THE THEORY OF NUTATION AND THE VARIATION OF LATITUDE

Harold Jeffreys and R. O. Vicente

(Received 1956 October 23)

Summary

The theory of the bodily tide and of the various nutations is developed. Elasticity of the shell and fluidity of the core are taken into account. The model used for the shell is Takeuchi's Model 2, based on one of Bullen's. The core is replaced by a homogeneous incompressible fluid, with an additional particle at the centre chosen to make the mass and moment of inertia of the core correct. Effects of the ocean are neglected, but can be allowed for at a later stage.

The period of the free nutation is found to be 392 days (which would be increased by the ocean). The bodily tide numbers for semidiurnal and long-period tides are: $h=0.58$, $k=0.29$, $l=0.082$. They have other values for diurnal tides, the greatest observable differences from the statical values being for the lunar tide $O$. The correcting factor for the 19-yearly nutation is $0.9964$.

1. The treatment of periodic changes of the Earth's rotation given in most books regards the Earth as a rigid body. It has been known since Newcomb that elasticity lengthens the free period; by itself its effects on the precession and nutations are negligible. On the other hand actual fluidity of the core shortens the free period, and in the simplified case of a rigid shell and a homogeneous incompressible core it produces a reduction of the amplitude of the 19-yearly nutation that is too great to agree with the facts.

Allowance for elasticity of the shell has been found to reduce the effect of fluidity of the core on the nutation. The first treatment (H. Jeffreys, 1949) used a Wiechert-Herglotz model, with the shell and core homogeneous and incompressible. With this no more than a rough agreement was to be expected. Recent work on the velocities of elastic waves and on the density distribution has made a detailed discussion by numerical integration possible, and such a study has been carried out by H. Takeuchi (1950). His solutions are based on two of K. E. Bullen's models (1936, 1940) but the models used actually differ in some minor respects from Bullen's. Three simultaneous differential equations of the second order are solved by a Taylor series method, and the calculated values for the bodily tide numbers $h$, $k$, $l$ are in good agreement with observation.

Takeuchi used a statical theory for both shell and core. It is known that this is valid for the shell for disturbances with the periods in question, and for the core for the semidiurnal, fortnightly, and semiannual tides. For disturbances tending to alter the axis of rotation, and therefore for all diurnal tides and the variation of latitude, a hydrodynamical theory of the core is necessary. His work, however, greatly assists such a treatment because it gives the general statical theory of the shell in terms of six adjustable constants; thus the only change needed for the shell is in the expression of the boundary conditions.

In a first attempt to use his solution the core was still treated as homogeneous and incompressible, but this was found to be unsatisfactory (H. Jeffreys, 1951b).
In the present paper, accordingly, this assumption is modified. The actual core is too complicated for detailed treatment, and simplified models are used.

We use a Lagrangian specification as before. The axis of $z$ is in a fixed direction, while those of $x$ and $y$ rotate about it with a constant angular velocity $\omega$. If $x_i$ are the coordinates of a particle in its mean position, $\xi_i$ in its actual position, where we shall exchange $x_1, x_2, x_3$, with $x, y, z$, and $\xi_1, \xi_2, \xi_3$, with $\xi, \eta, \zeta$ as convenient, the kinetic energy will be

$$\frac{1}{2} \int \int \int \rho_0 ((\dot{\xi} - \omega \eta)^2 + (\dot{\eta} + \omega \xi)^2 + \dot{z}^2) \, d\tau,$$

where $\rho_0$ is the undisturbed density and the integral is through the undisturbed position. There is actually a second-order variation of the angular velocity about $z$, but its reaction on the displacements in $x$ and $y$ will be of the third order.

The possibility of adapting the statical solution for the shell depends on the fact that if the displacements arose from a pure rotation $\xi \dot{\eta} - \dot{\xi} \eta$ would be of the form $\zeta (\dot{m} - \eta l)$ and the terms in $\omega$ and $\omega^2$ could be represented by an addition to the potential function. This requires care, however, since $\xi$ and $\eta$ are not small and second-order terms in $\xi, \eta$ need attention. We suppose the particle at $x_i$ displaced to $x_i + u_i'$, and then to be carried to $\xi_i$ by a rotation, so that

$$\begin{align*}
\xi_1 &= x_1(1 - \frac{1}{2}l^2) + u_1' + l(x_3 + u_3' - \frac{1}{2}lmx_2), \\
\xi_2 &= x_2(1 - \frac{1}{2}m^2) + u_2' + m(x_3 + u_3' - \frac{1}{2}lnx_1), \\
\xi_3 &= x_3(1 - \frac{1}{2}l^2 - \frac{1}{2}m^2) + u_3' - l(x_1 + u_1') - m(x_2 + u_2').
\end{align*}$$

The $\omega^2$ terms are equivalent to an addition $\frac{1}{2} \omega^2 (\xi'^2 + \eta'^2)$ to the potential function, and can therefore be included in the work function $W$. To develop this we first consider a displacement $u_i$. The gravitational potential consists of the undisturbed part $U_0$, a small external disturbance $U_1$, and a part $U_2$ due to displacements of the body. Then in a small variation of $u$, the work done by gravity is

$$\delta W_g = \int \int \rho_0 \frac{\partial}{\partial x_i} (U_0 + U_1 + U_2) \delta u_i \, d\tau,$$

where the derivative must be evaluated at $x_i + u_i$. To the second order

$$\delta W_g = \int \int \rho_0 \left( \frac{\partial U_0}{\partial x_i} + u_l \frac{\partial U_0}{\partial x_k} \frac{\partial}{\partial x_i} + \frac{\partial U_1}{\partial x_i} + \frac{\partial U_2}{\partial x_i} \right) \delta u_i \, d\tau,$$

all derivatives being now taken at $x_i$.

Since $u_i \frac{\partial U_0}{\partial x_i}$ is quadratic in the displacements this gives

$$W_g = \int \int \rho_0 \left( u_i \frac{\partial U_0}{\partial x_i} + \frac{1}{2} u_k u_k \frac{\partial U_0}{\partial x_k} \frac{\partial}{\partial x_i} + u_i \frac{\partial U_1}{\partial x_i} + \frac{1}{2} u_i \frac{\partial U_2}{\partial x_i} \right) \, d\tau.$$

The stress is

$$-p_0 \delta_{ik} + \lambda \Delta \delta_{ik} + 2\mu \epsilon_{ik},$$

where $p_0$ is the initial pressure at $x_i$,

$$\Delta = \frac{\partial u_i}{\partial x_m}, \quad 2\epsilon_{ik} = \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}.$$

The relative increase of volume of an element is, to the second order,

$$\| \frac{\partial (x_i + u_i)}{\partial x_k} \| = \frac{\partial u_i}{\partial x_i} + \frac{\partial u_2}{\partial x_2} \frac{\partial u_3}{\partial x_3} - \frac{\partial u_2}{\partial x_3} \frac{\partial u_3}{\partial x_2} + \ldots$$

$$= \frac{\partial u_i}{\partial x_i} + \frac{1}{2} \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i} - \frac{1}{2} \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i}.$$
Thus $p_0$ does work in the expansion equal to

$$W_p = \iiint \rho_0 \left( \frac{\partial u_i}{\partial x_i} + \frac{1}{2} \frac{\partial u_i}{\partial x_i} \frac{\partial u_k}{\partial x_k} - \frac{1}{2} \frac{\partial u_k}{\partial x_k} \frac{\partial u_i}{\partial x_i} \right) d\tau$$

$$= \iiint \rho_0 \left( \frac{1}{2} u_i + \frac{1}{2} u_i \frac{\partial u_k}{\partial x_k} - \frac{1}{2} u_k \frac{\partial u_i}{\partial x_i} \right) dS$$

$$- \iiint \left\{ u_i \frac{\partial \rho_0}{\partial x_i} + \frac{1}{2} u_i \frac{\partial (\rho_0 \frac{\partial u_i}{\partial x_i})}{\partial x_i} \right\} d\tau. \quad (9)$$

The surface integral is zero. The part over the outer surface vanishes because $p_0=0$. Over the inner boundary $p_0$ is constant and is continuous. To the first order the total volume of matter carried through $dS$ is $l u_i dS$. The second order terms might differ according as they are evaluated just inside or just outside the core. Consider the displacement imposed gradually; then $u_i$ is the displacement of the particle that starts at $x_i$, while the particle that finishes at $x_i$ starts, to the first order, at $x'_i$ such that

$$x_i = x'_i + u_i(x'_i) = x'_i + u_i(x' - x) \frac{\partial u_i}{\partial x_k} \quad (10)$$

and its displacement is $u_i - u_k \frac{\partial u_i}{\partial x_k}$. Thus the mean displacement of the particles that cross $dS$ is $u_i - \frac{1}{2} u_k \frac{\partial u_i}{\partial x_k}$. With an interchange of suffixes this gives the first and third terms of the surface integral. The second term gives the increase of volume due to expansion of the matter transferred. Hence the integral is zero, since, apart from $p_0$, it represents the difference of the calculated volumes within the core boundary according as the calculation is based on core or shell values of the displacements, and we are supposing that the core and shell continue to fit everywhere.

