THE STABILITY AND VIBRATIONS OF A GAS OF STARS

D. Lynden-Bell

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Summary

The initial value problem for a small disturbance from an equilibrium of an encounterless stellar system is discussed. In a spherical cluster small wave-length disturbances damp as the stars become well mixed. In the artificial infinite uniform rotating system we discuss disturbances of all wave-lengths. Jeans' condensational instability is the same as in a collision dominated gas. Landau damping occurs for wave-lengths smaller than the critical one.

Introduction.—Any approach to equilibrium involves some randomizing process. This process is usually so complicated that it is incalculable and a probability calculation is made instead. Justification for this replacement of a determined problem by one governed by chance lies in the fact that a very special “randomizing” law is required to invalidate the probability calculation. We have an unproven belief that we can always spot such special laws and label them “non-randomizing.”

Consider a system of many non-interacting particles bound in a given one-dimensional potential well. Particles of different energies have in general different periods of oscillation. Thus in time the particles become well mixed so that the assumption that they are equally likely to be at any phases of their oscillations becomes a sufficient approximation in the following sense. The value of any approximation depends on what one wishes to calculate with it; provided that we only wish to calculate properties of the system defined in terms of averages of functions which are smooth and continuous in phase space (e.g. “thermodynamic” parameters) the above approximation is valid. Thus the macroscopic characteristics of such a system approach some equilibrium although it is not claimed that there is any energy exchange or Maxwellization of the distribution function. The approach to equilibrium of an encounterless stellar system is analogous to that just described. Knowing now in what sense the system is expected to achieve an equilibrium we turn to discuss the mathematics of this process.

Notation and equations of the problem.—We use

\[ \mathbf{r} = (x, y, z) \] as our position coordinate,

\[ \mathbf{c} = (u, v, w) \] as our velocity coordinate,

\[ \mathbf{\omega} = (\omega_x, \omega_y, \omega_z) = \mathbf{r} \times \mathbf{c}, \]

\[ \nabla = \frac{\partial}{\partial \mathbf{r}} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \]

\[ \frac{\partial}{\partial \mathbf{c}} = \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w} \right). \]
$f(r, c) d^3r d^3c$ is the mass of stars at $r \rightarrow r + d r$ moving with velocities in the range $c \rightarrow c + d c$. The equation of continuity (in this case mass conservation) is

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + c \cdot \frac{\partial f}{\partial r} + \frac{\partial \psi}{\partial r} \cdot \frac{\partial f}{\partial c} = 0 \tag{1}$$

where we have used Newton's law to express the acceleration in terms of the gravitational potential $\psi$. We ignore star-star interactions and assume that each star moves independently in the field of the whole system. Then $\psi$ varies smoothly and the Poisson equation

$$\nabla^2 \psi = -4\pi \rho = -4\pi \gamma \int f \, d^3c \tag{2}$$

demands that the gravitational field arises from the stars themselves.

Consider for example a steady spherical system; it is readily checked that $f= f_0 (c^2/2 - \psi_0, \varpi)$ is a solution of (1) for the steady spherical potential $\psi_0$.

We shall discuss small perturbations from some such equilibrium state. We write $f= f_0 + f_1$, $\psi= \psi_0 + \psi_1$, and linearize (1) to obtain

$$\frac{\partial f_1}{\partial t} + c \cdot \frac{\partial f_1}{\partial r} + \frac{\partial \psi_0}{\partial r} \cdot \frac{\partial f_1}{\partial c} + \frac{\partial \psi_1}{\partial r} \cdot \frac{\partial f_0}{\partial c} = 0 \tag{3}$$

while from (2)

$$\nabla^2 \psi_1 = -4\pi \gamma \int f_1 \, d^3c. \tag{4}$$

For a special cylindrical system we shall solve (3) and (4) exactly. However, we first discuss small wave-length perturbations in a spherical system. We look at the orders of magnitude of the terms in (3). Writing $\lambda_1, c_1$ for the typical wave-length and velocity of a disturbance, $R_0$ for the radius of the cluster and $c_0$ for its typical velocity we find that the last three terms are in the ratios

$$\frac{1}{\lambda_1^2} \frac{c_0^2}{R_0^2} : \frac{\lambda_1^2 c_1^2}{R_0^2 c_0^2} : \frac{\lambda_1^2 c_1^2}{R_0^2 c_0^2}.$$  

