1^0$</td>
</tr>
<tr>
<td>1876.7</td>
<td>A. Hall</td>
<td>$+1^3$</td>
<td>$+0^2$</td>
</tr>
<tr>
<td>1880.2</td>
<td>A. Hall</td>
<td>$+7^9$</td>
<td>$+3^0$</td>
</tr>
<tr>
<td>1885.6</td>
<td>H. Struve</td>
<td>$+0^6$</td>
<td>$+0^2$</td>
</tr>
<tr>
<td>1917.2</td>
<td>H. Struve and Bernewitz</td>
<td>$+1^1.7$</td>
<td>$-0^7$</td>
</tr>
<tr>
<td>1918.2</td>
<td>H. Struve and Bernewitz</td>
<td>$+1^0$</td>
<td>$+0^4$</td>
</tr>
<tr>
<td>1926.4</td>
<td>Alden</td>
<td>$-3^0$</td>
<td>$-0^6$</td>
</tr>
<tr>
<td>1927.4</td>
<td>Alden</td>
<td>$+0^1$</td>
<td>$-0^4$</td>
</tr>
</tbody>
</table>

The residuals for the second period are substantially greater than the apparent observational uncertainties would suggest. Since a similar result was found for the inner satellites it looks as if some additional source of disturbance affects secular motions.

It has been indicated above that uncertainty of the mass of Rhea contributes one of about $0.3$ per cent to the mass of Titan. The uncertainty of the orbital plane of Titan is about $0.8$, compared with a mutual inclination of $10^5$, and therefore contributes about $0.1$ per cent. The effects of the uncertainty of Saturn's equatorial plane and of changes in the plane of its orbit would be still smaller. Thus the above uncertainty is probably valid and we have from Iapetus

$$I/m_{NI} = 4243 \pm 93.$$  

This value is consistent with Eichelberger's. If, as suggested above, Woltjer's value from Hyperion should be revised to about $\frac{1}{4120}$, it also would be consistent, but a value near $\frac{1}{1093}$ would be hard to reconcile with either the present value or Eichelberger's.

G. Struve gives also data for the eccentricity and pericentre of Iapetus. The pericentre moved $28^5$ in the interval, with reference to a fixed equinox, but
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inspection of the residuals left by Struve's linear solution (and indeed the uncertainty that he himself gives) shows that the standard error of \( \frac{d\sigma}{dt} \) is not less than 1 part in 30. Since only about a third of the movement is due to Titan it follows that these data cannot give \( m_{Ti} \) with an uncertainty under 10 per cent, and detailed computation would be useless. The reason for the lower accuracy is that \( e \) for Iapetus is much less than the sines of the inclinations of the orbital planes.

4. Combination of data.—The solutions for \( m_{Ti} \) are equivalent to the following values of \( \mu_2 \).

<table>
<thead>
<tr>
<th>Inner satellites</th>
<th>( +0.9 \pm 0.1 )</th>
<th>0.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iapetus</td>
<td>(-5.5 \pm 2.1)</td>
<td>0.8</td>
</tr>
<tr>
<td>Hyperion</td>
<td>(-4.1 \pm 1.3)</td>
<td>0.2</td>
</tr>
<tr>
<td>Eichelberger</td>
<td>(-2.9 \pm 1.0)</td>
<td>0.4</td>
</tr>
</tbody>
</table>

\( \chi^2 = 2.0 \) on 3 d.f.

The weighted mean is \(-3.54 \pm 0.74\). The uncertainty given to the adapted value from Woltjer's results is, of course, wholly provisional; but it would be impossible to reject these results and no better course suggests itself. The best method of all, of course, would be to extend the theory of Hyperion to take account of \( e^2 \) and possibly higher terms in \( m_{Ti} \), but it is already clear that this would be an undertaking comparable with the lunar theory.

The above value of \( \mu_2 \) was substituted in the normal equations derived from the secular movements of the inner satellites. The solution was

\[ \mu'_0 = +0.2 \pm 1.9; \quad \mu'_1 = +0.58 \pm 0.28; \quad \mu_2 = +0.15 \pm 1.5. \]

Comparing with the data that contain these three parameters gives \( \chi^2 = 4.9 \) on 4 degrees of freedom. Then

\[
\frac{k}{a_0^2} = 0.024305 \pm 0.000048, \\
\frac{l}{a_0^4} = (1.04 \pm 1.96) \times 10^{-4}, \\
\frac{m_{Rh} = (0.4 \pm 3.8) \times 10^{-4},}{m_{Ti} = (2.41 \pm 0.0185) \times 10^{-4},} \\
\frac{1/m_{Ti} = 4.147 \pm 32.}{\]}

The theoretical values of the secular motions of the inner satellites were finally computed directly from the original formulae with these values. The details are as follows; the separate contributions are given to indicate their relative importance. The third decimal was kept in the calculation in case small positive values might add up.

