VEILED ABSORPTION LINES

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Summary

It has been suggested that some apparently faint Fraunhofer lines may not be faint in the theoretical sense, but may be formed in low photospheric layers and 'veiled' by a higher emitting atmosphere.

This paper contains a theoretical investigation of this hypothesis. An exact solution is found for the residual intensity subject to certain standard though rather restrictive conditions. In the concluding section line profiles are drawn for veiled lines and for lines without veiling to show the effect of veiling on a line profile.

1. Introduction.—In a letter to The Observatory C. de Jager and L. Neven (9) wrote: "It is well known to many workers in the field of stellar spectroscopy that some faint Fraunhofer lines need not necessarily be 'faint' in the theoretical sense, which means that they should not necessarily be situated on the 45° part of the curve of growth. There are lines that are caused by strong absorption in deep layers of a stellar atmosphere and which are veiled by the radiation from the non-selectively absorbing higher layers. Such lines may apparently be weak, but nevertheless behave as a strong or medium strong line, and show saturation effects since they are caused by strong absorption in the relevant parts of the stellar photosphere, where the line is principally formed."

In view of this suggestion, it seems desirable to use a model in which the star is surrounded by a continuously absorbing region, which will be called the atmosphere, and to compare the exact solution with the corresponding solution in the standard case in which there is no such atmosphere. This comparison is effected by computing the profiles of veiled lines at various points of the disk in the two cases. More information about the centre to limb variation of the profiles of weak lines, thought to be veiled, will be required before adequate comparison with observation will be possible.

The model used (see Fig. 1) is that of a plane parallel atmosphere of thickness $x_1$, lying above a plane parallel photosphere of infinite extent. Throughout both the atmosphere and the photosphere there is continuous emission according to a linear Planck function. Absorption lines are formed in the photosphere and the atmosphere is non-scattering. There is no radiation falling on the upper surface of the atmosphere. It is required to find the emergent intensity at any point of the line profile considered.

In the first section of this paper an exact solution is found for the emergent intensity subject to certain standard conditions which will be enumerated later. The method used is one, developed by I. W. Busbridge, which was originally given by Ambartsumian for semi-infinite atmospheres. With this method it is

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comparatively simple to find an exact solution in terms of $H$ functions, their moments, and integrals of expressions containing $H$ functions. The coefficient $\varepsilon$ is introduced to allow for thermal emission and, in the numerical results, values for the emergent intensity are found in the case of lines formed by pure scattering ($\varepsilon = 0$) and pure absorption ($\varepsilon = 1$).

A method of dealing with the variation of $\eta$ (the ratio of the line absorption coefficient to the continuous absorption coefficient) is then developed in Section 6 for the case of lines formed by pure absorption ($\varepsilon = 1$), and a formula capable of quite wide application derived*.

Finally diagrams are drawn to show the effects of veiling on line profiles.

2. Derivation of the Milne equation of the problem.—The problem is solved subject to certain conditions. Throughout both the atmosphere and the photosphere it is assumed that

(i) the variations of the continuous absorption coefficient $\kappa$ over the range of frequencies considered may be neglected;

(ii) the variation of the Planck function $B(\nu, T)$ with the depth $x$ below the surface of the atmosphere is such that it can be represented in the form

$$B(\nu, T) = b_0 + b_1 \tau,$$

where

$$\tau = \int_0^x \kappa \rho \, dx$$

and where the variations of the coefficients $b_0$ and $b_1$ over the range of frequencies considered may be neglected.

* Similar work has been done by Eddington (6) using a Milne-Eddington approximation and for a particular type of variation of $\eta$ with optical depth.
Throughout the photosphere it is assumed, in addition, that
(iii) the ratio $\eta_v$ of the line scattering coefficient to $\kappa$ is independent of $\tau$;
(iv) the scattering in the line frequencies is isotropic, and also coherent, i.e. the frequency re-emitted within the line depends only on the radiation absorbed at the same frequency;
(v) the coefficient $\epsilon$, introduced to allow for thermal emission within the line, is independent of both $\nu$ and $\tau$.

Let $\tau = \tau_1$ when $x = x_1$ (see Fig. 1). In the atmosphere, ($0 \leq \tau \leq \tau_1$), we denote the intensity in the line considered by $I_s(\tau, \mu)$, and in the photosphere, ($\tau > \tau_1$), we denote it by $I_s(\tau, \mu)$, a prime indicating the photosphere. In the atmosphere the intensity $I_s(\tau, \mu)$, at an optical depth $\tau$ and at an angle $\cos^{-1}\mu$ with the outwards drawn normal, satisfies the transfer equation
$$\mu(d/d\tau)I_s(\tau, \mu) = I_s(\tau, \mu) - (b_0 + b_1 \tau)$$
with the boundary conditions
$$I_s(0, -\mu) = 0 \quad (0 \leq \mu \leq 1), \quad (2.2)$$
$$I_s(\tau_1, +\mu) = I_s(\tau_1, -\mu) \quad (0 \leq \mu \leq 1). \quad (2.3)$$