Thus, provided the disturbed stars have not speeds much greater than the mean, the last term may be negligible for small-scale disturbances. To omit this term is to neglect the gravitational effects of the perturbation itself. We assume therefore that this field is never so strong that it can significantly bend a significant number of the orbits of the original system. Now even an infinitesimal disturbance of large enough scale can cause Jeans's instability (x) (which we shall show occurs in the collisionless gas just as in the collision-dominated gas) so we shall consider disturbances of a much smaller scale. It could be argued that any disturbance spreads around the cluster and that the wave-length is always of the same order as the size of the cluster. However this spreading time turns out to be as large as the time in which the density perturbation dies out. All that happens in this approximation that we adopt is a spreading of the perturbation along the orbits of stars in the undisturbed cluster. This mere propagation leads to a distribution function which converges in the mean to that of a well-mixed steady-state.
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Mathematical discussion of the mean convergence of the mixing process.—Neglecting the last term in (3) gives

$$\frac{\partial f_1}{\partial t} + c \cdot \frac{\partial f_1}{\partial r} + \frac{\partial \psi_0}{\partial r} \frac{\partial f_1}{\partial c} = 0.$$  \hspace{1cm} (5)

As Jeans (2) pointed out, the associated ordinary differential equations

$$\frac{dt}{I} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{du}{\partial \psi_0} = \frac{dw}{\partial \psi_0},$$

may be written more familiarly as $dx/dt = u$ etc. and $du/dt = \partial \psi_0 / \partial x$ etc. Then $c^2/2 - \psi_0 = \epsilon$ and $r \times c = \omega$ are four first integrals. The remaining two integrals* are the constants of integration in the equations which serve to determine fully the motion at given $c, \omega$:

$$\frac{r^2}{2} + \frac{\omega^2}{2r^2} - \psi_0(r) = \epsilon \quad \text{and} \quad r^2 \theta = \omega,$$  \hspace{1cm} (6)

where $\theta$ is the azimuthal angle in the plane of the motion of the star under consideration. From (6),

$$\left(\frac{dt}{dr}\right)^2 = \frac{I}{2(\epsilon + \psi_0) - \omega^2 r^{-2}},$$

$r(t)$ is periodic and this gives $t$ as an inverse periodic function of $r$. The period is given by

$$\tau(\epsilon, \omega) = 2 \int_{r_1}^{r_2} \frac{dr}{\sqrt{2(\epsilon + \psi_0) - \omega^2 r^{-2}}}.$$

where $r_1$ and $r_2$ are the values of $r$ at the apses, i.e., the roots for $r$ of

$$r^2 = 2(\epsilon + \psi_0) - \omega^2 r^{-2} = 0.$$  

It is convenient to use the phase of the oscillation as a coordinate. The phase $\phi$ is defined by

$$\phi(\epsilon, \omega, r) - \phi(\epsilon, \omega, r_0) = \frac{2\pi}{\tau} \int_{r_0}^{r} \frac{dr}{\pm \sqrt{2(\epsilon + \psi_0) - \omega^2 r^{-2}}} = \frac{2\pi t}{\tau}. \hspace{1cm} (7)$$

To define the zero of $\phi$ we take $\phi(\epsilon, \omega, r_1) = \pm 2n\pi$ (all $\epsilon, \omega$) where $n$ is an integer. We leave $\phi$ undefined up to a multiple of $2\pi$. We must decide which sign is to be taken for the integral in (7). We actually require that $\phi$ determine not only the value of $\dot{r}$ but also the sign of $\dot{r} = \epsilon r$. Then strictly $\phi$ is not just a function of $\epsilon, \omega, r$ but it also depends on the sign of $c \cdot r$. We take the signs $\pm$ in (7) according as $c \cdot r$ is $>0$ or $<0$. Then $0 < \phi < \tau \mod 2\pi$ if $c \cdot r > 0$.

The zero of $\theta$ gives trouble, as we cannot measure azimuths in the orbital planes in such a way that points close together in phase space always have small differences in azimuths in their respective orbits. We take some reference half-plane and define $\theta$ as the azimuthal angle from the intersection of the plane of the orbit with this reference half-plane. The sense of $\theta$ is taken so that $\dot{\theta} = |\dot{\omega}|$ rather than $-|\dot{\omega}|$. All this is no good as a definition of $\theta$ in the reference plane itself but we may neglect any orbits exactly in this plane without affecting the dynamics of the whole cluster.

* There must be six independent integrals and this is the simplest way of specifying two more independent of the four we have.

