THE FIGURES OF ROTATING PLANETS

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Summary

The theory of the figure of the Earth on the hydrostatic hypothesis, developed by Darwin and de Sitter, is applied to models for Jupiter and Saturn proposed by W. H. Ramsey and B. Miles. It is found that the integral equation for the coefficient of the fourth harmonic is more easily solved directly than by conversion into a differential equation. The results show that the ratio $D/J$, which is equal to $35l/12k^2$ in the treatment of H. and G. Struve, is substantially larger than for a homogeneous body in a hydrostatic state, and in satisfactory agreement with observation for Saturn. Comparison with a model given by Darwin indicates that an inequality stated by de Sitter for the coefficient of the fourth harmonic is not general.

1. Darwin, in a classical paper, developed the hydrostatic theory of a rotating planet to the second order in the ellipticity.* His definitions of the quantities concerned led to considerable algebraic complexity; part of this was removed by de Sitter†, but de Sitter published only a few of the formulae and made several errors, some of which have been corrected previously, especially by Bullard.‡ The theory has been applied hitherto only to the Earth, but can now be usefully applied also to Jupiter and Saturn, since the fourth harmonic in their gravitational potentials has an observable effect on the motions of the nearest satellites. Further, the work of Ramsey and Miles on these planets makes it possible to give a theoretical determination of these terms and compare with observation. Application of the first-order theory to rotating stars is perhaps not altogether out of the question.

The first-order terms in the equation satisfied by the ellipticity lead to Clairaut’s differential equation, the numerical treatment of which for the Earth is greatly simplified by Radau’s approximation. The second-order terms were included by Darwin and de Sitter, who found that they could be treated by a simple modification of the equation. For Jupiter and Saturn, however, Radau’s approximation is seriously wrong and a differential equation has still to be solved numerically; and the equation satisfied by the fourth harmonic has so far been solved only by numerical integration, except in a few special cases.

The equations arise in the first place as integral equations, which are converted into differential equations and a pair of boundary conditions. It will appear in what follows that this is unnecessary, and that for the fourth harmonic at least it is easier to apply numerical methods directly to the integral equation.

In the theory of Darwin and de Sitter it was found convenient to define a modified ellipticity, the relation of which to the original ellipticity appears to

‡ In addition, in (1), $128/105$ should be $32/105$; in (11), $32/3$ should be $8/3$. In the set of equations at the foot of the page, $S_1 = T/J$ should read $S_1 = 2J(1 + 2c)$. (11) is stated to agree with Darwin, but Darwin’s form is correct.

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contain the coefficient of the fourth harmonic. This surprising result is due to
the definition of the ellipticity in terms of the ratio of the equatorial and polar
axes. I have proposed for geophysical purposes that the standard ellipsoid
should be so defined that, with the outer surface a level surface, it should lead to
the correct value of the second harmonic in the external field. Then allowance
for an additional fourth harmonic will not affect this relation. The reasons for
this course are stronger when the data are purely astronomical.

The following treatment is much shorter than Darwin’s; on the other hand,
de Sitter’s is condensed to an extent that makes the intermediate steps very
difficult to reconstruct, and I think a rather fuller account than his is needed.

The axes are taken to be \(a, a, a(1-e)\); the sphere of equal volume has
radius \(b\), satisfying

\[
b^3 = a^2 (1-e).
\]

The parameter \(m\) (de Sitter’s \(\rho_1\)) is defined by

\[
m = \omega^2 b^3 fM.
\]

I use the spherical harmonics in the form

\[
L_2 = \frac{1}{3} \sin^2 \phi' = -\frac{3}{2} P_2,
\]

\[
L_4 = \sin^4 \phi' - \frac{9}{2} \sin^2 \phi' + \frac{27}{35} = \frac{3}{5} P_4.
\]