Now consider the radiation in the photosphere in the line frequencies. Here the intensity $I_s(\tau, \mu)$ satisfies the transfer equation
$$\mu(d/d\tau)I_s(\tau, \mu) = (1 + \eta_v)I_s(\tau, \mu) - \frac{1}{2}(1 - \epsilon)\eta_v \int_{-1}^{1} I_s(\tau, \mu')d\mu' - (1 + \epsilon\eta_v)(b_0 + b_1 \tau)$$
and the boundary condition
$$I_s(\tau_1, -\mu) = I_s(\tau_1, +\mu) \quad (0 \leq \mu \leq 1). \quad (2.5)$$

If we write
$$t_s = (1 + \eta_v)\tau, \quad t_1 = (1 + \eta_v)\tau_1, \quad (2.6)$$
$$\lambda_s = (1 + \epsilon\eta_v)/(1 + \eta_v), \quad \eta_v = 1/(1 + \eta_v), \quad (2.7)$$
and if we replace $I_s(\tau, \mu)$ by $I_s(t, \mu)$, then (2.4) becomes
$$\mu(d/dt_s)I_s(t, \mu) = I_s(t, \mu) - \frac{1}{2}(1 - \lambda_v)\int_{-1}^{1} I_s(t, \mu')d\mu' - \lambda_v(b_0 + b_1 n_s t_s)$$
and the boundary conditions (2.2), (2.3) and (2.5) become
$$I_s(0, -\mu) = 0 \quad (0 \leq \mu \leq 1), \quad (2.9)$$
$$I_s(t_1, +\mu) = I_s(t_1, -\mu) \quad (0 \leq \mu \leq 1), \quad (2.10)$$
$$I_s(t_1, -\mu) = I_s(t_1, +\mu) \quad (0 \leq \mu \leq 1). \quad (2.11)$$

We also need a condition limiting the rate of increase of $I_s(t, \mu)$ for large $t$. We make this rate so small that the homogeneous Milne equation will have no non-null solution. (This restriction is in fact necessary to obtain a result which satisfies the physical conditions of the problem. For if we take the non-null solution we would obtain a source function which was exponentially large for large $t$, and therefore a temperature which increased exponentially for large $t$.)

Using the condition (2.2) we first solve (2.1) for $I_s(\tau, -\mu)$ and obtain the following expression
$$I_s(\tau, -\mu) = b_0 + b_1(\tau - \mu) - (b_0 - b_1\mu) \exp(-\tau/\mu) \quad (0 \leq \mu \leq 1), \quad (2.12)$$
and then the boundary condition (2.11) becomes
\[
I'(t, -\mu) = b_0 + b_1(\tau_1 - \mu) - (b_0 - b_1\mu) \exp\left(-\frac{\tau_1}{\mu}\right)
\]
\[
= b_0 + b_1(n_1 t_1 - \mu) - (b_0 - b_1\mu) \exp\left(-n_1 t_1/\mu\right) \quad (0 \leq \mu \leq 1).
\]

(2.13)

We now solve (2.8) subject to (2.13). Write
\[
\mathcal{F}'(t_v) = \frac{1}{2}(1 - \lambda_v) \int_{-1}^{+1} I'(t, \mu) d\mu' + \lambda_v (b_0 + b_1 n_t);
\]
then, dropping the subscript \(v\) for the time being, (2.8) is
\[
\frac{d}{dt} I'(t, \mu) - \frac{1}{\mu} I'(t, \mu) = -\frac{1}{\mu} \mathcal{F}'(t),
\]

(2.15)

giving, for \(0 \leq \mu \leq 1\),
\[
I'(t, +\mu) \exp\left(-\frac{t}{\mu}\right) = \int_t^\infty \mathcal{F}'(x) \exp\left(-x/\mu\mu^{-1} x\right) dx.
\]

(2.16)

(This assumes \(I'(t, +\mu) \exp\left(-\frac{t}{\mu}\right) \rightarrow 0 \) as \(t \rightarrow \infty\).)

When \(\mu < 0\), write \(-\mu\) for \(\mu\) in (2.15). Then, for \(0 \leq \mu \leq 1\),
\[
I'(t, -\mu) \exp\left(\frac{t}{\mu}\right) - I'(t_1, -\mu) \exp\left(\frac{t_1}{\mu}\right) = \int_{t_1}^t \mathcal{F}'(x) \exp\left(x/\mu\mu^{-1} x\right) dx,
\]

(2.17)