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Apart from the singularity associated only with orbits close to the reference plane for $\theta$, the values of $\varepsilon, \varpi, \phi, \theta$ form continuous and differentiable functions of $r, c$. They can be taken as a new complete independent set of coordinates in phase space. We re-express $f(r, c, t)$ in terms of them:

$$f(r, c, t) = F(\varepsilon, \varpi, \phi, \theta, t).$$

(8)

Our problem is to express $f(r, c, t)$ in terms of $f_0 = f(r, c, o)$ or equivalently to find $F$ in terms of $F_0 \ (\equiv F_{|t=0})$. Stars which initially had coordinates $\varepsilon, \varpi$ still have those coordinates. Stars which initially had phases $\phi$ now have phases $\phi + 2\pi t/\tau$ while azimuths $\theta$ have become

$$\theta + \int_0^t \frac{\varpi}{r^2} dt = \theta + \int_{r_0}^r \frac{\varpi}{r^2} \sqrt{2(\varepsilon + \psi_0)} - \varpi^2 r^{-2} dr.$$

Thus

$$F_0 \left( \varepsilon, \varpi, \phi - \frac{2\pi t}{\tau}, \theta - \int_0^t \frac{\varpi}{r^2} dt \right) d^3r_0 d^3c_0 = F(\varepsilon, \varpi, \phi, \theta, t) d^3r d^3c,$$

i.e.,

$$F = F_0 \left( \varepsilon, \varpi, \phi - \frac{2\pi t}{\tau}, \theta - \int_0^t \frac{\varpi}{r^2} dt \right) \frac{\partial(x_0 y_0 z_0 u_0 v_0 w_0)}{\partial(x, y, z, u, v, w)},$$

where the last factor is the Jacobian of the initial coordinates w.r.t. their values at time $t$. Since we have canonical coordinates $x, y, z, u, v, w$ and the system is conservative, this is 1 by Liouville's theorem. Thus

$$F = F_0 \left( \varepsilon, \varpi, \phi - \frac{2\pi t}{\tau}, \theta - \int_0^t \frac{\varpi}{r^2} dt \right)$$

is in a sense the solution to our problem.

Since $r, c$ are periodic in $\phi, \theta$, so will $F_0$ be (from (8)). Thus the above expression for $F$ tells us that it vibrates and will go on oscillating into the infinite future—just as the position of each star does. In this sense the system never settles down. However, what is meant by an equilibrium state in statistical mechanics is not a state in which exact measurement of any theoretically defined variable yields time-independent results but rather one in which measurements of the "macroscopic" variables have this property. Macroscopic variables are the averages over phase space of functions that are smoothly varying there. For our system such variables do tend to equilibrium values.

Let $Q(r, c)$ be such a smooth function giving a macroscopic variable

$$\overline{Q}(t) = \int Q d^3c d^3r.$$

$Q$ may of course be rewritten as some new smooth continuous function $\overline{Q}(\varepsilon, \varpi, \phi, \theta)$ over our new coordinates (excepting in the region where $\varpi$ is perpendicular to the $\theta$ reference plane, where it will vary rapidly)

$$\overline{Q}(t) = \int \int \int F_0 \left( \varepsilon, \varpi, \phi - \frac{2\pi t}{\tau}, \theta - \int_0^t \frac{\varpi}{r^2} dt \right) Q(\varepsilon, \varpi, \phi, \theta) d\varpi d\phi d\theta.$$

* e.g. the distribution function of the particles as it would have been 10 years ago may be expressed as a complicated function of the positions and velocities of the particles now. For a system started away from equilibrium it is clear that such a variable would take more than 10 years to relax.
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where the Jacobian

\[ J(\epsilon, \varpi) = \frac{\partial(x, y, z, u, v, w)}{\partial(\epsilon, \varpi, x, y, z, u)} \]

is independent of \( \theta \) and \( \phi \), since no values of these are preferred over any others in the geometry of phase space.