<table>
<thead>
<tr>
<th>Mimas</th>
<th>Enceladus</th>
<th>Tethys</th>
<th>Dione</th>
<th>Rhea</th>
<th>Titan</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>360.78</td>
<td>150.75</td>
<td>71.40</td>
<td>30.02</td>
<td>9.33</td>
</tr>
<tr>
<td>( l )</td>
<td>43.36</td>
<td>11.11</td>
<td>0.34</td>
<td>0.09</td>
<td>0.01</td>
</tr>
<tr>
<td>Mi</td>
<td>...</td>
<td>0.02</td>
<td>0.00</td>
<td>0.00</td>
<td>...</td>
</tr>
<tr>
<td>En</td>
<td>0.04</td>
<td>...</td>
<td>0.04</td>
<td>0.00</td>
<td>...</td>
</tr>
<tr>
<td>Te</td>
<td>0.08</td>
<td>0.33</td>
<td>...</td>
<td>0.15</td>
<td>0.02</td>
</tr>
<tr>
<td>Di</td>
<td>0.04</td>
<td>0.09</td>
<td>0.28</td>
<td>...</td>
<td>0.08</td>
</tr>
<tr>
<td>Rh</td>
<td>0.00</td>
<td>0.00</td>
<td>0.01</td>
<td>0.02</td>
<td>...</td>
</tr>
<tr>
<td>Ti</td>
<td>0.09</td>
<td>0.14</td>
<td>0.20</td>
<td>0.31</td>
<td>0.63</td>
</tr>
<tr>
<td>Total</td>
<td>365.40</td>
<td>152.43</td>
<td>72.272</td>
<td>30.60</td>
<td>10.06</td>
</tr>
<tr>
<td>Obs.</td>
<td>( 365.62 \pm 0.20 )</td>
<td>( 365.23 \pm 0.20 )</td>
<td>( 72.277 \pm 0.065 )</td>
<td>( 30.75 \pm 0.42 )</td>
<td>( 10.20 \pm 0.08 )</td>
</tr>
<tr>
<td>O-C</td>
<td>+0.20, -0.17</td>
<td>( -0.045 )</td>
<td>( +0.15 )</td>
<td>( +0.14 )</td>
<td>( +0.069 )</td>
</tr>
</tbody>
</table>
There are some end-figure changes, but recomputation was not thought necessary. G. Struve's values of $k$ and $l$ (p. 54) correspond to
\[ \frac{k}{a_0^2} = 0.024398; \quad \frac{l}{a_0^4} = 7.013 \times 10^{-4}. \]
These do not appear to have been derived from a least squares solution; the difference in $k$ is significant and that in $l$ is great. He gets a residual of $-0.26/\pi^2$ for Tethys, in comparison with an apparent "probable error" of $0.021/\pi^2$, but passes it with no comment except that increasing the mass of Rhea would make it worse. In the present solution this is avoided by having $k$ slightly smaller and $l$ larger, thus fitting both Mimas and Tethys.

In comparison with the discussion of 2.2, the calculated $\beta n$ for Enceladus gives $\delta(\beta n) = -0.34 \pm 0.05$. Correcting the masses of Enceladus and Dione accordingly, we have
\[ m_{En} = (1.27 \pm 0.53) \times 10^{-7}; \]
\[ m_{Di} = (1.825 \pm 0.061) \times 10^{-8}. \]

With regard to the still unsatisfactory determination of the mass of Rhea, this may be interpreted as saying that there is a probability of about \( \frac{2}{3} \) that it is less than \( \frac{1}{27000} \) of that of Saturn, one of about \( \frac{1}{3} \) that it exceeds this value and \( \frac{1}{20} \) that it exceeds \( \frac{1}{15000} \). As Rhea is brighter than Tethys and Dione a value near \( \frac{1}{27000} \) would be plausible.

160 Huntingdon Road,
Cambridge:
1953 January 9.