where \(I'(t_1, -\mu)\) is given by (2.13). From (2.16), (2.17) and (2.14) we obtain
\[
\mathcal{F}'(t) = \frac{1}{2}(1 - \lambda) \int_{-1}^{+1} I'(t, \mu') d\mu' + \lambda (b_0 + b_1 n_t)
\]
\[
= \frac{1}{2}(1 - \lambda) \left\{ \int_0^1 I'(t, +\mu') d\mu' + \int_0^1 I'(t, -\mu') d\mu' \right\} + \lambda (b_0 + b_1 n_t)
\]
\[
= \frac{1}{2}(1 - \lambda) \int_0^1 d\mu' \int_t^\infty \mathcal{F}'(x) \exp\left[-(x-t)/\mu'\mu^{-1} x\right] dx
\]
\[
+ \frac{1}{2}(1 - \lambda) \int_0^1 d\mu' \int_{t_1}^t \mathcal{F}'(x) \exp\left[-(t-x)/\mu'\mu^{-1} x\right] dx
\]
\[
+ \frac{1}{2}(1 - \lambda) \int_0^1 I'(t_1, -\mu') \exp\left[-(t-t_1)/\mu'\mu^{-1} x\right] d\mu' + \lambda (b_0 + b_1 n_t)
\]
i.e.
\[
\mathcal{F}'(t) = \frac{1}{2}(1 - \lambda) \int_t^\infty \mathcal{F}'(x) dx \int_1^\infty \exp\left[-u(x-t)\right] u^{-1} du
\]
\[
+ \frac{1}{2}(1 - \lambda) \int_{t_1}^t \mathcal{F}'(x) dx \int_1^\infty \exp\left[-u(t-x)\right] u^{-1} du + B(t),
\]

(2.18)

where \(u = t/\mu'\) and
\[
B(t) = \frac{1}{2}(1 - \lambda) \int_0^1 I'(t_1, -\mu') \exp\left[-(t-t_1)/\mu'\mu^{-1} x\right] d\mu' + \lambda (b_0 + b_1 n_t).
\]

(2.19)

Equation (2.18) is
\[
\mathcal{F}'(t) = \frac{1}{2}(1 - \lambda) \int_t^\infty \mathcal{F}'(x) E_1(x-t) dx + \frac{1}{2}(1 - \lambda) \int_{t_1}^t \mathcal{F}'(x) E_1(t-x) dx + B(t),
\]

(2.20)

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where
\[ E_n(x) = \int_{1}^{\infty} x^{-n} \exp(-sx) \, dx. \]

The right hand side of (2.20) is not quite
\[ (1-\lambda) \Lambda_n[\mathcal{F}(x)] + B(t), \]
where
\[ \Lambda_n[J(t)] \equiv \frac{1}{2} \int_{0}^{\infty} J(t) E_1(|t-\tau|) \, dt. \]

Write \( t = t_1 + y \); \( x = t_1 + u \), so that \( u = y \) when \( x = t \).

Then (2.20) is
\[ \mathcal{F}'(t_1 + y) = \frac{1}{2}(1-\lambda) \int_{y}^{\infty} \mathcal{F}'(t_1 + u) E_1(u-y) \, du \]
\[ + \frac{1}{2}(1-\lambda) \int_{0}^{u} \mathcal{F}'(t_1 + u) E_1(y-u) \, du + B(t_1 + y) \]
and this is the Milne equation
\[ \mathcal{F}'(t_1 + y) = (1-\lambda) \Lambda_n[\mathcal{F}'(t_1 + u)] + B(t_1 + y), \] (2.21)
where, from (2.19) and (2.13),
\[ B(t) = \frac{1}{2}(1-\lambda) \int_{0}^{1} \{ b_0 + b_1 nt - (b_0 - b_1 \mu') \exp(-nt_1/\mu') \} \exp[-(t-t_1)/\mu'] d\mu' \]
\[ + \lambda(b_0 + b_1 nt) \]
\[ = \frac{1}{2}(1-\lambda) \{ (b_0 + b_1 nt_1) E_2(t-t_1) - b_1 E_3(t-t_1) - b_0 E_2(t-t_1 + nt_1) \]
\[ + b_1 E_3(t-t_1 + nt_1) \} + \lambda(b_0 + b_1 nt). \] (2.22)

Define \( B_1(y) \) by
\[ B_1(y) = B(t_1 + y). \] (2.23)

Then on putting \( t = t_1 + y \) in (2.22) we obtain
\[ B_1(y) = \frac{1}{2}(1-\lambda) \{ (b_0 + b_1 nt_1) E_2(y) - b_1 E_3(y) - b_0 E_2(y + nt_1) + b_1 E_3(y + nt_1) \}
\[ + \lambda(b_0 + b_1 nt_1 + b_1 ny). \] (2.24)

On writing \( \mathcal{F}'(t_1 + y) = F(y), \)
(2.21) becomes
\[ F(y) = (1-\lambda) \Lambda_y[F(u)] + B_1(y). \] (2.25)

From (2.16)
\[ F'(t_1, +\mu) = \int_{t_1}^{\infty} \mathcal{F}'(x) \exp[-(x-t_1)/\mu] \mu^{-1} \, dx, \]
and on writing \( t_1 + y = x \) this becomes
\[ F'(t_1, +\mu) = \int_{0}^{\infty} \mathcal{F}'(t_1 + y) \exp[-y/\mu] \mu^{-1} \, dy, \] (2.27)
i.e.
\[ F'(t_1, +\mu) = \int_{0}^{\infty} F(y) \exp[-y/\mu] \mu^{-1} \, dy. \] (2.28)

The next section will be devoted to finding results which are needed to find \( F'(t_1, +\mu) \) when \( F(\mu) \) is a solution of (2.26).
3. Preliminary mathematics.—We begin by solving the equation

\[(1 - \alpha \Lambda) \{F(t)\} = E_n(\tau + a)\]  