We have the following identities:

\[
\sin^2 \phi' = \frac{1}{3} - L_2; \quad \sin^4 \phi' = L_4 - \frac{6}{5} L_2 + \frac{1}{5};
\]

\[
\sin^2 \phi' \cos^2 \phi' = -L_4 - \frac{1}{5} L_2 + \frac{2}{15}; \quad L_2^2 = L_4 - \frac{4}{21} L_2 + \frac{4}{45}.
\]

The equation of the ellipsoid, to order \(e^2\), is

\[
r/b = 1 - \frac{4}{45} e^2 + (e + \frac{28}{45} e^2) L_2 + \frac{3}{2} e^2 L_4.
\]

A term \(-\kappa \sin^2 2\phi\) would give one in \(4\kappa L_4\); taking also, with de Sitter,

\[
e' = e - \frac{5}{42} e^2,
\]

we have

\[
r/b = 1 - \frac{4}{45} e^2 + (e' + \frac{28}{45} e^2) L_2 + \frac{3}{2} e^2 L_4,
\]

differing from (6) only by a fourth harmonic. The quantity \(e'\) is identical with
de Sitter’s, but, with this definition of \(e, \kappa\) does not appear in the definition of \(e'\).

The internal and external potentials and their derivatives are continuous
at the surface. These relations have to be converted into a pair valid at \(r=b\).
Suppose that \(f, \partial f/\partial r\) vanish at \(r=b+h\). Then to the order retained

\[
f(r) = \frac{1}{2} (r-b-h)^2 f'' (b+h) + \frac{1}{6} (r-b-h)^4 f''' (b+h),
\]

\[
f'(r) = (r-b-h) f'' (b+h) + \frac{1}{2} (r-b-h)^2 f''' (b+h)
\]

\[
= (r-b-h) f'' (b) + h (r-b-h) f''' (b) + \frac{1}{2} (r-b-h)^2 f''' (b+h),
\]

to the second order in \(h\), with \(r=b\),

\[
f(b) = \frac{1}{2} h^2 f'' (b); \quad f'(b) = -h f''' (b) - \frac{1}{2} h^2 f''' (b).
\]

The external and internal potentials of a homogeneous body are taken as

\[
\overline{U}_0 = fM \left\{ \frac{1}{r} + (J + J_0) \frac{b^2}{r^3} L_2 + D_0 \frac{b^4}{r^5} L_4 \right\}
\]

and

\[

\overline{U}_1 = \frac{fM}{2b^3} \left( 3b^2 - r^2 \right) + fM \left\{ (J + J_1) \frac{b^2}{r^3} L_2 + D_1 \frac{b^4}{r^5} L_4 \right\}.
\]

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Their difference and its first derivative vanish when \( r = b \) by (11), and adapting these conditions to \( J = b \) by (11), and treating \( J \) as of the first order in \( e \), we find

\[
J = \frac{2}{3} e^2; \quad J_0 = \frac{2}{3} e^2; \quad J_1 = \frac{1}{3} e^2; \quad J_2 = \frac{2}{3} e^2 + \frac{1}{3} \kappa; \quad J_3 = \frac{1}{3} \kappa.
\]

(14)

Darwin gets equivalent formulae by using the exact potential of an ellipsoid of revolution. The adaptation to a heterogeneous body in a hydrostatic state with external mean radius \( b \) is as usual; if \( r_1 \) is the mean radius of the surface of equal density through the point considered,

\[
U = \frac{1}{2} \int_{r_1}^{b} \rho d\left(3a'^2 + \frac{1}{r} \right) \rho_0 + \frac{3}{5} \left( \frac{S'}{r^3} + \frac{r^2}{r} \right) L_2 + \left( \frac{3}{5} \frac{Pr_1}{r^3} + \frac{4}{3} \frac{Q r_1^2}{r^3} \right) L_4,
\]