Since the initial stress is hydrostatic,

$$\frac{\partial p_0}{\partial x_i} = \rho_0 \frac{\partial U_0}{\partial x_i}, \quad (11)$$

and the first order terms in $W_p$ and $W_p$ cancel. The second order terms give, in all,

$$W = \iiint \rho_0 \left( \frac{1}{2} u_i u_k \frac{\partial^2 U_0}{\partial x_i \partial x_k} + u_i \frac{\partial U_1}{\partial x_i} + \frac{1}{2} u_i \frac{\partial U_2}{\partial x_i} - \frac{1}{2} u_i \frac{\partial U_0}{\partial x_i} \frac{\partial u_k}{\partial x_k} + \frac{1}{2} u_i \frac{\partial U_0}{\partial x_i} \frac{\partial u_k}{\partial x_k} \right) d\tau$$

$$- \frac{1}{2} \iiint (\lambda \Delta \delta_{ik} + 2 \mu \epsilon_{ik}) \frac{\partial u_i}{\partial x_k} d\tau. \quad (12)$$

If we make small variations $\delta u_i$, apply Green's lemma, and make some interchanges of dummy suffixes, we get the equations of equilibrium

$$\frac{\partial}{\partial x_k} (\lambda \Delta \delta_{ik} + 2 \mu \epsilon_{ik}) + \rho_0 \frac{\partial}{\partial x_i} (U_1 + U_2) + \rho_0 \left( u_k \frac{\partial^2 U_0}{\partial x_i \partial x_k} - \frac{\partial U_0}{\partial x_i} \frac{\partial u_k}{\partial x_k} + \frac{\partial U_0}{\partial x_i} \frac{\partial u_k}{\partial x_k} \right) = 0. \quad (13)$$

The terms containing $U_0$ are expressible as

$$- \frac{\partial}{\partial x_k} (\rho_0 u_k) \frac{\partial U_0}{\partial x_i} + \frac{\partial}{\partial x_i} \left( \rho_0 u_k \frac{\partial U_0}{\partial x_k} \right),$$

and when we use the relation

$$\frac{\partial (\rho_0, U_0)}{\partial (x_i, x_k)} = 0 \quad (14)$$
these reduce to the terms depending on $X_0$, $Y_0$, $Z_0$ in the equations (esp. (20))
previously found by the Eulerian method (Jeffreys, 1929, p. 163).

We take first $u_i = u'_i$. In calculating $W$ for $u'_i$ we must replace $U_0$ by

$$\Psi = U_0 + \frac{1}{2} \omega^2 (x_1^2 + x_2^2).$$  \hspace{1cm} (16)

We then impose a rotation as in (2). In this the internal reactions depending on
the mutual attractions and internal stresses do no work and do not alter $W$. The
initial pressure, as modified by the inclusion of the rotation terms in $\Psi$, still does
no work, but the extra term added to $U_0$ does some. The contribution is the
difference between $\iint \rho \frac{1}{2} \omega^2 \xi_1^2 + \xi_2^2 \, d\tau$ in the positions specified by $x_i + u'_i$ and
$\xi_i$ and hence is

$$\sum_{i=1,2} \frac{1}{2} \omega^2 \iint \rho (\xi'_1 - x_i - u'_i)(\xi'_1 + x_i + u'_i) \, d\tau.$$  \hspace{1cm} (17)

Evaluating this to the second order and remembering that the axes are principal
axes of the body in its standard position, we find that it is

$$-\frac{1}{2} \omega^2 (C - A)(l^2 + m^2) + \iint \rho_0 \omega^2 u'_i \frac{\partial}{\partial x_i} x_3 (l x_1 + m x_2) \, d\tau.$$  \hspace{1cm} (18)

The whole displacement has been separated into the displacement $u'_i$ followed
by a rotation, and so far it has been arbitrary how much of it is interpreted as
belonging to $u'_i$. We can remove the ambiguity by introducing the condition
that $u'_i$ in the shell makes no contribution to the angular momenta; this is
equivalent, to the first order, to

$$\iint_{\text{shell}} \rho_0 \varepsilon_{km} u'_k x_m \, d\tau = 0.$$  \hspace{1cm} (19)

Dropping exact derivatives with regard to the time we have for the terms in
$\xi'_1 - \xi_1$

$$\omega \iint \rho_0 \{ (u'_1 + lx_3)(u'_2 + mx_3) - (u'_2 + mx_3)(u'_1 + lx_3) \} \, d\tau$$
$$= \omega \iint \rho_0 (l m - l m) x_3^2 \, d\tau + \omega \iint \rho_0 \{ (u'_1 u'_2 - u'_2 u'_1) + x_3 (l u'_2 - m u'_1)$$
$$+ m u'_1 - l u'_2) \} \, d\tau.$$  \hspace{1cm} (20)

We can replace $l u'_2 - m u'_1$ by $-l u'_2 + m u'_1$, and then by (19) we can further
replace the resulting integral in the shell by

$$\iint \rho_0 (-l x_3 u'_3 + m x_3 u'_3) \, d\tau.$$  \hspace{1cm} (21)

Then (20) is replaced by

$$\omega (A - \frac{1}{2} C) (l m - l m) + \omega \iint \rho_0 (u'_1 u'_2 - u'_2 u'_1) \, d\tau$$
$$+ \omega \iint_{\text{shell}} \rho_0 \frac{\partial}{\partial x_i} \{ x_3 (m x_1 - l x_2) \} \, d\tau + 2 \omega \iint_{\text{core}} \rho_0 x_3 (m u'_1 - l u'_2) \, d\tau.$$  \hspace{1cm} (22)

The transformation for the shell makes the second integral equal to that due to
an addition to the gravitation potential. If we extend this integral to the whole
Earth, the core integral is replaced by

$$\omega \iint_{\text{core}} \rho_0 (x_3 (m u'_1 - l u'_2) - u'_3 (m x_1 - l x_2)) \, d\tau.$$  \hspace{1cm} (23)
The integrals through the whole Earth containing $\omega u_1$ and $\omega^2 u_1$ can be taken into account by replacing $U_1$ by

$$U_1' = U_1 + \omega^2 x_3(lx_1 + mx_2) + \omega x_3(mx_1 - lx_2).$$  \hspace{1cm} (24)

We can write to the first order

$$u_i = u_i' + u_{oi},$$  \hspace{1cm} (25)

$$u_{oi} = (lx_3, mx_3, -lx_1 - mx_2).$$  \hspace{1cm} (26)

Then

$$\frac{1}{2} \iiint \rho_0 (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2) d\tau = \frac{1}{2} \iiint \rho_0 (l^2 + m^2)(x_1^2 + x_2^2) d\tau + \iiint \rho_0 (\dot{u}_i' \dot{u}_{oi} + \frac{1}{2} \dot{u}_i''^2) d\tau. \hspace{1cm} (27)$$

The first term is $\frac{1}{2} A(l^2 + m^2)$.

If

$$U_1 = x_0(c_1 x_1 + c_2 x_2),$$  \hspace{1cm} (28)

$$\iiint \rho_0 U_{oi} \frac{\partial U_1}{\partial x_i} d\tau = - (C - A)(l c_1 + m c_2). \hspace{1cm} (29)$$

Then in all (integrals being through the whole Earth unless the contrary is stated)

$$L = \frac{1}{2} A(l^2 + m^2) + \omega(A - \frac{1}{3} C)(\dot{m} \dot{m} - \dot{lm}) - \frac{1}{3} \omega^2(C - A)(l^2 + m^2) - (C - A)(l c_1 + m c_2)$$

$$+ \iiint \rho_0 (\dot{u}_i' \dot{u}_{oi} + \frac{1}{2} \dot{u}_i''^2) d\tau + \iiint \rho_0 \omega(u_i' \dot{u}_i' - u_i'' \dot{u}_i') d\tau$$

$$+ \iiint \rho_0 \dot{u}_i' \frac{\partial U_1}{\partial x_i} d\tau$$

$$+ \iiint \rho_0 \omega(x_0(\dot{m}u_i' - \dot{u}_i') - u_i''(mx_1 - lx_2)) d\tau - \frac{1}{3} \iiint \lambda \Delta \delta_{ik} + 2 \mu e_{ik} \frac{\partial u_i'}{\partial x_k} d\tau$$

$$+ \iiint \rho_0 \dot{u}_i' \left( \frac{1}{2} u_k'' \frac{\partial \Psi}{\partial x_i \partial x_k} + \frac{1}{2} \frac{\partial U_1}{\partial x_i} \frac{\partial u_i'}{\partial x_k} + \frac{1}{2} \frac{\partial \Psi}{\partial x_i} \frac{\partial u_k'}{\partial x_i} \right) d\tau. \hspace{1cm} (30)$$

The choice of axes and angular velocities in problems of deformable bodies has been discussed often (see especially Tisserand, 1891). The usual choice is to make the axes rotate so that the actual angular momentum at any moment agrees with that of a rigid body with the same inertia tensor as the actual body and rigidly attached to the axes. If the axes are initially principal axes, they do not in general remain so throughout the motion.