Since \( F_0 \) and \( Q \) are periodic in \( \theta \) and \( \phi \) of period \( 2\pi \) it is convenient to expand them in Fourier series

\[ F_0(\phi) = \sum_{n=-\infty}^{\infty} F_0^{(n)} e^{in\phi}, \]

therefore

\[ F_\ell \left( \phi - \frac{2\pi t}{\tau} \right) = \sum_{n=-\infty}^{\infty} F_\ell^{(n)} e^{in\phi} e^{-2\pi i n \tau}, \]

\[ Q = \sum_{m=-\infty}^{\infty} Q^{(m)} e^{im\phi}. \]

Now

\[ \int_0^{2\pi} \sum_{n=-\infty}^{\infty} F_0^{(n)} e^{in\phi} \sum_{m=-\infty}^{\infty} Q^{(m)} e^{im\phi} d\phi = 2\pi \sum_{n=-\infty}^{\infty} F_0^{(n)} Q^{(-n)}. \]

Thus

\[ \dot{Q}(t) = \sum_{-\infty}^{\infty} \int_0^{2\pi} \int_0^{2\pi} F_0^{(n)} Q^{(-n)} e^{-2\pi i n \tau} 2\pi J d\theta d\varpi d\epsilon. \]

Similarly, putting

\[ F_0^{(n)}(\theta) = \sum_{m=-\infty}^{\infty} F_0^{(m)}(m) e^{im\theta}, \]

\[ Q^{(-n)}(\theta) = \sum_{m=-\infty}^{\infty} Q^{(m)}(-n) e^{im\theta}, \]

we obtain

\[ \dot{Q}(t) = 4\pi^2 \sum_{m, n=-\infty}^{\infty} \int_0^{2\pi} \int_0^{2\pi} F_0^{(m)} Q^{(-n)}(-n) e^{-2\pi i n \tau} \exp \left[ -im \int_0^{\frac{t}{\tau}} \frac{\varpi}{r^2} dt \right] J d\varpi d\epsilon. \]

Now, except possibly when \( \varpi \) is perpendicular to the reference plane for \( \theta \), the coefficients \( F_0^{(m)} Q^{(-n)}(-n) J \) vary smoothly with \( \epsilon \) and \( \varpi \) whereas \( e^{-2\pi i n \tau} \) and

\[ \exp \left( i \int_0^{\frac{t}{\tau}} \frac{\varpi}{r^2} dt \right) \]

are violently fluctuating functions of \( \epsilon, \varpi \) whenever \( t \gg \tau \) and

\[ \int_0^{\frac{t}{\tau}} \frac{\varpi}{r^2} dt \gg 2\pi. \]

Thus by dividing up the range of integration into small regions in such a way that \( F_0^{(m)} Q^{(-n)}(-n) J \) is effectively constant over a region whereas the periodic terms have many fluctuations* we see that in each region the fluctuating terms may be replaced by their averages. These averages are all zero for \( n \neq 0 \) since both \( e^{-2\pi i n \tau} \) and

\[ \exp \left( -im \int_0^{\frac{t}{\tau}} \frac{\varpi}{r^2} dt \right) \]

*Note that if over some region \( \tau \) is independent of \( \epsilon \) and \( \varpi \) there can be no fluctuations of \( e^{2\pi i n \tau} \) in that region, however large \( t \) may be. (However, although all small oscillations have period independent of amplitude it is not in general independent of energy.)
have both real and imaginary parts equally positive and negative for large $t$
and small ranges in $\epsilon$ and $\varpi$. Thus for large $t \gg \tau$ and

$$\int_0^t \frac{\varpi}{r^2} dt \gg 2\pi$$

(10)

the only surviving term in (9) is that with $m = n = 0$, i.e.

$$Q(t) \rightarrow 4\pi^2 \int \int F_{00}^0 Q_{00}^0 J d^3\varpi de,$$

(11)

which is independent of time. We have still to deal with our mathematical
difficulty when $\varpi$ is perpendicular to the reference plane for $\theta$. Physically
it seems unlikely that this should change the argument; we shall dismiss it as
follows: Leave out the contribution to the integral when $\varpi$ lies close to one of
these "poles". Now choose a new reference plane for $\theta$ and evaluate just these
contributions—no such difficulties emerge.

Now $r/4\pi^2 F_{00}$ is the average of $F_0$ over the phases of the radial oscillation
and over azimuth while from (11)

$$Q = \int \int \int F_{00}^0 Q_{00}^0 J d\phi d\theta de d^3\varpi = \int \int F_{00}^0 Q d^3r d^3c.$$

Thus all mean values of continuous smooth functions in phase space tend to the
values they would have if the distribution function tended to $F_{00}^0$. The
distribution function $f$ is said to converge in the mean to the function
$F_{00}^0(\epsilon, \varpi) \equiv F_{00}^0(c^2/2 - \psi, r \times c)$, that is, to the initial distribution function
averaged over the phases of the oscillations in $r$ and over azimuth.* The
time taken to approach this equilibrium is qualitatively estimable from (10) but
different macroscopic variables approach their equilibrium values at somewhat
different rates depending on the gradients of the corresponding $Q$ in phase space.
The time is roughly the time it takes for the oscillations of the perturbed stars
to get out of step with one another.