(3.1)

in which \(\alpha = 1 - \lambda\), \(\alpha > 0\) and \(n = 2, 3, \ldots\). First we consider the auxiliary equation [see (2), section 3]

\[(1 - \alpha \Lambda) \{F(t, \delta)\} = \exp(-\delta \tau).\]  

(3.2)

We can multiply by \(\exp(-a\delta)d\delta/d\delta^n\) and integrate with respect to \(\delta\) over \((1, \infty)\) and this gives

\[\int_1^\infty \exp(-a\delta) F(t, \delta) \delta^{-n} d\delta = \int_1^\infty \exp(-\delta(\tau + a)) \delta^{-n} d\delta = E_n(\tau + a).\]

Thus

\[F_1(t) = \int_1^\infty \exp(-a\delta) F(t, \delta) \delta^{-n} d\delta\]  

(3.3)

is a solution of (3.1). It is known [see (2), theorem iv] that

\[\mathcal{L}_{1\mu} \{F(t, \delta)\} = \int_1^\infty F(t, \delta) \exp(-t/\mu) \mu^{-1} dt = \frac{H(\mu)H(\delta^{-1})}{1 + \mu \delta},\]  

(3.4)

where \(H(\mu)\) is the solution of

\[H(\mu) = 1 + \frac{1}{\mu} H(\mu)\sigma \int_0^1 \frac{H(\xi)}{\xi + \mu} d\xi\]  

(3.5)

which is continuous for \(0 < \mu < 1\) [see (4), Chapter 2]. Hence, by (3.3)

\[I_1(t_1, +\mu) = \int_0^\infty F_1(y) \exp(-y/\mu) \mu^{-1} dy\]

\[= \int_1^\infty \exp(-y/\mu) \mu^{-1} dy \int_1^\infty F(y, \delta) \exp(-a\delta) \delta^{-n} d\delta\]

\[= \int_1^\infty \exp(-a\delta) \delta^{-n} d\delta \int_1^\infty F(y, \delta) \exp(-y/\mu) \mu^{-1} dy\]

and therefore, by (3.4),

\[I_1(t_1, +\mu) = \int_1^\infty \frac{H(\mu)H(\delta^{-1})}{1 + \mu \delta} \exp(-a\delta) \delta^{-n} d\delta.\]

On putting \(\delta = 1/\xi\), this becomes

\[I_1(t_1, +\mu) = H(\mu) \int_1^\infty \frac{\xi^{-n-1}}{\xi^2 + \mu} H(\xi) \exp(-a\xi) d\xi.\]  

(3.6)

In (2), theorem v (with \(\chi = 1 - \lambda\)), it is shown that the Laplace transform of the solution of

\[F_2(y) = (1 - \lambda) A_y \{F_2(t)\} + a_0 \tau + a_1\]  

(3.7)

is

\[I_2(t_1, +\mu) = \int_0^\infty F_2(y) \exp(-y/\mu) \mu^{-1} dy\]

\[= H(\mu)(\lambda^{-1} a_0 \mu + a_1) + \frac{1}{2} a_0 \lambda^{-1} (1 - \lambda) a_1,\]  

(3.8)
where

\[ \alpha_n = \int_0^1 H(\xi) \xi^n d\xi. \quad (3.9) \]

4. The solution for the emergent intensity.—The results of Section 3 can now be used to find the Laplace transform \( I'(t_1, +\mu) \) of the solution of (2.26) with \( B_1(y) \) given by (2.24). As the operator \( \Lambda_r \) is linear, this solution is

\[
I'(t_1, +\mu) = \frac{1}{2}(1 - \lambda) \left\{ \left( b_0 + b_1 n t_1 \right) H(\mu) \int_0^1 \frac{\xi}{\xi + \mu} H(\xi) d\xi \\
- b_1 H(\mu) \int_0^1 \frac{\xi^2}{\xi + \mu} H(\xi) d\xi \\
- b_0 H(\mu) \int_0^1 \frac{\xi}{\xi + \mu} H(\xi) \exp\left( -n t_1/\xi \right) d\xi \\
+ b_1 H(\mu) \int_0^1 \frac{\xi^2}{\xi + \mu} H(\xi) \exp\left( -n t_1/\xi \right) d\xi \right\} \\
+ H(\mu) \left\{ \lambda^{1/2} (b_1 n \mu + b_0 + b_1 n t_1) + \frac{1}{2}(1 - \lambda) \alpha b_1 n \right\}. \quad (4.1) \]

By using (3.5) and the relations

\[ \sigma = 1 - \lambda, \quad \frac{1}{2}(1 - \lambda) \alpha_0 = 1 - \lambda^{1/2} \]

[see (2), equation (4.15)] the following expressions are found which simplify equation (4.1):

\[
H(\mu) \int_0^1 \frac{\xi}{\xi + \mu} H(\xi) d\xi = H(\mu) \int_0^1 \frac{\xi + \mu}{\xi} H(\xi) d\xi \\
= H(\mu) \alpha_0 - 2(1 - \lambda)^{-1} [H(\mu) - 1] \\
= 2(1 - \lambda)^{-1} [1 - \lambda^{1/2} H(\mu)],
\]

\[
H(\mu) \int_0^1 \frac{\xi^2}{\xi + \mu} H(\xi) d\xi = H(\mu) \int_0^1 \frac{\xi(\xi + \mu)}{\xi} H(\xi) d\xi - \frac{\mu}{\xi + \mu} H(\xi) d\xi \\
= H(\mu) \alpha - 2\mu (1 - \lambda)^{-1} [1 - \lambda^{1/2} H(\mu)].
\]