We have defined $l, m$ so that they give the actual angular momentum of the shell, not of the whole body. It is this choice that reduces the terms in $u_i' \dot{u}_{oi}$ to the work done by a potential as in (22), and this is part of the standard treatment of problems of elastic rotating bodies. Its convenience is that the $u_i'$ motion can be expressed in terms of the normal coordinates of the shell vibrating by itself. The longest period of free vibration is about 40 minutes, while we are concerned with periods of about a day or longer. Hence $\dot{u}_i''$ can be neglected in the shell in comparison with the elastic terms in $u_i'$. The same applies to $\omega(u_i' \dot{u}_i'' - u_i'' \dot{u}_i')$. The term in $\dot{u}_i' \dot{u}_{oi}$ vanishes in consequence of (19).

This is not true for the core. It is obvious that if the core boundary was exactly spherical, rotations of the shell would not affect the core at all, so that $u_i' = -u_{oi}$; and this is in fact approximately true for slow speeds. But the
slowest gravitational oscillation of a fluid sphere with the density of the core
would have a period of about 1 hour (Lamb, 1932, p. 457). A longitudinal wave
passes through the core in about 10 minutes. Hence in the core we can neglect
the terms in \( \dot{u}_i^2 \) that arise from compression and from motions normal to the
level surfaces. We cannot however neglect those for motions along the level
surfaces, since for them the free periods are very long.

2. Lemmas on small oscillations.—The following principles are often used in the
theory.

2.1. Quasi-statical coordinates.—If the Lagrangian function is given by

\[
2L = a_{rs} \dot{q}_r \dot{q}_s + e_{rs} q_r q_s - b_{rs} q_r q_s + 2c_r q_r \quad (r, s = 1, 2 \ldots n)
\]  

where \( \dot{q}_r \) does not appear in \( L \) for \( r = k+1 \) to \( n \), we write

\[
L' = L - \frac{1}{2} \sum_{s=k+1}^{n} q_s \frac{\partial L}{\partial q_s}.
\]

Lagrange's equations give

\[
\frac{\partial L}{\partial q_s} = 0 \quad s = k + 1, \ldots, n,
\]

which determine the \( q_s \) (\( s = k+1 \) to \( n \)) as functions of the \( q_r \) and \( \dot{q}_r \) (\( r = 1 \) to \( k \)). Consider small variations of the \( q_r \) (\( r = 1 \) to \( k \)), which we shall now denote by \( q_i \),

\[
\delta \frac{\partial L}{\partial q_j} = 0, \quad \delta \frac{\partial L}{\partial q_i} = 0, \quad \delta L' = \delta L.
\]

Hence if we use (3) to eliminate \( q_j \) from \( L' \), \( L' \) can be used as the Lagrangian for
the \( q_i \). This result is exact. In the formation of \( L' \) terms quadratic in the \( q_j \)
disappear and linear terms in \( q_j \) are halved. The result will ordinarily be
used as an approximation when some of the free periods are very short compared
with that of the motion being considered. It would, for instance, provide an
easy way of estimating the correction of pendulum observations for elasticity of
the rod (Jeffreys, 1956).

2.2. Exact derivatives with regard to the time.—We often make use of the
following principle. Dr T. J. I' A. Bromwich mentioned it in lectures, and it is
probably well known, but we have not found it in print after a good deal of search.
If \( L \) contains a batch of terms expressible as

\[
\frac{d}{dt} f(q_1, \ldots, q_n, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_r} \dot{q}_r,
\]

the contribution to Lagrange's equation for \( q_s \) is

\[
\frac{d}{dt} \left( \frac{\partial f}{\partial q_s} \right) - \frac{\partial^2 f}{\partial q_s \partial t} - \frac{\partial^2 f}{\partial q_r \partial q_s} \dot{q}_r,
\]

all terms of which are immediately seen to cancel. Thus such a set of terms
may be omitted from the Lagrangian.

3. Now consider small variations of \( u_i' \) in the shell. We write

\[
U_2 = U_s + U_c
\]
corresponding to contributions due to \( u'_i \) in the shell and core respectively. Then an alteration of \( u'_i \) in the shell alone does not alter \( U_a \), but by reciprocity

\[
\iint \rho_0 u_i' \frac{\partial U^2}{\partial x_i} \, d\tau = \iint \rho_0 u_i' \frac{\partial U}{\partial x_i} \, d\tau + \iint \rho_0 u_i' \frac{\partial U}{\partial x_i} \, d\tau + 2 \iint \rho_0 u_i' \frac{\partial U}{\partial x_i} \, d\tau.
\]

(2)

The first integral is quadratic in shell displacements. We also put

\[
U'_i = U_1 + \omega x_3 ((\dot{m} + \omega l)x_1 - (\dot{\omega} - \omega m)x_2) = x_3 (c_1' x_1 + c_2' x_2).
\]

(3)

Then, all integrals being through the shell,

\[
\delta L = \iint \rho_0 \delta u_i' \left( \frac{\partial U'_1}{\partial x_i} + \frac{\partial U}{\partial x_i} + \frac{\partial U}{\partial x_i} \right) \, d\tau - \iint (\lambda \Delta \delta_{ik} + 2 \mu e_{ik}) \frac{\partial}{\partial x_k} \delta u_i' \, d\tau
\]

\[
+ \iint \rho_0 \left\{ u'_k \frac{\partial \Psi}{\partial x_k} \delta u_i' - \frac{1}{3} \frac{\partial \Psi}{\partial x_k} \delta u_i' - \frac{1}{3} \frac{\partial \Psi}{\partial x_k} u'_i \frac{\partial}{\partial x_k} \delta u_i' \right\} \, d\tau.
\]

(4)

This gives, by application of Green’s lemma and some interchanges of dummy suffixes, the equations of equilibrium (1(13)), with \( U_0 \) replaced by \( \Psi \) and \( U_1 \) by \( U'_1 \).

Now suppose that \( u'_i \) and its variations are restricted to satisfy the equations of equilibrium. Multiply these by \( \frac{1}{2} u_i' \), subtract from \( L \), and apply Green’s lemma. Then the contribution of \( u'_i \) in the shell becomes

\[
\frac{1}{2} \iint \rho_0 u_i' \frac{\partial}{\partial x_i} (U'_1 + U_c) \, d\tau - \frac{1}{2} \iint l_i u_i' (\lambda \Delta \delta_{ik} + 2 \mu e_{ik}) \, dS.
\]

(5)

The halving of the terms in \( U'_1 \) and \( U_c \) is an instance of the general theorem of 2.1. The surface integral depends only on surface values of \( u_i' \) and its derivatives. It represents work done by the stresses over the boundaries and can be simplified by means of the condition that there is no tangential stress.

The core integrals in lines 3, 5 and the second in line 4 of 1(30) can also be simplified. The part arising from \( U_0 \) has already been included in the shell integrals. Green’s lemma gives

\[
\iint l_i \rho_0 u_i' \left\{ U'_1 + \frac{1}{3} u'_k \frac{\partial \Psi}{\partial x_k} + \frac{1}{3} U_c \right\} \, dS
\]

\[- \iint \left\{ U'_1 + \frac{1}{3} u'_k \frac{\partial \Psi}{\partial x_k} + \frac{1}{3} U_c \right\} \left( \rho_0 \frac{\partial u_i'}{\partial x_i} + u_i' \frac{\partial \rho_0}{\partial x_i} \right) \, d\tau - \frac{1}{2} \iint \rho_0 u_i' \frac{\partial \Psi}{\partial x_i} \frac{\partial u'_k}{\partial x_k} \, d\tau
\]

(6)

the surface integral being over the core boundary. We write \( u_n' \) for the normal displacement; and if gravity at the core boundary is \( g_1 \),

\[
\frac{\partial \Psi}{\partial x_k} = -l_{kg_1},
\]

(7)

whence the surface integral is

\[
\iint \rho_0 u_n' \left( U'_1 - \frac{1}{3} g_1 u_n' + \frac{1}{3} U_c \right) \, dS.
\]

(8)

The volume integral would vanish for a uniform incompressible core and can be treated as a correction.