Gravitating perturbations of an infinite uniform rotating stellar system.—Landau
(3), treating a classical problem in plasma physics, has given a way of solving
equations similar to (3) and (4). In his case $\psi_0$ was constant and $f_0$
independent of position. Bernstein (4) has treated the problem of a plasma in a magnetic
field by a development of the same method. Both of these treatments depend
on Fourier and Laplace transformations in space and time whose usefulness
depends on the fact that the coefficients in (3) of $f_1, \psi_1$, and their derivatives,
are independent of $r, t$. This cannot be so in a stellar system since $\partial \psi_0 / \partial r$
must then be independent of $r$ which at once yields $- \nabla^2 \psi / 4\pi r = \rho = 0$. However,
in axes rotating with angular velocity $\Omega$ (3) becomes, using $R$ as the vector from the axis of $\Omega$,

$$\frac{\partial f_1}{\partial t} + c \cdot \frac{\partial f_1}{\partial r} + \left( \frac{\partial \psi_0}{\partial r} - 2 \Omega \times c + \Omega^2 R \right) \cdot \frac{\partial f_1}{\partial c} + \frac{\partial \psi_1}{\partial c} \cdot \frac{\partial f_0}{\partial c} = 0;$$

(12)

* The effect of mixing is to make $f$ converge in the mean to a function of only isolating constants of the motion. If we use a physically measurable $f$ obtained from $f$ by smoothing over some cell size in phase space large enough to contain many stars, then $f$ genuinely converges. Defining entropy

$$S = \int \int f \log f d^3c d^3r$$

we find that entropy increases as $f$ becomes mixed on a scale smaller than the cell size. The loss of information corresponding to this entropy increase is the loss of knowledge of the values of the time-dependent integral on which $f$ depends less and less as the mixing continues.
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if \( \partial \psi_0 / \partial \mathbf{r} = -\Omega^2 \mathbf{R} \) the coefficient of \( \partial f_1 / \partial \mathbf{c} \) is independent of \( \mathbf{r}, t \). Evidently

\[
\nabla^2 \psi_0 = -2 \Omega^2, \quad \therefore \quad \rho_0 = \frac{\Omega^2}{2 \pi \gamma}.
\]

(13)

This system is the limit of an infinite cylindrical system rotating at equilibrium as the radius of the cylinder \( \to \infty \). It is the medium of stars of constant density rotating at equilibrium. For the equilibrium distribution function we have

\[
\mathbf{c} \cdot \frac{\partial f_0}{\partial \mathbf{r}} - 2 \Omega \times \mathbf{c} \cdot \frac{\partial f_0}{\partial \mathbf{c}} = 0 \quad (i), \quad \frac{\Omega^2}{2 \pi \gamma} = \int f_0 d^3 \mathbf{c} \quad (ii)
\]

(14)

which are satisfied by any

\[
f = f_0(c^2, \varphi)
\]

(15)

subject to (14) (iii).

We shall now discuss the stability of systems of this type. The method is similar to that of Bernstein (4) but there are a number of changes in that where we can use a general distribution function of type (15) and that gravity is attractive whereas electricity is repulsive.

We here note one peculiarity of the special system we shall discuss. The unperturbed orbits (in our rotating axes) are helices parallel to the \( z \) axis while the projection of the motions on to \( z=0 \) gives uniform circular motions all of the same period. Thus any disturbance independent of \( z \) whose gravity is not significant will exactly reconstruct itself after one period. This is one of the exceptional cases, mentioned earlier, in which mixing does not occur due to the periods of the individual stars being independent of energy. We may therefore expect strange behaviour for wave vectors \( \mathbf{k} \) perpendicular to \( \Omega \) even for disturbances of significant gravity. In our analysis we do find that the case \( \mathbf{k} \cdot \Omega \cdot \mathbf{0} \) is singular and cannot be treated along with the other wave vectors. I do not believe that any physical results can depend on such waves since any actual disturbance is a superposition of \( \infty^3 \) modes of different \( \mathbf{k} \) while we are only excepting an \( \infty^3 \) set of these. We shall of course discuss waves with \( \mathbf{k} \cdot \Omega \) small but non-zero.