On substituting these in (4.1) and writing \( \tau_1 = n t_1 \) and

\[
S(\tau_1, \mu) = \frac{1}{2}(1 - \lambda) H(\mu) \left\{ b_0 \int_0^1 \frac{\xi}{\xi + \mu} H(\xi) \exp\left( -\tau_1/\xi \right) d\xi \\
- b_1 \int_0^1 \frac{\xi^2}{\xi + \mu} H(\xi) \exp\left( -\tau_1/\xi \right) d\xi \right\}, \quad (4.2)
\]

we obtain

\[
I'(\tau_1, +\mu) = (b_0 + b_1 \tau_1) [1 - \lambda^{1/2} H(\mu)] - \frac{1}{2}(1 - \lambda) b_1 \alpha_1 H(\mu) \\
+ b_1 \mu [1 - \lambda^{1/2} H(\mu)] + \lambda^{1/2} (b_1 n \mu + b_0 + b_1 \tau_1) H(\mu) \\
+ \frac{1}{2}(1 - \lambda) \alpha b_1 n H(\mu) - S(\tau_1, \mu),
\]

i.e.,

\[
I'(\tau_1, +\mu) = (b_0 + b_1 \tau_1 + b_1 \mu) - \lambda^{1/2} b_1 \mu (\lambda - n) H(\mu) \\
- \frac{1}{2}(1 - \lambda) b_1 \alpha_1 (\lambda - n) H(\mu) - S(\tau_1, \mu). \quad (4.3)
\]
The solution of (2.1) subject to (2.3) is

\[ I(\omega, +\mu) = \exp(-\tau_1/\mu) I'(\tau_1, +\mu) + \mu^{-1} \int_0^\tau_1 \exp(\mu b_0 + b_1 \tau) d\tau \]

\[ = \exp(-\tau_1/\mu) I'(\tau_1, +\mu) + b_0 + b_1 \mu \]

\[ - \exp(-\tau_1/\mu) (b_0 + b_1 \tau + b_1 \mu) \]

\[ \text{(4.4)} \]

where \( I'(\tau_1, +\mu) \) is given by (4.3).

For emphasis \( I(\omega, +\mu) \) will be written as \( I_{\tau, \nu}(\omega, +\mu) \), and the corresponding quantity in the standard case in which there is no non-selectively absorbing atmosphere will be written as \( I_{0, \nu}(\omega, +\mu) \). The intensity in the continuum (which is the same in both cases) will be denoted by \( I^*(\omega, +\mu) \). Let \( r_{\tau, \nu} \) and \( r_{0, \nu} \) be respectively the percentage residual intensities for the veiled line and for the line in the standard case.

Then

\[ r_{\tau, \nu} - r_{0, \nu} = \frac{100[I_{\tau, \nu}(\omega, +\mu) - I_{0, \nu}(\omega, +\mu)]}{I^*(\omega, +\mu)} \]

\[ \text{(4.5)} \]

The emergent intensity in the standard case in which there is no non-selectively absorbing atmosphere is given by *

\[ I_{0, \nu}(\omega, +\mu) = H(\mu) \{\lambda^{1/2}(b_0 + b_1 \eta \mu) + \frac{1}{2}(1-\lambda) \alpha_2 \beta_2 \mu \} \]

Hence, from (4.5), (4.4) and (4.3),

\[ \frac{I^*(\omega, +\mu)}{100} [r_{\tau, \nu} - r_{0, \nu}] = \exp(-\tau_1/\mu)(b_0 + b_1 \tau + b_1 \mu - \lambda^{1/2} b_1 \mu (1 - n) H(\mu)) \]

\[ - \frac{1}{2}(1-\lambda) b_1 \alpha_1 (1-n) H(\mu) - S(\tau_1, \mu)] + b_0 + b_1 \mu \]

\[ - \exp(-\tau_1/\mu)(b_0 + b_1 \tau + b_1 \mu) \]

\[ - \lambda^{1/2}(b_0 + b_1 \eta \mu) H(\mu) - \frac{1}{2}(1-\lambda) \alpha_2 \beta_2 \mu \]

\[ \text{Hence from (2.4) we get} \]

\[ \frac{I^*(\omega, +\mu)}{100} [r_{\tau, \nu} - r_{0, \nu}] = - \exp(-\tau_1/\mu) H(\mu) \{\lambda^{1/2}(b_0 + b_1 \mu + \frac{1}{2} b_1 (1-\lambda) \alpha_2 \}

\[ - H(\mu) [1 - \exp(-\tau_1/\mu)] \}

\[ \begin{aligned}
  &+ \frac{1}{2} (1-\lambda) b_1 \eta \alpha_1 + b_0 + b_1 \mu \\
  &- \frac{1}{2} (1-\lambda) \exp(-\tau_1/\mu) H(\mu) \left\{ b_0 \int_0^1 \frac{\xi}{\xi + \mu} H(\xi) \exp(-\tau_1/\xi) d\xi \right\} \\
  &- b_1 \int_0^1 \frac{\xi^2}{\xi^2 + \mu} H(\xi) \exp(-\tau_1/\xi) d\xi \right\}
\end{aligned} \]

\[ \text{(4.6)} \]

where \( I^*(\omega, +\mu) = b_0 + b_1 \mu \) [see (5)]†.