The normal to a boundary is as usual always taken outward from the region considered, so that at the core boundary \( l_i \) has opposite signs according as we are
considering a shell or a core integral.  With these simplifications $2L$ is reduced to

$$2L = A(\hat{l}^2 + \hat{m}^2) + \omega(2A - C)(\hat{m} \cdot \hat{m}) - \omega^2(C - A)(\hat{l}^2 + \hat{m}^2) - 2(C - A)(\hat{l}c_1 + mc_2)$$

$$+ \int \int \int \rho_0 u_i^{'2} + 2u_i u_o \, d\tau + 2 \int \int \int \rho_0 \omega (u_i^{'u} - u_o^{'u}) \, d\tau$$

$$+ 2 \int \int \rho_0 \omega (u_3 (mu_i^{' - lu_o^{'}) - u_3 (mx_1 - lx_2)) \, d\tau$$

$$+ \int \int \int \rho_0 u_i^{' \frac{\partial}{\partial x_i}(U_1 + U_c) \, d\tau - \int \int l_k u_i^{'(\lambda \Delta \delta_{ik} + 2\mu\epsilon_{ik})} \, dS$$

$$+ \int \int \rho_0 u_i^{'(2U_1 - g_1 u_n^{' + U_c)} \, dS$$

$$- \int \int \int \left( 2U_1^{' + u_k \frac{\partial \Psi}{\partial x_k} + U_c \right) \left( \rho_0 \frac{\partial u_i^{' - u_i^{' \rho_0}}}{\partial x_i} \right) \, d\tau$$

$$- \int \int \int \rho_0 u_i^{' \frac{\partial \Psi}{\partial x_i} \frac{\partial u_k}{\partial x_k} \, d\tau - \int \int \int \lambda \Delta^2 d\tau. \quad (9)$$

4. Form of solution for the shell.—We take briefly

$$U_1 = cK_2 = cr^2 S_2, \quad (1)$$

where $c$ is a constant and $K_2$ a solid harmonic of degree 2.  At present we ignore
the difference between $U_1$ and $U_1'$, which can be adjusted later when required.
We also take

$$U_1 + U_2 = KcK_2, \quad (2)$$

$K$ is a function of $r$ and is a number.  We introduce the variable

$$\xi = r/a, \quad (3)$$

where $a$ is the outer radius, and write

$$x_i u_i^' = q = q(\xi)K_2. \quad (4)$$

$q(\xi)$ is a number.  $K'(\xi)$ is discontinuous at the boundaries, and we have

$$U_2 = (K(\xi) - 1)cK_2 \quad \xi < 1$$

$$= (K(1) - 1)(a/r)^5 cK_2 \quad \xi > 1, \quad (6)$$

and at the outer surface

$$\left[ \frac{\partial U_2}{\partial r} \right]_{+}^{1-} = -4\pi \rho_0(1) \frac{q(1)}{a} K_2. \quad (7)$$

This leads to, at $\xi = 1 -$,

$$c \left[ \frac{\partial K}{\partial \xi} + 5\{K(1) - 1\} \right] = 4\pi \rho_0(1)q(1). \quad (8)$$

We can also take

$$U_c = P(\frac{aq}{\alpha})^5 cK_2 \quad \xi > a, \quad (9)$$

$$U_s = QcK_2 \quad \xi < a. \quad (10)$$

Then we have

$$\left( \frac{\partial U_c}{\partial r} \right)_{+} = \frac{-3}{a\alpha} P cK_2; \quad \left( \frac{\partial U_c}{\partial r} \right)_{-} = \left( \frac{\partial U_c}{\partial r} \right)_{+} + 4\pi \rho_0(\alpha -) \frac{q(\alpha)}{a\alpha} K_2, \quad (11)$$

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
\[
(\frac{\partial U_g}{\partial r})_{\alpha-} = \frac{2}{a} \alpha QcK_2, \quad (\frac{\partial U_g}{\partial r})_{\alpha+} = (\frac{\partial U_g}{\partial r})_{\alpha-} + 4\pi f\rho_0(\alpha+) \frac{q(\alpha)}{a\alpha} K_2 \tag{12}
\]

and by comparing the values of \(U_g\) and \(\partial U_g/\partial r\) at \(\xi = \alpha+\) we have a pair of equations for \(P\) and \(Q\), the solution of which is

\[
5Pc = -\alpha cK'(\alpha+) + 4\pi f\rho_0(\alpha+)q(\alpha) \tag{13}
\]

\[
5Qc = \alpha cK'(\alpha+) + 5\{K(\alpha)-1\}c - 4\pi f\rho_0(\alpha+)q(\alpha). \tag{14}
\]

Also

\[
\alpha c \left[\frac{dK}{d\xi}\right]_{\alpha-} = 4\pi f\rho_0(\alpha+)\{\rho_0(\alpha+) - \rho_0(\alpha-)\}. \tag{15}
\]

For the displacements we take

\[
u^i_t = F(\xi) \frac{\partial K_2}{\partial x_i} + \frac{G(\xi)}{a^2} \xi_i K_2. \tag{16}
\]

This is of the form of the statical solution and has the property of orthogonality with rotations used in Section 2. \(F\) and \(G\) are numbers. Let accents denote derivatives with regard to \(\xi\). Then

\[
q(\xi) = 2F(\xi) + \xi^2G(\xi). \tag{17}
\]

Again, with \(l_k = x_k/r\), we have

\[
l_k(\lambda \Delta \delta_{ik} + 2\mu \epsilon_{ik}) = \frac{\mu}{a^2} \frac{\partial K_2}{\partial x_i}(2F + F'\xi + G\xi^2)
\]

\[
+ \frac{x_i K_2}{a^2} \left[\lambda(2F' + \xi^2G' + 5\xi G) + \mu(2F' + 2\xi^2G' + 4\xi G)\right]. \tag{18}
\]

Since there is no tangential stress at the boundaries

\[
\xi F' + 2F + \xi^2G = 0, \quad \xi = \xi, \alpha \tag{19}
\]

and the elasticity contribution to \(2L\) is

\[
- a^2 M[\lambda(2F' + \xi^2G' + 5\xi G) + \mu(2F' + 2\xi^2G' + 4\xi G)]q(\xi), \tag{20}
\]

where

\[
M = \iint S^2_{2d\Omega} = \frac{4}{15} \pi. \tag{21}
\]

when \(K_2\) is the solid harmonic \(x\) or \(y\).

For \(\xi = 1\), (18) is correct. For \(\xi = \alpha\), it must be taken with the opposite sign.

For integration over a sphere we have the identity

\[
\iint \left(\frac{\partial K_n}{\partial x_i}\right)^2 dS = \frac{n(2n+1)}{r^2} \iint K_n^2 dS. \tag{22}
\]

Now

\[
U_1 + U_\alpha = c(1 + P(\alpha/\xi)^5)K_2, \tag{23}
\]

whence the volume integral in \(2L\) is

\[
Ma^2 \epsilon \int_\alpha^1 \rho_0\left(10F + 2\xi^2G - 3GP\frac{\xi^8}{\xi^2}\right)\xi^4 d\xi. \tag{24}
\]

(18) is already expressed in terms of boundary values of \(F\), \(G\) and their first
derivatives. (24) still contains internal values of $F$ and $G$, and the corresponding expression requires a numerical integration. Takeuchi solves the differential equations for $F$, $G$ and $K$ (the last being Poisson's equation) in terms of their values and their first derivatives at $\xi = \alpha \pm$. His functions, however, are multiples of those used here; in particular he takes $c = 1$. In view of the pair of boundary conditions (19) we can eliminate two of the constants; and on account of the importance of $q(1)$ and $q(\alpha)$ it is convenient to eliminate two others in favour of them. Further, $c$ is a datum, and we have the relation (8) connecting $K'(1)$ and $K(1)$. One constant remains arbitrary, and on account of the prominence of $P$ it seems best to take $K'(\alpha \pm)$, which is wholly determined by the displacements of the core. When this is done the internal values of $u_i'$ in the shell will have been eliminated.

If we indicate Takeuchi's functions by the suffix $T$, the correspondence between his and ours is

$$\begin{align*}
F(\xi) &= ca^2 F_T(\xi); \\
G(\xi) &= ca^2 G_T(\xi); \\
q(\xi) &= ca^2(2F_T + G_T); \\
K'(\xi) &= 4\pi a^2 K_T(\xi).
\end{align*}$$

(25)

Since his equations are all homogeneous the factor $a^2$ can be omitted. We write $c/4\pi f = D$ and take the correspondence to be

$$\begin{array}{cccc}
F & G & q & DK \\
F_T & G_T & q_T & K_T
\end{array}$$

$D$ and $\rho$ are densities, and where they appear in the numerical solution they must be interpreted as $(D, \rho)/(1\ g/cm^3)$.

Then (8) can be written

$$D(K'(1) + 5K(1)) = 5D + \rho(1)q(1).$$

(26)

Also

$$2F(1) + G(1) = q(1),$$

(27)

$$F'(1) = -q(1),$$

(28)

and with $\alpha = 0.5447$,

$$2F(\alpha) + 0.2967 G(\alpha) = q(\alpha),$$

(29)

$$0.5447 F'(\alpha) = -q(\alpha),$$

(30)

$$DK'(\alpha) = cK_T'(\alpha).$$

(31)

Dr Takeuchi very kindly gave us a copy of his matrices for his Model 2 connecting $F(\xi)$, $G'(\xi) \ldots K(\xi)$ with their values at $\xi = \alpha$, which are not given in detail in his paper. These were transformed to the four constants used here, by means of the six equations above. There are discrepancies between the last lines in his (198) and (205); the second entry should be $0.072887$, the fourth $0.23256$ in both cases.