Denote the Fourier transforms of \( f_1, \psi_1 \) by \( f_k, \psi_k \). Then taking Fourier transforms of (12) (for the \( f_0, \psi_0 \) discussed) and (4) we have

\[
\frac{\partial f_k}{\partial t} + i \mathbf{k} \cdot \mathbf{f}_k - 2 \Omega \times \mathbf{c} \cdot \frac{\partial f_k}{\partial \mathbf{c}} + \psi_k i \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{c}} = 0.
\]

(16)

\[
-k^2 \psi_k = -4 \pi \gamma \int f_k d^3 \mathbf{c}.
\]

(17)

Now take Laplace transforms by multiplying by \( e^{-pt} \) and integrating \( \int_0^\infty dt \). We denote the transformed functions by a \( \sim \) (note: \( \text{Re}(p) \) must be taken large enough to ensure that \( f_k \) converges)

\[
-g(\mathbf{c}, \mathbf{k}) + pf_k + i \mathbf{k} \cdot \mathbf{c} f_k - 2 \Omega \times \mathbf{c} \cdot \frac{\partial f_k}{\partial \mathbf{c}} + \psi_k i \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{c}} = 0.
\]

(18)

\[
\psi_k = \frac{4 \pi \gamma}{k^2} \int f_k d^3 \mathbf{c}
\]

(19)

where \( f_k |_{i=0} \) is \( g(\mathbf{c}, \mathbf{k}) \).
We now solve (18) for $f_k$ in terms of $f_0$, $g$ and $\tilde{\psi}_k$. We will then substitute in (19) and by taking all the terms in $\tilde{\psi}_k$ on to the L.H.S. we shall obtain $\tilde{\psi}_k$ in terms of $f_0$ and $g$. We have then only to invert the transformations to obtain $\psi_1(r, t)$. We shall call the behaviour of the system with respect to a certain perturbation stable or unstable according as $\psi_1$ behaves. (Actually $f_1$ is never unstable when $\psi_1$ is stable or vice versa.)

Write $c = (u, v, w) = (c_\perp \cos \phi, c_\perp \sin \phi, w)$ then

$$\Omega \times c \cdot \frac{\partial}{\partial c} = \Omega \frac{\partial}{\partial \phi}$$

(where the $z$-axis is oriented along $\Omega$). Then orienting the $y$-axis along $k_\perp$ (the component of $k$ perpendicular to $Oz$) (18) reads

$$\left( p + ik_\perp z \omega + ik_\perp c_\perp \sin \phi - 2 \Omega \frac{\partial}{\partial \phi} \right) f_k = g - ik \cdot \frac{\partial f_0}{\partial c} \tilde{\psi}_k.$$  

The integrating factor is $e^{a \phi + i \kappa \cos \phi}$ where

$$\alpha = -\frac{p + ik_\perp \omega}{2 \Omega}, \quad \kappa = \frac{k_\perp c_\perp}{2 \Omega},$$

and the solution is

$$f_k = e^{-a \phi - i \kappa \cos \phi} \left[ \int_{\phi_0}^{\phi + 2 \pi} \frac{g - ik \cdot \frac{\partial f_0}{\partial c} \tilde{\psi}_k}{-2 \Omega} \right] e^{i (\phi + \kappa \cos \phi) + M},$$

where $M$ is a function of integration (independent of $\phi$), which must be so chosen that $f_k$ has period $2\pi$ in $\phi$. Thus

$$e^{2 \pi a} f_k = e^{-a \phi - i \kappa \cos \phi} \left[ \int_{\phi_0}^{\phi + 2 \pi} \frac{g - ik \cdot \frac{\partial f_0}{\partial c} \tilde{\psi}_k}{-2 \Omega} \right] e^{i (\phi + \kappa \cos \phi) + M},$$