(15)

where

\[
\rho_0 = \frac{1}{r_1} \int_{a' = 0}^{r_1} \rho d\alpha'^3; \quad S = \frac{1}{r_1} \int_{a' = 0}^{r_1} \rho d\left\{ \alpha'^5 \left( e' + \frac{9}{5} \alpha'^2 \right) \right\};
\]

\[
T = \frac{1}{r_1} \int_{a' = r_1}^{r} \rho d\left\{ e' + \frac{9}{5} \alpha'^2 \right\}; \quad P = \frac{1}{r_1} \int_{a' = 0}^{r_1} \rho d\left\{ \alpha'^7 \left( e' + \frac{9}{5} \alpha'^2 \right) \right\};
\]

\[
Q = \frac{1}{r_1} \int_{a' = 0}^{r} \rho d\left( \frac{\kappa}{a'^3} \right);
\]

(16)

where in the integrals \( \rho, e' \) and \( \kappa \) are regarded as functions of the variable of integration \( a' \). The difference between \( e' \) and \( e \) does not affect the terms of order \( \alpha'^2 \), in which the accent is therefore omitted. To the same accuracy the rotation term is

\[
\frac{1}{2} \omega^2 r^2 \cos^2 \phi' = \frac{2}{5} \pi f \bar{\rho} m r^2 (\frac{3}{5} + L_2).
\]

(17)

The sum of (16) and (17) must be constant over a surface of constant \( r' \). Then using (8), with \( b \) replaced by \( r_1 \), and expressing \( L_2 \) in terms of \( L_2 \) and \( L_4 \) by (5), we find

\[
-(e' + \frac{9}{5} \alpha'^2)\rho_0 + (\frac{3}{5} + \frac{1}{5} \alpha'^2) S + (\frac{2}{5} - \frac{1}{5} \alpha'^2) T + \frac{1}{5} \bar{\rho} m + \frac{4}{7} \bar{\rho} m e = 0,
\]

(18)

\[
-(\frac{1}{5} \alpha'^2 + 4\alpha) \rho_0 - \frac{9}{5} \alpha' S + \frac{9}{5} \alpha' T + \frac{4}{5} P + \frac{4}{5} Q + \bar{\rho} m e = 0.
\]

(19)

Multiplying (18) by \( (1 - \frac{1}{5} \alpha' e) \) gives

\[
(e' + \frac{3}{5} \alpha'^2) \rho_0 - \frac{9}{5} (S + T) - \frac{1}{5} \bar{\rho} m = \frac{4}{21} (e(\bar{\rho} m - 3 T)),
\]

(20)

and multiplying (18) by \( 2e' \) and subtracting from (19) gives

\[
(3 \alpha'^2 - 8e) \rho_0 - 6 \alpha S + 3 P + \frac{4}{5} Q = 0.
\]

(21)

(20) and (21) are de Sitter's equations, except for the correction of an algebraical error. The latter can be checked by comparison with the first-order theory of the fourth harmonic.

2. It is of some interest to show how some of the main results derived from the differential equations can be found more easily directly from the integral equations. I take the first-order equation, for a harmonic of any order, in the form

\[
\rho \equiv \frac{1}{2n+1} \left\{ \frac{1}{r^{n+3}} \int_{a' = 0}^{r} \rho d(a'^n + \alpha e) + \frac{1}{r^{n-2}} \int_{a' = r}^{b} \rho d \left( \frac{e}{a'^{n-2}} \right) \right\} + (0, \frac{1}{5} \bar{\rho} m).
\]

(1)