The solution is as follows:

$$\begin{align*}
F'(\alpha) &= -1.8359 q(\alpha), \\
G'(\alpha) &= 24.6632 q(1) - 10.2459 q(\alpha) - 0.00013 DK'(\alpha) + 1.8392 D, \\
DK'(\alpha) &= DK'(\alpha), \\
F(\alpha) &= 0.31942 q(1) + 0.18614 q(\alpha) - 0.00062 DK'(\alpha) + 0.03546 D, \\
G(\alpha) &= -2.1524 q(1) + 2.11499 q(\alpha) + 0.004145 DK'(\alpha) - 0.23895 D, \\
DK(\alpha) &= 0.00078 q(1) + 0.3555 q(\alpha) - 0.01110 DK'(\alpha) + 1.1020 D,
\end{align*}$$

(32)
\[ F'(1) = -q(1), \]
\[ G'(1) = 3.50477 q(1) - 2.09580 q(x) - 0.00325 DK'(x) - 0.1476 D, \]
\[ DK'(1) = 1.08059 q(1) - 0.59626 q(x) + 0.02873 DK'(x) - 0.22403 D, \]
\[ F(1) = 1.12191 q(1) - 0.06591 q(x) - 0.00058 DK'(x) + 0.04254 D, \]
\[ G(1) = 0.76229 q(1) + 0.1378 q(x) + 0.00118 DK'(x) - 0.08488 D, \]
\[ DK(1) = 0.38367 q(1) + 0.11938 q(x) - 0.00578 DK'(x) + 1.04472 D. \]

Some features of the solutions are worth special mention. The coefficient of \( D \) in \( K(\xi) \) never deviates far from 1. If motions of the core and the boundaries were prevented, the displacements of the shell would have little effect on the potential. The coefficient of \( q(1) \) in \( DK(x) \) is very small. With a homogeneous incompressible shell and core, and with \( q(x) = 0 \) and no external field, \( K(\xi) \) would be constant. This is the problem of distortion by loading over the outer surface, with a constraint preventing core motions. But here the coefficient of \( q(1) \) in \( K(\xi) \) varies by a factor of 50. In a uniform sphere the effects of compressibility are usually not large, but it appears that in this problem they are. The large coefficient of \( q(1) \) in \( G'(x) \) corresponds to a large disturbance of density just outside the core, and it appears that the reduction of potential due to this cancels most of the increase due to the elevation of the outer surface. The coefficients of \( q(x) \) and \( DK'(x) \) in \( DK(\xi) \) decrease greatly outward; this is mainly due to the factor \( \xi^{-5} \) in the contribution from \( P \).

Extra figures were calculated in some of the coefficients because of cancellation in the checking.

The solutions for other values of \( \xi \) were calculated on the EDSAC machine at the Mathematical Laboratory, Cambridge, by Miss M. Lewin (now Mrs Mutch). As Takeuchi's last two figures are mainly guard figures they are rounded off at this stage, and the solutions are in Table I. The values of \( \xi \) are as follows:

\[ \xi_0 = \infty, \quad \xi_1 = 0.54147, \quad \xi_2 = 0.59184, \quad \xi_3 = 0.63893, \quad \xi_4 = 0.68603, \quad \xi_5 = 0.73313, \]
\[ \xi_6 = 0.78022, \quad \xi_7 = 0.82732, \quad \xi_8 = 0.87441, \quad \xi_9 = 0.92151, \quad \xi_{10} = 1.00000. \]

The intervals for \( r \), measured from the outer surface, are 250 km to 500 km, then 300 km to 2000 km. The data assume a discontinuous change of properties at a depth of 500 km. The rows in the matrices give \( F, G, DK, F, G, K \) for each \( \xi \).

In the integration we need only \( F \) and \( G \), but some of the coefficients vary too rapidly for accurate integration by the Gregory formula. Since, however, \( F' \) and \( G' \) were also available, the interval was halved by the formula (Jeffreys, 1953)

\[ f(\frac{1}{2}h) = \frac{1}{2}f(o) + \frac{1}{2}f(h) - \frac{1}{2}h[f'(h) - f'(o)]. \]

This formula is more accurate than the Newton–Bessel formula with second differences and can be used at the end intervals. When this was done integration by Gregory's formula was accurate enough.

After integration it was convenient to use (13) to eliminate \( DK(x+) \); with the values adopted \( \rho_0(x+) = 5.53 \text{ g/cm}^3 \),

\[ DK(x+) = -9.1787 DP + 10.152 q(x). \]

Then the surface integrals give in 2L (lines 4, 5 of 3(9))

\[ -Ma^3 \times 10^{12} q(1)[8.415 q(1) - 3.168 q(x) - 0.00670 DP - 0.7698 D], \]
\[ -Ma^3 \times 10^{12} q(x)[-3.157 q(1) + 1.644 q(x) + 0.0631 DP - 0.1047 D]. \]
and the volume integral (line 4) gives

\[
Ma^3 \times 10^{12}(0.7401 \, q(1) \, D + 0.1003 \, q(\alpha)D + 0.0164 \, D^2P + 0.0744 \, D^2 \nonumber \\
+ 0.0068 \, q(1) \, DP - 0.0621 \, q(\alpha) \, DP + 0.0017 \, D^2P^2).
\] (38)

C.g.s. units are understood.

We have two useful checks. \(q(1)\) and \(q(\alpha)\) are Lagrangian coordinates, and by a reciprocity theorem the coefficients of \(q(1)\) \(q(\alpha)\) in the two surface integrals should be equal. They agree to about 1 part in 300. Also \(D\) and \(DP\) represent external forces, acting on the shell, and if \(q(1)\) and \(q(\alpha)\) are put zero the terms in \(D\) and \(DP\) will represent work done by the external forces with the boundaries fixed. Then the amounts of work done by \(D\) on the displacements due to \(DP\) and done by \(DP\) on those due to \(D\) should be equal, by another reciprocity theorem, and the coefficient 0.0164 is in fact the sum of two of 0.0082. These comparisons are checks both on the present work and Takeuchi’s.

**Table I**

<table>
<thead>
<tr>
<th>(q(1))</th>
<th>(q(\alpha))</th>
<th>(DK'(\alpha))</th>
<th>(D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y_1)</td>
<td>0.069</td>
<td>1.206</td>
<td>0.0002</td>
</tr>
<tr>
<td>15.198</td>
<td>-8.770</td>
<td>-0.009</td>
<td>1.056</td>
</tr>
<tr>
<td>0.486</td>
<td>-0.468</td>
<td>0.611</td>
<td>-0.042</td>
</tr>
<tr>
<td>0.317</td>
<td>0.115</td>
<td>-0.0006</td>
<td>0.036</td>
</tr>
<tr>
<td>-1.238</td>
<td>1.663</td>
<td>0.0039</td>
<td>-0.173</td>
</tr>
<tr>
<td>0.020</td>
<td>0.343</td>
<td>-0.0074</td>
<td>1.101</td>
</tr>
<tr>
<td>(Y_2)</td>
<td>-0.211</td>
<td>-0.892</td>
<td>0.0003</td>
</tr>
<tr>
<td>9.829</td>
<td>-6.918</td>
<td>-0.011</td>
<td>0.631</td>
</tr>
<tr>
<td>0.723</td>
<td>-0.620</td>
<td>0.388</td>
<td>-0.073</td>
</tr>
<tr>
<td>0.310</td>
<td>0.065</td>
<td>-0.0006</td>
<td>0.036</td>
</tr>
<tr>
<td>-0.662</td>
<td>1.293</td>
<td>0.0034</td>
<td>-0.134</td>
</tr>
<tr>
<td>0.049</td>
<td>0.316</td>
<td>-0.0051</td>
<td>1.100</td>
</tr>
<tr>
<td>(Y_3)</td>
<td>-0.297</td>
<td>-0.661</td>
<td>0.0002</td>
</tr>
<tr>
<td>6.613</td>
<td>-5.206</td>
<td>-0.010</td>
<td>0.390</td>
</tr>
<tr>
<td>0.847</td>
<td>-0.634</td>
<td>0.255</td>
<td>-0.097</td>
</tr>
<tr>
<td>0.298</td>
<td>0.029</td>
<td>-0.0006</td>
<td>0.037</td>
</tr>
<tr>
<td>-0.282</td>
<td>1.007</td>
<td>0.0029</td>
<td>-0.111</td>
</tr>
<tr>
<td>0.086</td>
<td>0.287</td>
<td>-0.0036</td>
<td>1.094</td>
</tr>
<tr>
<td>(Y_4)</td>
<td>-0.371</td>
<td>-0.503</td>
<td>0.0001</td>
</tr>
<tr>
<td>4.610</td>
<td>-4.054</td>
<td>-0.008</td>
<td>0.249</td>
</tr>
<tr>
<td>0.922</td>
<td>-0.593</td>
<td>0.173</td>
<td>-0.117</td>
</tr>
<tr>
<td>0.282</td>
<td>0.001</td>
<td>-0.0006</td>
<td>0.038</td>
</tr>
<tr>
<td>-0.021</td>
<td>0.789</td>
<td>0.0025</td>
<td>-0.096</td>
</tr>
<tr>
<td>0.128</td>
<td>0.258</td>
<td>-0.026</td>
<td>1.089</td>
</tr>
<tr>
<td>(Y_5)</td>
<td>-0.437</td>
<td>-0.394</td>
<td>0.0000</td>
</tr>
<tr>
<td>3.369</td>
<td>-3.139</td>
<td>-0.007</td>
<td>0.161</td>
</tr>
<tr>
<td>0.972</td>
<td>-0.540</td>
<td>0.120</td>
<td>-0.134</td>
</tr>
<tr>
<td>0.263</td>
<td>-0.020</td>
<td>-0.0006</td>
<td>0.039</td>
</tr>
<tr>
<td>0.163</td>
<td>0.621</td>
<td>0.0022</td>
<td>-0.087</td>
</tr>
<tr>
<td>0.172</td>
<td>0.231</td>
<td>-0.0019</td>
<td>1.083</td>
</tr>
<tr>
<td>(Y_6)</td>
<td>-0.514</td>
<td>-0.310</td>
<td>-0.0000</td>
</tr>
<tr>
<td>2.599</td>
<td>-2.467</td>
<td>-0.005</td>
<td>0.099</td>
</tr>
<tr>
<td>1.009</td>
<td>-0.403</td>
<td>0.085</td>
<td>-0.149</td>
</tr>
<tr>
<td>0.240</td>
<td>-0.036</td>
<td>-0.0006</td>
<td>0.040</td>
</tr>
<tr>
<td>0.302</td>
<td>0.490</td>
<td>0.0019</td>
<td>-0.081</td>
</tr>
<tr>
<td>0.219</td>
<td>0.207</td>
<td>-0.015</td>
<td>1.076</td>
</tr>
</tbody>
</table>
5. Models for the core.—The actual core is too complex for detailed treatment, but useful simplifications are possible. It has appeared probable that so long as the moments of inertia of the core and the ellipticity of its boundary are correct, the detailed distribution of density is of secondary importance. But if the Radau approximation is valid, the mean moment of inertia is enough to determine the other quantities. Now Bullen (1942) found that extreme hypotheses gave for the core