* See (5) Section 84, equation (66). This is also obtained from (4.1) on putting \( \tau_1 = \infty \).

† The value of \( I^*(\omega, +\mu) \) can also be obtained by letting \( \tau_1 \to \infty \) in (4.4), viz. \( I^*(\omega, +\mu) = b_0 + b_1 \mu \).
In the case of lines formed by pure absorption, \((\epsilon = 1)\), (4.6) can be reduced to a much simpler expression.

On putting \(\epsilon = 1\) in (2.7) this gives \(\lambda = 1\) and, from (3.5) with \(w = 1 - \lambda = 0\), we see that

\[
H(\mu) = 1.
\]

Hence (4.6) reduces to

\[
\frac{I(\alpha + \mu)}{100} \left[ r_{\nu, \nu} - r_{\nu, \nu} \right] = b_1 \mu \left[ 1 - \exp \left( -\tau_1/\mu \right) \right] (1 - n), \tag{4.7}
\]

and the expression for the percentage residual intensity in the veiled line becomes

\[
\frac{I(\alpha + \mu)}{100} r_{\nu, \nu} = b_0 + b_1 \mu (1 - (1 - n) \exp \left( -\tau_1/\mu \right)). \tag{4.8}
\]

5. Numerical results.—A plane parallel model is not a good representation near the limb and numerical results are therefore obtained only for \(0.2 \leq \mu \leq 1\). However it is interesting to note that in every case

\[
r_{\nu, \nu} \frac{I(\alpha + \mu)}{100} \rightarrow (b_0 + b_1 \mu)_{\mu=0} \text{ as } \mu \rightarrow 0.
\]

Since this is \(I(\alpha + \mu)_{\mu=0}\), the lines fade out completely at the limb.

As any line is formed by a mixture of absorption and scattering it was decided to consider two extreme cases, one in which the lines are formed by pure scattering \((\epsilon = 0)\), and the other in which the lines are formed by pure absorption \((\epsilon = 1)\). In both these cases values of \(r_{\nu, \nu} - r_{\nu, \nu} \) and \(r_{\nu, \nu} \) have been calculated for various values of \(\eta_\nu\) and for two thicknesses of the atmosphere, using the new tables of \(H\) functions prepared by Stibbs and Weir [see (10)]. In the former case the electronic computer at the University Computing Laboratory was used in obtaining numerical results and my thanks are due to the Director for allowing use of these facilities. The results are given in Tables I–IV.

**Table I**

The veiling effect of the atmosphere \((\epsilon = 0)\)

<table>
<thead>
<tr>
<th>(\eta)</th>
<th>(\mu)</th>
<th>0.9</th>
<th>0.8</th>
<th>0.7</th>
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<th>0.5</th>
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<th>0.3</th>
<th>0.2</th>
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### Table II

The centre to limb variation of the veiled lines ($\epsilon = 0$)

<table>
<thead>
<tr>
<th>$\eta$</th>
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<th>$\circ$</th>
<th>$\circ$</th>
<th>$\circ$</th>
<th>$\circ$</th>
<th>$\circ$</th>
<th>$\circ$</th>
<th>$r_{0,9,7}$</th>
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</thead>
<tbody>
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<td>75.10</td>
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### Table III

The veiling effect of the atmosphere ($\epsilon = 1$)

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<th>$\mu$</th>
<th>$\circ$</th>
<th>$\circ$</th>
<th>$\circ$</th>
<th>$\circ$</th>
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### Table IV

The centre to limb variation of the veiled lines ($\epsilon = 1$)

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$\mu$</th>
<th>$\circ$</th>
<th>$\circ$</th>
<th>$\circ$</th>
<th>$\circ$</th>
<th>$\circ$</th>
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<th>$r_{0,9,7}$</th>
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</table>

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6. Variation of $\eta_\nu$ with optical depth.—Since the objection may be raised that the variation of $\eta_\nu$ with optical depth has been neglected, a method will now be developed which will take into account this variation in the case of lines formed by pure absorption ($\epsilon = 1$).

The same notation will be used as in Section 2, except that $\eta_\nu$ will be denoted by $\eta_\nu(\tau)$ as it is a function of optical depth. The problem will be solved subject to conditions (i), (ii), (iv) and (v) of Section 2.