This is essentially an integral equation (written by means of Stieltjes integrals) with fixed termini. For the general case (zero on the right, \( n \neq 1 \)) we can without loss of generality take \( e_n \) positive at its extreme value; let this be \( e \), taken at \( r = r' \) (\( e_n \) is continuous by the physical conditions). In (1) replace \( r \) by \( r' \) and integrate
by parts, which is permissible because \( \epsilon_n \) is continuous and \( \rho \) non-increasing. Then the right side is

\[
\frac{3}{2n+1} \left\{ \int_{r'}^{b} \frac{\rho(b') \epsilon_n(b')}{b'^{n-2}} \, dp - \frac{1}{r'^{n+3}} \int_{a'}^{b'} a'^{n+3} \epsilon_n \, dp - \int_{a'}^{b} a'^{n-2} \frac{\epsilon_n}{a'-r'} \, dp \right\},
\]

which is not decreased if \( \epsilon_n \) is replaced by \( \epsilon \) everywhere (since \( \rho \) is non-increasing). Make this substitution and integrate by parts again; then (2) is

\[
\leq \frac{3 \epsilon}{2n+1} \left\{ \frac{I}{r'^{n+3}} \int_{a'}^{b'} \rho \, da'^{n+3} + r'^{n-2} \int_{a'}^{b} \rho \, d\left( \frac{I}{a'-r'} \right) \right\}.
\]

The second integral is zero for \( n=2 \), or \( r'=b \), otherwise negative for \( n>2 \). \( \rho_0(r') \) is not a function of \( a' \); hence the right side is

\[
\leq \frac{3 \epsilon}{2n+1} \, r'^{n+3} \int_{a'}^{b} \rho \, da'^{n+3} + \int \{ \rho - \rho_0(r') \} \, da'^{n+3}.
\]

Unless \( \rho \) is constant there is an \( r'' \) such that \( \rho(a') \leq \rho_0(r') \) for \( a'>r'' \), \( \geq \rho_0(r') \) for \( a'<r'' \); then the integral is

\[
\leq \frac{n+3}{3} \int \{ \rho - \rho_0(r') \} (a'^{n} - r''^n) \, da'^{3} < 0,
\]

and finally the right side of (1) is

\[
\leq \frac{3 \epsilon}{2n+1} \rho_0.
\]

But this is less than the left side unless \( n=1 \) or \( \epsilon = 0 \). Hence all harmonics other than those with \( n=1 \) and the main ellipticity term have zero coefficients.

An argument on these lines is well known to give a proof that \( \epsilon_n(b)=0 \), after it has been shown from the differential equation that \( \epsilon_n \), if not zero, must be monotonic. The above generalization of the argument does not need the preliminary use of the differential equation.

If \( n=1 \), take \( \epsilon_1 = \frac{1}{r} \). Then the right side of (1) is

\[
\frac{I}{r} \int_{a'}^{b} \rho \, da'^{3} + \frac{I}{r} \int_{a'}^{b} \rho \, d(\frac{I}{a'-r}) = \frac{\rho_0}{r},
\]

agreeing with the left side. Hence \( \frac{1}{r} \) is a solution.

The above argument cannot be used directly to prove that it is the only solution, since \( \epsilon_1 \) may be unbounded. But put \( r_1 = \eta \). Then \( \eta \) is continuous even at \( r=0 \). Let its maximum be \( \zeta \) at \( r' \); then the equation (1) is

\[
\rho_0 \eta = \rho(b) \eta(b) - \frac{I}{r_3} \int_{a'}^{b} a'^{2} \eta \, dp - \int_{a'}^{b} \eta \, dp
\]

\[
\leq \left( \rho(b) - \frac{I}{r_3} \int_{a'}^{b} a'^{2} \, dp - \int_{a'}^{b} \, dp \right) \zeta
\]

\[
= \left( \frac{I}{r_3} \int_{a'}^{b} \rho \, da'^{3} \right) \zeta = \rho_0 \zeta.
\]

Hence \( \eta(r') = \zeta \) is false unless the equality sign can be taken in the second line, that is, unless \( \eta \) is a constant. Hence \( \epsilon_1 \propto \frac{1}{r} \) is the only solution (with the trivial exception that in intervals where \( \rho \) is constant \( \eta \) might vary without there being any dynamical consequences).
For the main ellipticity term and for the second-order term containing a fourth harmonic, the terms of lowest order take the form

$$\rho_0 e_n = \frac{3}{2n+1} \left\{ \frac{\rho(b)e_n(b)}{a^{n-2}} \right\} - \frac{1}{r^{n+3}} \int_{\alpha'=0}^{r} a' \epsilon_n d\rho' - \frac{\epsilon_n}{\alpha'=r} \int_{a'=0}^{b} a' d\rho' \right\} + f(r).$$