\[
\frac{I}{Ma^2} = 0.389 \text{ and } 0.373
\]

in comparison with 0.400 for a homogeneous core. Now it has been shown in a former paper (Jeffreys, 1951 a), in which \(K = \frac{2}{3}\) corresponds to \(I/Ma^2 = \frac{2}{3}\), that the ratio \(e/m\) for a Wiechert model varies over the whole range of possible core radii by only 2.5 per cent. For \(I/Ma^2 = 0.4\), the body is homogeneous and the ratio does not depend at all on the radius that we might choose to be that of the core. Thus it seems likely that for the Earth’s core (in which the inner core would play the part of the core of a Wiechert model, and the potential due to the figure of the shell would be added to that due to rotation) the Radau approximation will always give solutions valid within 1 per cent.

Through most of the core the variation of density is mainly due to compression. If this held for the whole of the core a smooth distribution of Laplace’s or Roche’s
type would fit fairly well. It is desirable also to have an estimate of the effect of the inner core, but for simplicity we shall treat this as a point mass at the centre.

Bullard’s (1946) density distribution makes the moment of inertia of the core $0.107$ of that of the whole Earth, and the ellipticity $0.002567$. The dynamical ellipticity is $0.002562$; this was computed from his solution by numerical integration. We shall adopt these values in all models. The reason for the close agreement is that the dynamical ellipticity is not a weighted mean of the internal ellipticities in the usual sense. In computing it we compare integrals of the forms $\int \rho d(r^5 e)$ and $\int \rho dr^5$, whose ratio would be $e$ if $e$ was constant. With variable $e$, the smaller $e$ near the centre reduces the first integral, but this is compensated by the fact that the consequent reduction of $r^5 e$ at intermediate depths has to be made up by increased values of its steps as the boundary is approached, so that the dominant part of the integrals still comes from values near the surfaces.

We consider two extreme models for the core. In the first the core is treated as homogeneous and incompressible, with a point mass added at the centre. In the second we take a Roche density law

$$\rho = k_0 - \frac{\xi_c^2}{a_1^3},$$

where $a_1 = a \alpha$ is the radius of the core and $\xi_c = r/a_1$. This makes the mean density

$$\bar{\rho} = k_0 - \frac{3}{5} k_1,$$

and the moment of inertia is that of a homogeneous sphere of density

$$\rho_c = k_0 - \frac{5}{3} k_1.$$

Bullard’s values for the density lead to

$$\bar{\rho} = 10.70, \quad \rho_c = 10.28 \text{ g/cm}^3,$$

whence

$$k_0 = 12.91 \text{ g/cm}^3, \quad k_1 = 3.68 \text{ g/cm}^3, \quad k_1/k_0 = 0.285.$$  

For a distribution of density $\rho_c$ with a central particle of mass $m_1$ the mean density is

$$\bar{\rho} = \rho_c + \frac{m_1}{\frac{4}{3} \pi a_1^3},$$

whence

$$\frac{m_1}{\frac{4}{3} \pi \rho_c a_1^3} = 0.041.$$  

We note that a 28 per cent variation of density in the Roche model is equivalent to putting 4 per cent of the mass in a central particle, so far as the hydrostatic theory is concerned.

More can be said; for Radau’s approximation is good for any Roche model, and it is easy to show that for any model satisfying it the surface ellipticity of a free body is given by

$$\frac{e}{\frac{4}{3}m} = \frac{1}{1 - \frac{5}{2} \left( \frac{5I}{2Ma^2} - 1 \right)},$$

to the first order in the small quantity $5I/2Ma^2 - 1$. But for the central particle model we can easily show that the same is true. Also to the same order the
dynamical and geometrical ellipticities are equal. There can therefore be no inconsistency in adopting Bullard’s ellipticity of the core, namely $0.002567$, in both models.

For Roche’s models we take the undisturbed density over a level surface to be

$$\rho_0 = k_0 - k_1 \xi_c^2,$$

where $a_1 \xi_c$ is now the mean radius of the surface, and the corresponding geopotential is

$$\Psi = -4\pi f a_0^2 \left( \frac{1}{k_0} \xi_c^2 - \frac{1}{20} k_1 \xi_c^4 \right).$$

(10)

By Radau’s approximation the ellipticity of a stratum of uniform density is

$$\epsilon = \epsilon_1 \left( 1 - \frac{\xi_c}{k_0} \right),$$

(11)

where $\epsilon_1$ refers to the boundary. Then the equation of the stratum is

$$r/a_1 = \xi_c \left( 1 + \epsilon \left( \frac{1}{3} - \cos^2 \theta \right) \right),$$

(12)

and we put

$$W_2 = r^2 \left( \frac{1}{3} - \cos^2 \theta \right) = \frac{1}{3} (x^2 + y^2 - 2z^2).$$

(13)

Then to orders $k_1$ and $\epsilon_1$

$$\rho = k_0 - k_1 \frac{r^2}{a^2} + \frac{2k_1 \epsilon_1}{a^2} W_2,$$

(14)

$$\Psi = -\frac{3}{8} \pi f \left( k_0 r^2 - \frac{k_1}{10} r^4 \right) + \frac{3}{8} \pi f \left( k_0 - \frac{k_1}{5} \right) \epsilon_1 W_2 - \frac{3}{2} \pi f k_1 \epsilon_1 \frac{r^2}{a^2} W_2.$$

(15)

We shall not need the ellipticities of surfaces of equal density again, and shall therefore denote the ellipticity of the core boundary simply by $\epsilon$.

6. Solutions for central particle models: statical case.—In this case the volume integrals in 3(9) disappear; and

$$U_c = \frac{4}{5} \pi f \rho_c q(x) \left( \frac{ax}{r} \right)^5 K_2,$$

(1)

whence

$$DP = \frac{1}{2} \rho_c q(x) = 2.056 q(x).$$

(2)

Also if the normal displacement is

$$u_n^r = q(x) K_2 / ax,$$

(3)

$$\iint \rho_0 u_n^r \left( 2 U_1 + U_c + u_k \frac{\partial \Psi}{\partial x_k} \right) dS$$

$$= \iint \rho_0 u_n^r (2 U_1 + U_c - \frac{m_1}{4 \pi a_1^2} q(x)) dS$$

$$= 4 \pi f a^2 M a^3 \cdot \rho_0 \left( 2 D - \frac{h}{10} \rho_c q(x) - \frac{m_1}{4 \pi a_1^2} q(x) \right) q(x)$$

$$= 0.1677 \times 10^{12} M a^3 \{ 2 D q(x) - 1.51 q^2(x) \}.$$  

(4)

We add this to 3(36) (37) (38) to give the potential terms in $2 L$. We take first the statical case. We have in all

$$\frac{2 L}{Ma^3 \times 10^{12}} = -8.415 \ q(1)^2 + 6.353 \ q(1) q(x) - 2.147 \ q(x)^2$$

$$+ 1.510 \ q(1) D + 0.574 \ q(x) D + 0.0744 D^2,$$

(5)

whence

$$q(1) = 0.317 D, \quad q(x) = 0.603 D, \quad K(1) = 1.289, \quad F(1) = 0.0444 D.$$

(6)
No. 2, 1957  

Theory of nutation and the variation of latitude

Love's numbers $h$ and $k$ are

$$h = \frac{g}{a} q(1) = \frac{3\beta}{2} \frac{q(1)}{D} = 0.585$$

(7)

$$k = K(1) - 1 = 0.289$$

(8)

and the Shida–Lambert number is

$$l = \frac{3\beta}{2} \frac{F(1)}{D} = 0.082.$$  

(9)

For comparison Takeuchi's values are

<table>
<thead>
<tr>
<th>Model</th>
<th>$h$</th>
<th>$k$</th>
<th>$l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.587</td>
<td>0.290</td>
<td>0.068</td>
</tr>
<tr>
<td>2</td>
<td>0.610</td>
<td>0.281</td>
<td>0.082</td>
</tr>
</tbody>
</table>

Agreement is good, but as the present model for the crust is nearer to Takeuchi's Model 2 than to Model 1, it is rather surprising that two of the numbers agree better with Model 1.