In this case equation (2.4) becomes (with $\epsilon = 1$)

$$\mu (d/d\tau) I'(\tau, \mu) = [1 + \eta_\nu(\tau)] I'(\tau, \mu) - [1 + \eta_\nu(\tau)][b_0 + b_1 \tau]$$

(6.1)

and from (2.1) and (2.10) the expression for the emergent intensity is

$$I_{\nu, \nu} (0, + \mu) = I'(\tau_1, + \mu) \exp (-\tau_1/\mu) + b_0 + b_1 \mu - (b_0 + b_1 \tau_1 + b_1 \mu) \exp (-\tau_1/\mu) \ (0 \leq \mu \leq 1).$$

(6.2)

Equation (6.1) has the integrating factor

$$\exp \left\{ - \int_{\tau_1}^\tau [1 + \eta_\nu(t)] \mu^{-1} dt \right\},$$

(6.3)

and on writing

$$n_\nu(\tau) = 1/[1 + \eta_\nu(\tau)], \quad y(\tau) = \int_{\tau_1}^\tau [n_\nu(t)]^{-1} dt,$$

(6.4)

(6.3) becomes

$$\exp \left[ - y(\tau)/\mu \right].$$

(6.5)

On multiplying (6.1) by $\exp [-y(\tau)/\mu]$ and integrating with respect to $\tau$ over $(\tau, \infty)$, the following expression is obtained when $0 \leq \mu \leq 1$:

$$\mu I'(\tau, + \mu) \exp [-y(\tau)/\mu] = \int_{\tau_1}^\infty (b_0 + b_1 t) [1 + \eta_\nu(t)] \exp [-y(t)/\mu] dt.$$

(6.6)

[This assumes $I'(\tau, + \mu)$ does not increase exponentially for large $\tau$.]

Since $\eta_\nu(\tau) > 0$, therefore

$$1/n_\nu(\tau) = 1 + \eta_\nu(\tau) \geq 1$$

and hence, for $\tau \geq \tau_1$,

$$y(\tau) = \int_{\tau_1}^\tau [n_\nu(t)]^{-1} dt \geq \tau - \tau_1.$$

(6.7)

Thus $y(\tau_1) = 0$ and $y(\tau) \to \infty$ as $\tau \to \infty$. On putting $\tau = \tau_1$ in (6.6) this becomes

$$\mu I'(\tau_1, + \mu) = \int_{\tau_1}^\infty (b_0 + b_1 t) [1 + \eta_\nu(t)] \exp [-y(t)/\mu] dt.$$

(6.8)

This expression can be simplified since, from (6.4),

$$(d/d\tau) y(\tau) = 1/n_\nu(\tau) = 1 + \eta_\nu(\tau),$$

(6.9)

and therefore

$$(d/d\tau) \exp \left\{- y(\tau)/\mu \right\} = - \mu^{-1} [1 + \eta_\nu(\tau)] \exp \left\{- y(\tau)/\mu \right\}.$$

(6.10)

Also, from (6.7),

$$(b_0 + b_1 \tau) \exp \left\{- y(\tau)/\mu \right\} \to 0 \quad (\tau \to \infty).$$

22
Hence, on integrating (6.8) by parts, we obtain
\[
\mu I'(\tau_1, + \mu) = \left[-\mu(b_0 + b_1 t) \exp\left(-y(t)/\mu\right)\right]_{\tau_1}^{\infty} \\
+ \mu b_1 \int_{\tau_1}^{\infty} \exp\left[-y(t)/\mu\right] dt \\
= \mu \left(b_0 + b_1 \tau_1\right) + \mu b_1 \int_{\tau_1}^{\infty} \exp\left[-y(t)/\mu\right] dt. 
\]  
(6.11)

Then from (6.2) the emergent intensity is given by
\[
I_{\tau_1,\nu}(0, + \mu) = \left(b_0 + b_1 \tau_1\right) \exp\left(-\tau_1/\mu\right) \\
+ b_1 \int_{\tau_1}^{\infty} \exp\left[-y(t)/\mu\right] dt \\
+ b_0 + b_1 \mu - \left(b_0 + b_1 \tau_1 + b_1 \mu\right) \exp\left(-\tau_1/\mu\right) \\
= b_0 + b_1 \mu \left[1 - \exp\left(-\tau_1/\mu\right)\right] \\
+ b_1 \int_{\tau_1}^{\infty} \exp\left[-y(t)/\mu\right] dt. 
\]  
(6.12)

The expression (6.12) is an exact solution for the emergent intensity in lines formed by pure absorption, and no particular law of variation of \(\eta_r\) with optical depth is presupposed.

Numerical values for the percentage residual intensity were obtained in the case in which the photosphere is divided into two layers, in the upper of which \(\eta_r(\tau)\) was a non-zero constant and in the lower \(\eta_r(\tau)\) was zero. The values chosen for \(\eta_r(\tau)\) in the upper layer were some of those used in the first part of this paper, viz. 1, 4, 9 and 19, and for these the optical thicknesses of the layer were taken to be 0.5, 0.2, 0.1 and 0.05 respectively. The values 0.2 and 0.3 were again used for \(\tau_1\). By comparing the results thus obtained with similar ones obtained by I. W. Busbridge for \(\tau_1 = 0\) (see (3)), values for the veiling effect of the atmosphere were also obtained. Since, however, all my results for this case are similar to those obtained previously (see Section 5), it seems unnecessary to insert tables of them.