(8)

Take as a trial solution

$$\epsilon_n = \eta_n = \frac{f(r)}{\rho_0}$$

(9)

and substitute on the right; then the modulus of the first set of terms, by the previous argument, is

$$\leq \frac{3}{2n+1} \rho_0 |\eta_n|_{\text{max}};$$

(10)

hence the correction is everywhere numerically \(\leq \frac{3}{2n+1} (\eta_n)_{\text{max}}\) and the results of successive approximations converge like \(\sum_{m} \left(\frac{3}{2n+1}\right)^m\). Since the trial solution is continuous the corrections are continuous, and the limit, being the sum of a uniformly convergent series of continuous functions, is continuous; hence all the required conditions are satisfied.

I have not succeeded in finding a direct proof from the integral equation that \(\epsilon_2\) is monotonic for the main ellipticity term. Clearly \(\epsilon_2\) is everywhere \(>0\), since the first approximation and all corrections are positive.

The series may converge too slowly for direct computation. In fact if the upper bound of the first approximation is \(\epsilon\), that of the true solution may be as large as \(\frac{2n+1}{2(n-1)} \epsilon\), which is \(\frac{5}{6}\epsilon\) for \(n=2\) and \(\frac{7}{8}\epsilon\) for \(n=3\). The former is actually attained for a homogeneous body (for which the first approximation is \(\frac{1}{2}m\)). This difficulty may be circumvented as follows. Write the equation as

$$\epsilon_n = \frac{f(r)}{\rho_0} + K\epsilon_n,$$

(11)

where \(K\) is a linear operator. The first approximation is \(\epsilon_n = u_1 = f(r)/\rho_0\). Substitution of \(u_1\) on the right gives a second approximation \(u_2\). The essence of the matter is that \(\left\{u_m - \frac{f(r)}{\rho_0} - Ku_m\right\}\) forms a sequence of functions, decreasing approximately geometrically. With two values, say \(m = 1, 2\), we can choose a constant \(\mu\) so that the function

$$\left(1 + \mu\right) \left(u_2 - \frac{f(r)}{\rho_0} - Ku_2\right) - \mu \left(u_1 - \frac{f(r)}{\rho_0} - Ku_1\right)$$

is everywhere as small as possible, and this will be done by taking \(\mu\) so that the greatest positive and negative values of the expression under the modulus sign are equal in modulus. But this amounts to saying that, judged by the residuals left in the original equation,

$$(1 + \mu)u_2 - \mu u_1 \approx \frac{f(r)}{\rho_0} + K\{(1 + \mu)u_2 - \mu u_1\}$$

(12)

with the smallest possible error. Then \((1 + \mu)u_2 - \mu u_1\) can be used as the first of a second pair of approximations, and solution should be rapid. The method
is essentially one of relaxation. In contrast to the use of such methods for differential equations, the "point adjustment" needed to smooth out rapid fluctuations of the errors does not arise, being automatically taken into account in the integration process; all the approximations are "block adjustments".

On the other hand, it may not be so convenient for most integral equations as with the present ones, because these have the feature that the kernel can be written in the form

\[ g_1(\alpha')h_1(r) + g_2(\alpha')h_2(r), \]

thus containing four functions that can be tabulated once for all; a more general type of kernel would have to be fully tabulated as a function of two variables.