The factor $M$ has different values for different second harmonics, but cancels in the statical case, which is applicable to the semiannual and long-period tides.

7. Central particle model: dynamical case.—When $l$, $m$ are not zero $u'_i$ can no longer be neglected in the core. The statical solution for the shell must be modified by restoring $U'_1$.

$u'_i$ can be broken into two parts,

$$u'_i = v_{i1} + v_{i2},$$

(1)

where

$$v_{i1} = \frac{l_2 a_1}{c_1}, \quad v_{i2} = \frac{m_2 a_1}{c_1}, \quad \frac{c_1}{a_1} \left( \frac{l_2 x + m_2 y}{c_1} \right) c_i,$$

(2)

This makes $u'_i$ linear in the coordinates; this is justified because the obstruction due to the central particle will be of no importance since $u'_i \to 0$ at the centre.

The normal displacement at the boundary is

$$u'_n = \frac{2R}{a_1 c_1} x(l_2 x + m_2 y),$$

(4)

where

$$\frac{1}{R^2} = \frac{x^2 + y^2}{a_1^4} + \frac{z^2}{c_1^4}.$$  

(5)

$u'_n$ must be continuous with the normal displacement of the shell; since this is an elastic displacement $R$ can be replaced by $a_1$.

We introduce the constants

$$A_{11} = \iint \int \rho_0 x^2 d\tau, \quad A_{33} = \iint \int \rho_0 z^2 d\tau; \quad C_1 = 2 A_{11}, \quad A_1 = A_{11} + A_{33}, \quad F = 2 \frac{a}{c} A_{33} = 2 \frac{c}{a} A_{11}.$$  

(7)

Then

$$F^2 = 4 A_{11} A_{33} = C_1 (2 A_1 - C_1).$$

(8)

The ellipticity of the core being small, we have nearly

$$C_1 = A_1 (1 + e), \quad 2 A_1 - C_1 = A_1 (1 - e)$$

(9)
and hence
\[ F = A_1 (1 + O(e^2)). \] (10)

We find
\[ \int \int \int \rho u_t \, d\tau = A_1 (I_1^2 + \dot{I}_2^2 + \dot{m}_1^2 + \dot{m}_2^2) + 2(C_1 - A_1)(\dot{I}_1 \dot{I}_2 + m_1 \dot{m}_2) \] (11)
\[ 2 \int \int \int \rho u_{\omega} u_t \, d\tau = 2 F (\dot{I}_1 + \dot{m}_1) \] (12)
\[ 2 \omega \int \int \int \rho_0 (u_1^t u_2^t - u_2^t u_1^t) \, d\tau = \omega C_1 (l_1 \dot{m}_1 - m_1 \dot{I}_1 + 2m_1 \dot{m}_1 - m_1 l_1 - l_1 \dot{m}_1 + l_1 \dot{m}_2) \]
\[ - m_2 \dot{I}_1 + l_2 \dot{m}_2 - m_2 \dot{I}_2) \]

which is equivalent to
\[ \omega C_1 (l_1 \dot{m}_1 - m_1 \dot{I}_1 + 2l_2 \dot{m}_1 - 2m_2 \dot{I}_1 + 2l_2 \dot{m}_2). \] (13)

The term in \( l_2 \dot{m}_2 \) can be dropped, since it is of the order of the kinetic energy of the elastic displacements. But those in \( l_2 \dot{m}_1 - m_1 \dot{I}_1 \) are of the order of those arising from the correcting terms in \( U_1' \), and must be retained.

\[ 2 \int \int \int \rho_0 \omega (x_3 (\dot{m}_1 - \dot{I}_1) - u_3 (\dot{m}_1 - \dot{I}_1)) \, d\tau = 2 F \omega (\dot{m}_1 - \dot{I}_1). \] (14)

The core integrals arising from the last three lines of 3(9) are the same as in the statical case except that \( U_1 \) has to be replaced by \( U_1' \), and with this change 6(4) will hold.

We take
\[ A_1 = 0.107 C, \quad C_1 = A_1 = e A_1, \quad e = 0.002562; \]
\[ a_1/a = \alpha = 0.5447; \quad \omega = 7.2921 \times 10^{-5} \text{ sec}; \quad C = 8.05 \times 10^{44} \text{ gm cm}^2; \]
\[ A = C(1 - H), \quad H = 0.003273; \]
\[ a = 6.371 \times 10^8 \text{ cm}. \]

Then
\[ C \omega^2 = 4.28 \times 10^{36} \text{ gm cm}^2/\text{sec}^2; \]
\[ M = \frac{4}{15} \pi, \quad Ma^3 = 2.16 \times 10^{26} \text{ cm}^3, \]
\[ 10^{12} Ma^3 \omega^2 = 50.61. \]

We keep \( H \) and \( e \) explicit for the present; this facilitates some checks. We take also
\[ q(1) K_2 = r_1 xx + r_2 yy \]
\[ q(x) K_2 = s_1 xx + s_2 yy; \quad l_2 = \frac{1}{2} s_1, \quad m_2 = \frac{1}{2} s_2 \]
\[ DK_2 = d_1 xx + d_2 yy \]
\[ = \frac{\omega^2}{4 \pi f} \left\{ \left( \frac{c_1}{\omega^2} + \frac{\dot{m}}{\omega} + l \right) xx + \left( \frac{c_2}{\omega^2} - \frac{\dot{I}}{\omega} + m \right) yy \right\} \] (15)

and
\[ \omega^2 / 4 \pi f = 0.006345. \]

It is convenient to use \( 2L/C \omega^2 \) instead of \( 2L \). Then
\[ \frac{2L}{C \omega^2} = -425.9 (r_1^2 + r_2^2) + 321.5 (r_1 s_1 + r_2 s_2) - 108.7 (s_1^2 + s_2^2) + 76.4 (r_1 d_1' + r_2 d_2') \]
\[ + 0.85 (s_1 d_1' + s_2 d_2') + 3.765 (d_1'^2 + d_2'^2). \]
\[ + \frac{1}{\omega^2} [(1 - H)(l^2 + \dot{m}^2) + \omega (1 - 2H)(\dot{m} - \dot{I}) - \omega^2 H(l^2 + \dot{m}^2) \]
\[ + 0.107 (l_1^2 + \dot{m}_1^2) + 0.107 e (\dot{I}_1 s_1 + \dot{m}_1 s_2) \]
\[ + 0.214 (\dot{I}_1 + \dot{m}_1) + 0.214 (1 + e) \omega (l_1 \dot{m}_1 - m_1 \dot{I}_1 + s_1 \dot{m}_1 - s_2 \dot{I}_1) \]
\[ + 0.214e (\dot{m}_1 l_1 - \dot{I}_1 l_1) - 2H (l_1 m_1 - m_1 \dot{I}_1). \] (16)
The terms in $e\ell s$, $wes\hat{m}$ can be dropped in comparison with those that do not contain $e$.

We put

$$c_1 + ic_2 = k, \quad l + im = \zeta, \quad l_1 + im_1 = \zeta_1,$$

(17)

Derivatives of $r_1$, $r_2$, $s_1$, $s_2$ do not appear in the Lagrangian; hence these coordinates can be eliminated.