7. Conclusions.—In the previous sections formulae have been obtained which can be used to find the percentage residual intensity in certain cases. It now remains to compare the results, so obtained, with relevant observational data and to show how a veiling atmosphere would affect a line profile*. In Fig. 2 the variation of the percentage residual intensity for \(\eta = 1\) is compared with observational data for weak lines obtained by M. G. Adam for faint Fraunhofer lines (x). The full line shows the variation of the percentage residual intensity from centre to limb for \(\epsilon = 0\) and \(\tau_1 = 0.3\) obtained from the formulae of Section 4. The scattered points show the values for the central intensity (expressed as a percentage) obtained from Miss Adam’s work. The filled circles are for the line \(\lambda 5102.975\), the triangles for \(\lambda 5145.470\) and the crosses for \(\lambda 5147.484\) (these lines have respectively Rowland Intensities of 1, 0, 0), where for each line the mean of the results from three plates is shown. These lines are weak and are chosen because they give points quite close to the full curve at \(\mu = 1\).

* For observational work on line profiles see, for example, (7) and (8). Computations have been made by de Jager and Neven (9a) for solar lines of N I and O I. Working from curves of growth for certain lines of N I and O I, computed with the aid of the Pecker saturation function, they showed that large errors are introduced by the usual weak line approximation.
In this paper the parameter $\eta_r$ has been used to denote the ratio of the line absorption coefficient to the continuous absorption coefficient. For a line produced by a certain atom, therefore, $\eta_r$ is the ratio of the atomic absorption coefficient to the continuous absorption coefficient. As we can take the variation with frequency of the continuous absorption coefficient over a particular line to be negligible, the coefficient $\eta_r$ varies as the atomic absorption coefficient over the line. In the centre of the line (Doppler core), therefore, $\eta_r$ is proportional to $\exp\left(-\Delta\lambda^2/\Delta\lambda_D^2\right)$ and in the wings (damping wing) $\eta_r$ is proportional to $1/\Delta\lambda^3$, where $\Delta\lambda$ is the distance from the centre of the line in units of wavelength and $\Delta\lambda_D$ is the Doppler half width (i.e. $\Delta\lambda_D$ is the distance from the centre of the line to the point at which the intensity has $e^{-1}$ of its value at the centre of the line).

In Fig. 3 the line profiles were drawn using $\eta = 399 \exp\left(-v^2\right)$ for the central part of the line and $\eta = 9/v^2$ for the wings, and those in Fig. 4 using $\eta = 19 \exp\left(-v^2\right)$ for the central part of the line and $\eta = 2.25/v^2$ for the wings, where $v = \Delta\lambda/\Delta\lambda_D$. In each case $\eta$ is taken to be independent of depth in the photosphere. These figures show how the line profile of a typical Fraunhofer line varies from centre to limb under certain circumstances. In each figure, the line profile at the centre of the Sun ($\mu = 1$) and at the limb ($\mu = 0.2$) are shown. Fig. 3 shows the case in which the lines are formed by pure scattering ($\epsilon = 0$). In Fig. 3(a) there is a
Fig. 3.—Line profiles at the centre ($\mu=1$) and at the limb ($\mu=0.2$) of a typical Fraunhofer line for the case in which the line is formed by pure scattering ($\varepsilon=0$). In Fig. 3(a) there is a veiling atmosphere of optical thickness 0.3 and in Fig. 3(b) there is no such atmosphere. In each case the distance from the centre of the line is measured in units of $\Delta \lambda / \Delta \lambda_D$, and the ordinates represent percentage residual intensity.

veiling atmosphere of optical thickness 0.3 and in Fig. 3(b), which is shown for comparison, there is no such veiling atmosphere. Fig. 4 shows the case in which the lines are formed by pure absorption ($\varepsilon=1$). In Fig. 4(a) there is a veiling atmosphere of optical thickness 0.3 and in Fig. 4(b), which is shown for comparison, there is no such veiling atmosphere.
FIG. 4.—Line profiles at the centre ($\mu=1$) and at the limb ($\mu=0.2$) of a typical Fraunhofer line for the case in which the line is formed by pure absorption ($\epsilon=1$). In Fig. 4(a) there is a veiling atmosphere of optical thickness 0.3 and in Fig. 4(b) there is no such atmosphere. In each case the distance from the centre of the line is measured in units of $\Delta \lambda/\Delta \lambda_D$, and the ordinates represent percentage residual intensity.
These figures show the behaviour of the line profiles in two extreme cases, viz.
lines formed by pure absorption and lines formed by pure scattering. As there is
some similarity between Figs. 3 (a) and 4 (a), it would not be difficult to visualize
the variation of the line profile of a veiled line which is formed by a mixture of
scattering and absorption.

In conclusion, I should like to thank Dr I. W. Busbridge for all her help and
encouragement, and also Professor H. H. Plaskett who made several very helpful
suggestions, Professor D. W. N. Stibbs who advised me about the best method
of integrating expressions containing $H$ functions, Dr W. R. Hindmarsh for very
useful discussions on the physics of absorption lines, and the Department of
Scientific and Industrial Research for awards, during the tenure of which this
work has been carried out.

University Observatory,
Oxford;
1960 September.

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