3. Possible variation of \( D/J^2 \).—The form of a homogeneous rotating body would be an exact ellipsoid of revolution. It was shown by H. Struve that his parameters \( k \) and \( l \) would lead to

\[ l/k^2 = \frac{10}{7}. \]

for all ellipticities, corresponding, in the present notation, to \( D/J^2 = \frac{25}{6} \). The value found in my discussion of Saturn’s satellites was \( l/k^2 = 1.87 \pm 0.33 \), and it is desirable to examine whether the doubtfully significant difference would be explicable by variation of density. At first sight this seemed unlikely. For if we consider an extreme case where the whole mass is concentrated in a homogeneous core, the outer part has no dynamical effect at all, the core would take a Maclaurin form, and the same relation would hold. Thus rather wide departures from homogeneity might not affect \( D/J^2 \). Examination of Bullard’s solution for the Earth, however, did show an appreciable departure, and direct examination of the data for the constitutions of Jupiter and Saturn was desirable.

3.1. Solution for Jupiter.—Possible distributions of density in Jupiter and Saturn have been given by Ramsey and Miles.* They worked out the second-order theory of the ellipticity, but not the fourth harmonic. The details were not published, but Ramsey and Miles kindly gave me a typed copy. Of their models, those denoted by \( J_3 \) and \( S_8 \) agreed best with the observational data; but those denoted by \( J_4, S_4 \) were only slightly inferior and have the advantage for the present purpose that the distributions are smoother. The object of the present work is to find out whether the ratio \( D/J^2 \), for given \( m \), can depart widely from the value \( \frac{25}{6} \) that it takes for a homogeneous body of any ellipticity. Accordingly I accepted their values of \( \eta \) derived by de Sitter’s method for \( J_4, S_4 \), derived the distribution of \( \epsilon \), and then used the integral equation 2 (21) to estimate \( \kappa \). I use their values for the mass, radius, and rate of rotation of each planet. The results for Jupiter are in Table I, where \( m = 0.083 \ 39 \).

\( \kappa \) at most is only about \( 0.1 \ \epsilon^2 \); this is mainly because the \( \epsilon^2 \) terms in the integral equation are (in much of the interval) nearly as \( 1, -2, 1 \). The corrections in the second approximation never exceeded \( 0.1 \), so that the method is even more successful than there was apparent reason to expect. This is chiefly because \( \kappa \) is small except where \( \rho \) is small compared with \( \tilde{\rho} \).

For the external field

\[ \tilde{\rho}b^b J = \frac{3}{5} \int_{r=0}^{b} \rho \, d\{r^5(\epsilon' + \frac{2}{3} \epsilon^2)\}, \]

\[ \tilde{\rho}b^b D = \frac{3}{2} \int_{r=0}^{b} \rho \, d\{r^7(\epsilon^2 + \frac{8}{3} \kappa)\}. \]

For comparison with a homogeneous body it seemed best to drop the \( e^2 \) term in \( J \).
This is because \( e^2 \) and \( \kappa e \) are neglected in \( D \), and we know that for a homogeneous body \( D/J^2 \) is independent of \( e \). It therefore appeared possible that inclusion of a higher term in \( J \) and not in \( D \) would actually decrease the accuracy. On this basis \( J = 0.0260, D = 0.00339 \), of which only \( \frac{1}{20} \) comes from \( \kappa \). Then
\[
D/J^2 = 5.0.
\]
Allowance for the \( e^2 \) term would reduce this to 4.8 or 4.9.

### Table I

<table>
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<th>( r(10^8 \text{ cm}) )</th>
<th>( \eta )</th>
<th>( e )</th>
<th>( \rho )</th>
<th>( \rho_0 )</th>
<th>( 10^4 \kappa ) (first approximation)</th>
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</table>

### Table II

<table>
<thead>
<tr>
<th>( r )</th>
<th>( \eta )</th>
<th>( e )</th>
<th>( \rho )</th>
<th>( \rho_0 )</th>
<th>( 10^4 \kappa )</th>
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<td>0.756</td>
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<td>0.0923</td>
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<td>0.0888</td>
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<td>0.0884</td>
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<td>0.0785</td>
<td>1.91</td>
<td>1.91</td>
<td>0.0</td>
</tr>
</tbody>
</table>