We find

$$\begin{align*}
(r_1, r_2) &= 0.3774(s_1, s_2) + 0.0897(d_1', d_2'), \\
(s_1, s_2) &= 0.607(d_1', d_2') + \frac{0.00115}{\omega}(m_1 - \hat{l}_1),
\end{align*}$$

(18)

and the modified Lagrangian is given by

$$\begin{align*}
\frac{2L'}{C} &= (1 - H + 0.00100)(\hat{l}^2 + \hat{m}^2) + \omega(1 - 2H + 0.00200)(\hat{m} - \hat{l}) \\
&\quad - \omega^2(H - 0.00100)(\hat{l}^2 + \hat{m}^2) \\
&\quad + 0.10706(\hat{l}_1^2 + \hat{m}_1^2) + 0.21441(\hat{l}_1 + \hat{n}_1)(\hat{m}_1 - \hat{l}_1) \\
&\quad + 0.2140\omega(\hat{m}_1 - \hat{l}_1) + 0.00041\omega(\hat{m}_1 - \hat{l}_1) \\
&\quad + (2H + 0.00200)(c_1 \hat{l} + c_2 \hat{m}) + \frac{0.00200}{\omega}(c_1 \hat{m} - c_2 \hat{l}) \\
&\quad + \frac{0.00001}{\omega}(c_1 \hat{m}_1 - c_2 \hat{l}_1) + \frac{0.00100}{\omega^2}(c_1^2 + c_2^2).
\end{align*}$$

(20)

We form the equations of motion and assume a time factor $e^{\gamma t}$. Two speeds of free motions are found to be $\gamma$ and $-\omega$. This gives a valuable check, since the speed $\omega$ corresponds to a statical displacement of the core without moving the shell and that of $-\omega$ gives a steady displacement of the axis in space. A further check is that for $\gamma = -\omega$ the rate of motion in space of the axis, given by $(\gamma + \omega)\zeta$, is found to be the same as for a rigid body. This represents the precession, which in all previous models has been found to be independent of the internal constitution.

For the other free motions we find

$$\begin{align*}
\gamma \omega &= 0.00255; \quad \frac{\omega}{\gamma} = 392.4; \quad \frac{\zeta_1}{\zeta} = -0.99931, \\
\frac{\gamma + \omega}{\omega} &= -0.00224; \quad \frac{\omega}{\gamma + \omega} = -447; \quad \frac{\zeta_1}{\zeta} = -9.33.
\end{align*}$$

(21)

(22)

The first of these is the variation of latitude. The period is shorter than any found from observation. It would be lengthened by the effect of the oceans; it remains to be seen whether compressibility of the core would make much difference. There is very little motion of the core, since apart from the effect of the ellipticity the displacements due to $\zeta$ and $\zeta_1$ nearly cancel.*

The second movement is not observed. The core displacements are opposite to the general rotation and much larger.

* A former solution (Jeffreys, 1951 b) by the method of undetermined multipliers gave 463 days for the period. A homogeneous core was assumed. This differs from the present model only by the central particle, and a great difference was not to be expected. The difference has been traced to a misplaced 2 in the earlier solution, which when corrected leads to 390 days.
For the forced motions we write $\gamma + \omega = n$. As a standard of comparison we take the value of $\zeta$ for a rigid Earth and call this $\zeta_0$. We have

$$Hk = -n^2\zeta_0(\omega - n + nH),$$

(23)

and the equations of motion become

$$(\omega - 0.99773n)\zeta + 0.10720(\omega - n)\zeta_1 = (\omega - 0.99673n)\left(1 - 0.3055 \frac{n}{\omega}\right)\zeta_0,$$

(24)

$$0.10720n\zeta + (0.000214\omega + 0.10706n)\zeta_1 = 0.0611n\left(1 - 0.3055 \frac{n}{\omega}\right)\zeta_0.$$

(25)

The coefficient $0.3055$ is $0.00100/H$, and is a consequence of elasticity of the shell, not of fluidity of the core.

Results for the most interesting speeds are as follows:

<table>
<thead>
<tr>
<th>$n/\omega$</th>
<th>$\zeta/\zeta_0$</th>
<th>$\zeta_1/\zeta_0$</th>
<th>Tidal component</th>
<th>Nutation component</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\frac{1}{13.7}$</td>
<td>1.0768</td>
<td>-0.509</td>
<td>O</td>
<td>Secondary fortnightly</td>
</tr>
<tr>
<td>$-\frac{1}{183}$</td>
<td>1.0895</td>
<td>-0.819</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{6800}$</td>
<td>0.9964</td>
<td>0.034</td>
<td>K$_1$ (companion)</td>
<td>Principal nutation</td>
</tr>
<tr>
<td></td>
<td>1.0000</td>
<td>0.000</td>
<td>K$_1$</td>
<td>Precession</td>
</tr>
<tr>
<td>$\frac{1}{6800}$</td>
<td>1.0036</td>
<td>-0.034</td>
<td>K$_1$ (companion)</td>
<td>Principal nutation (correction)</td>
</tr>
<tr>
<td>$\frac{1}{183}$</td>
<td>1.0350</td>
<td>-0.342</td>
<td>P</td>
<td>Semiannual</td>
</tr>
<tr>
<td>$\frac{1}{13.7}$</td>
<td>1.0269</td>
<td>-0.458</td>
<td>O</td>
<td>Fortnightly</td>
</tr>
</tbody>
</table>

For $n/\omega = -1/447$, $\zeta/\zeta_0$ and $\zeta_1/\zeta_0$ become infinite and change sign; and for $n/\omega = -1/479$, $\zeta$ vanishes and $\zeta_1$ does not. This is the explanation of the peculiar behaviour between $-1/183$ and $-1/6800$.

On the whole the results are fairly similar to those found for a rigid shell. The principal difference is that for a rigid shell, unless $|n/\omega|$ is small of order $e$, $\zeta_1$ is nearly $-\zeta$ and $\zeta/\zeta_0$ is about 1.12. This is not true with an elastic shell. Even for O and OO $\zeta_1$ is only about $-0.5 \zeta_0$, and for the principal lunar diurnal tide O the predicted amplitude of $\zeta$ is under 3 per cent in excess of the rigid body value. H. R. Morgan (1952) has made a thorough analysis of the observations and has finally decided that they will not determine any significant correction.

For a rigid shell the correcting factor to the principal 19-yearly nutation was 0.9940. The result here is 0.9964. For comparison, Newcomb's standard value of the nutation in obliquity is 9".210. The rigid body value based on Spencer Jones's values of the solar parallax and the lunar inequality is 9".2272 ± 0".0008 (p.e.) Rabe's parallax and my value of the lunar inequality lead to nearly the same results. Recent observed values have mostly been about 9".207. Corresponding to Spencer Jones's calculated value, the amplitude in longitude would be 6".872. A determination by E. P. Fedorov (1952) separates the corrections to the amplitudes in obliquity and longitude, and indicates that the former should be 9".190 to 9".200, which would agree with a correcting factor of 0.996 to 0.997. In longitude he gets 6".874 to 6".886, larger than the rigid body value. But theoretically both components should be reduced, that in longitude by the larger factor. There is therefore still an inconsistency.
8. The bodily tide numbers for diurnal tides.—To get these we have to express $\xi_0$ in terms of $k/\omega^2$ and then express $r_1 + ir_2, s_1 + is_2$ in terms of $c_1 + ic_2$. The results are as follows:

<table>
<thead>
<tr>
<th>$n/\omega$</th>
<th>$h$</th>
<th>$k$</th>
<th>$l$</th>
<th>$1-h+k$</th>
<th>$1+h-\frac{3}{2}k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{13.7}$</td>
<td>0.590</td>
<td>0.244</td>
<td>0.082</td>
<td>0.654</td>
<td>1.224</td>
</tr>
<tr>
<td>$\frac{1}{183}$</td>
<td>0.523</td>
<td>0.218</td>
<td>0.084</td>
<td>0.695</td>
<td>1.196</td>
</tr>
<tr>
<td>$\frac{1}{6800}$</td>
<td>0.490</td>
<td>0.205</td>
<td>0.086</td>
<td>0.715</td>
<td>1.182</td>
</tr>
<tr>
<td>0</td>
<td>0.492</td>
<td>0.206</td>
<td>0.086</td>
<td>0.714</td>
<td>1.183</td>
</tr>
<tr>
<td>$\frac{1}{6800}$</td>
<td>0.494</td>
<td>0.207</td>
<td>0.086</td>
<td>0.713</td>
<td>1.184</td>
</tr>
<tr>
<td>$\frac{1}{183}$</td>
<td>0.555</td>
<td>0.231</td>
<td>0.082</td>
<td>0.676</td>
<td>1.209</td>
</tr>
<tr>
<td>$\frac{1}{13.7}$</td>
<td>0.584</td>
<td>0.242</td>
<td>0.082</td>
<td>0.658</td>
<td>1.221</td>
</tr>
</tbody>
</table>

$l$ is practically the same as on the statical theory. $h$ and $k$ are substantially less, except that $h$ approaches the statical value for the tides O and OO. The statical values of $1-h+k$ and $1+h-\frac{3}{2}k$ are 0.704 and 1.152. These are the factors associated with observations of the bodily tide, and it appears that $K_1$ and its companions are likely to be nearly the same as on the statical theory, but there is some hope of detecting a difference for O.

St. John's College, Cambridge: Faculty of Sciences, The University, Lisbon.

1956 October.

References

H. Jeffreys, 1929, The Earth, Chapter IX.
H. Jeffreys, 1951b, Observatory, 71, 154.
H. Lamb, 1932, Hydrodynamics.
F. Tisserand, 1891, Traité de Mécanique Céleste, 2, ch. 30.