3.2. Solution for Saturn.—This is given, for Model \( S_4, m = 0.1353 \), in Table II. Further corrections were negligible. The results lead to
\[
J = 0.0438; \quad D = 0.0116; \quad D/J^2 = 6.0.
\]
The conjecture that \( D/J^2 \) for a body in hydrostatic state is nearly independent of the distribution of density is therefore seriously wrong for Jupiter and Saturn.
4. A model given by Darwin.—To investigate the possible range of values of $D/J^2$ I consider a case treated by Darwin, of a body of unit density with a fraction $\mu$ of its mass concentrated at the centre, the density otherwise being uniform. I take the radius also as $r$. Darwin determined the value of $\mu$ that makes $\kappa$ greatest, $m$ being taken as given, but did not explicitly attend to the external field. (My $\mu$ is Darwin's $\mu/(r + \mu)$.) The results above become

\[
\begin{align*}
\rho_0 &= \frac{1}{r^3} (\mu + (1 - \mu)r^2), \\
S &= (1 - \mu)r^2(e' + \frac{3}{2}e^2); \quad T = (1 - \mu)(e'_s + \frac{15}{61}e^2_s - e - \frac{10}{61}e^2), \\
P &= (1 - \mu)(e^2 - \frac{3}{8}e'), \quad Q = (1 - \mu)(\kappa_r^2 - \kappa).
\end{align*}
\]

To the first order

\[
e'_s = \frac{5m}{4(1 + \frac{3}{2}r^2)}, \quad e' = e'_s + \frac{r^2}{\mu + \frac{3}{8}(1 - \mu)r^2 - \frac{3}{8}(1 - \mu)r^2},
\]

and to the second order

\[
\kappa_s = \frac{9\mu e^2_s}{8(2 + \mu)}, \quad J = \frac{3}{5}S, \quad D = \frac{3}{2}P,
\]

\[
D = \frac{25}{6} \left(2 + 2\mu\right) \quad \text{and} \quad J^2 = \frac{25}{6} \left(2 + 2\mu\right).
\]

Hence this ratio actually tends to infinity as $\mu$ tends to $1$, and is never less than \(\frac{25}{6}\), the value for a homogeneous body. The reason is that as $\mu \to 1$, $J$ is of order $(1 - \mu)m$ and $D$ of order $(1 - \mu)m^2$.

This result is of some interest in relation to the models $J_3$ and $S_3$ of Miles and Ramsey, which they thought the most probable. In these the heavy elements are supposed concentrated near the centre. For $J_3$ the nucleus has about $\frac{1}{20}$ of the mass of the whole body, and we might expect $D/J^2$ to be increased by about 7 per cent above the value found for $J_4$. For Saturn the corresponding quantities are about $\frac{1}{14}$ and 10 per cent. It is therefore quite likely that the correct values are about $5$ for Jupiter and $6.6$ for Saturn.

5. Bullard's solution for the Earth.*—This applies de Sitter's method. Since the differential equation for $\kappa$ is derived from Darwin's solution, it is correct. The boundary condition (10) for $\kappa$ should read

\[
\frac{d\kappa}{d\beta} = -4\kappa - \frac{5}{4} \rho_1 \epsilon_1' + \frac{90}{16} \rho_1^2;
\]

the calculations have been carried out with the correct form. The easiest way of finding $D/J^2$ is to compare with the International Formula, which gives $J = 0.00164$, $D = 0.0000107$, $D/J^2 = 3.98$. Bullard finds on the hydostatic theory that $\kappa = 6.8 \times 10^{-7}$. This represents the distortion of the level surface, and since the International Formula is consistent with the outer surface being an exact ellipsoid, irrespective of whether hydostatic conditions subsist in the interior, $4\kappa$ is added directly to $D$ and makes $D/J^2 = 4.98$.

6. An inequality given by de Sitter.—De Sitter states without proof that for surface values

\[
\kappa \leq \frac{5}{12} m - \frac{1}{3} e^2.
\]

---

This relation is satisfied by Bullard's solution for the Earth and by the above solutions for J₄ and S₄. For Darwin's model, on the other hand, the right side is \( \frac{3}{2} \mu \kappa^2 \), which is less than \( \kappa \) except for \( \mu = 0 \) and \( \mu = 1 \). It appears, therefore, that the above inequality is not general.

7. **Comparison with observation for Saturn.**—The values for \( k \) and \( l \) for Saturn* made \( k/l^2 = 1.87 \pm 0.33 \); to reduce to \( J \) and \( D \) we must multiply by \( \frac{35}{12} \), giving \( D/J^2 = 5.45 \pm 0.96 \). Thus the theoretical results for Saturn are consistent with observation. There is no basis for an attempt to estimate the mass of the Ring, such as might have been made had the observational value exceeded the theoretical one.

On the other hand, the results of the present paper suggest that it is reasonable to take for Saturn \( D/J^2 = 6.3 \pm 0.3 \), which is equivalent to \( l/k^2 = 2.16 \pm 0.10 \). This leads to an additional equation of condition, in the notation of the previous paper,

\[
0.027 \mu_0' - 7.0 \mu_1' = -5.7 \pm 0.58.
\]

Combining this with the previous data I get

\[
k/a_0^2 = 0.024270 \pm 0.0000016; \quad l/a_0^4 = (12.42 \pm 0.52) \times 10^{-4};
\]

\[
m_{\text{Rh}} = (2.8 \pm 2.8) \times 10^{-6}; \quad m_{\text{Tt}} = (2.412 \pm 0.018) \times 10^{-4}.
\]

The comparison with observation is as follows.

<table>
<thead>
<tr>
<th>A satellite</th>
<th>Observation</th>
<th>Calc.</th>
<th>( \chi^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mimas apse node</td>
<td>366.60 ± 0.20</td>
<td>365.44</td>
<td>0.6</td>
</tr>
<tr>
<td>Enceladus</td>
<td>365.23 ± 0.20</td>
<td>365.44</td>
<td>1.1</td>
</tr>
<tr>
<td>Tethys node</td>
<td>72.227 ± 0.065</td>
<td>72.257</td>
<td>...</td>
</tr>
<tr>
<td>Dione apse</td>
<td>30.75 ± 0.42</td>
<td>30.69</td>
<td>...</td>
</tr>
<tr>
<td>Rhea node</td>
<td>10.20 ± 0.08</td>
<td>10.05</td>
<td>...</td>
</tr>
<tr>
<td>Titan apse node</td>
<td>0.5012 ± 0.0070</td>
<td>0.4976</td>
<td>0.3</td>
</tr>
<tr>
<td>( \mu_4 ) from Rhea</td>
<td>0.492 ± 0.029</td>
<td>0.4976</td>
<td>...</td>
</tr>
<tr>
<td>( 1.85 \mu_8 ) from</td>
<td>-2.8 ± 6.8</td>
<td>-3.5</td>
<td>0.0</td>
</tr>
<tr>
<td>Hyperion and Iapetus</td>
<td>-6.6 ± 1.4</td>
<td>-6.5</td>
<td>0.0</td>
</tr>
<tr>
<td>( l/k^2 ) (theory)</td>
<td>2.16 ± 0.10</td>
<td>2.11</td>
<td>0.2</td>
</tr>
</tbody>
</table>

5.9 (6 d.f.)

The general comparison is quite satisfactory. In so far as the theoretical value taken for \( l/k^2 \) is the mean of values for the models \( S_3 \) and \( S_4 \), and the other data indicate a probable value a shade less than this, it may be held that the result slightly favours \( S_4 \), but the evidence is weak.

* St. John's College,
  Cambridge :
  1953 January 9.