Subtracting (20) from this, and dividing by $e^{2 \pi a} - 1$, we obtain

$$f_k = \frac{e^{-a \phi - i \kappa \cos \phi}}{e^{2 \pi a} - 1} \left[ \int_{\phi_0}^{\phi + 2 \pi} \frac{g - ik \cdot \frac{\partial f_0}{\partial c} \tilde{\psi}_k}{-2 \Omega} \right] e^{i (\phi + \kappa \cos \phi) + M},$$

which demonstrates explicitly that $f_k$ is periodic in $\phi$. Substitution into (19) gives

$$\tilde{\psi}_k = \frac{4 \pi \gamma}{k^2} \int \frac{e^{-i \kappa \cos \phi}}{e^{2 \pi a} - 1} \int_{0}^{2 \pi} \frac{g}{2 \Omega} \left| e^{i (\phi + \kappa \cos \phi) + M} \right|$$

$$+ \frac{4 \pi \gamma}{k^2} \int \frac{e^{i \kappa \cos \phi}}{e^{2 \pi a} - 1} \int_{0}^{2 \pi} \frac{ik \cdot \frac{\partial f_0}{\partial c}}{2 \Omega} \left| e^{i (\phi + \kappa \cos \phi) + M} \right|$$
or denoting the integrals by \( I_1^* \) and \( I_2^* \)
\[
\bar{\psi}_k = \frac{4\pi\gamma}{k^2} I_1^* + \frac{4\pi\gamma}{k^2} \bar{\psi}_k I_2^*, \quad \ldots \quad \bar{\psi}_k = \frac{I_1^*}{\frac{k^2}{4\pi\gamma} - I_2^*}.
\] (22)

Insertion of (22) into (21) expresses \( \bar{f}_k \) in terms of the initial conditions. In principle (22) and (21) solve the problem since with the aid of an inverse Fourier Laplace transformation we can use them to express \( \psi_1 \) and \( f_1 \) at any time in terms of the initial conditions and the properties of the unperturbed state. The difference between solving the general case in principle and very special cases in practice is exemplified in the length of what follows.

We wish to invert the Laplace transformation in (22) to obtain \( \psi_k(t) \). The inversion formula is
\[
\psi_k(t) = \frac{1}{2\pi i} \int_{-\infty + \sigma}^{+\infty + \sigma} \bar{\psi}_k(p) e^{pt} \, dp
\] (23)
where \( \sigma \) must be taken so large that all the poles of \( \bar{\psi}_k(p) \) lie to the left of \( \text{Re}(p) = \sigma \) in the complex \( p \) plane.

![Fig. 1.](image.png)

Let the pole of \( \bar{\psi}_k \) which lies furthest to the right in the \( p \) plane be at \( p_k + iq_k \). Then provided \( \bar{\psi}_k \) is meromorphic we may replace the integral along \( C \) by that along \( C' \) (see Fig. 1). For large enough \( t \) the contribution to \( \psi_k \) from the line integral is small compared with that from the integral around the pole. This varies like \( e^{p_k t+\sigma t} \), so the system is stable or unstable according as \( p_k \) is negative or positive. Thus our stability problem reduces to that of finding the analytic continuation of \( \bar{\psi}_k \) for all \( p \) (it is only defined as yet for \( \text{Re}(p) \) large enough to make \( f_1 \) converge) and determining its poles.

To this end we study the analyticity of \( I_1^*, I_2^* \):
\[
I_1^* = -\frac{1}{2\Omega} \int_{-\infty}^{\infty} G \frac{1}{e^{\frac{1}{\text{Re}w}} - 1} \, dw
\] (24)
where
\[
G = \int_{c_1}^{\infty} \int_{0}^{2\pi} g|_{\phi + \Phi} e^{i\Phi + i\kappa (\cos \Phi + \theta - \cos \phi)} d\phi d\phi' c_1 dc_1
\]  
(25)

and as before
\[
\alpha = -\frac{i}{2\Omega} (p + ik_z w); \quad \kappa = \frac{i}{2\Omega} k_z c_1.
\]

Now for a restricted but large class of initial disturbances \(g\), \(G\) will be an analytic function of \(w\) for any \(\alpha\). Also provided the integrals are absolutely and uniformly convergent for all \(\alpha\) within some specified domain of the complex plane, \(G\) will be analytic in \(\alpha\) for values within that domain. This latter condition will be satisfied for any reasonable \(g\) and a domain in the \(\alpha\) plane as large as we please but bounded on the right. We cannot make \(G\) uniformly convergent for all \(\alpha\) but it will be uniformly convergent for all \(\alpha\) with real part less than any finite value we wish to choose. This is sufficient for our purposes. We hereafter restrict \(g\) to belong to this class of initial disturbances, i.e. those that make \(G(\alpha, w)\) analytic in \(\alpha\) and \(w\) for all bounded above ranges of \(Re(\alpha)\) and for all \(w\). Evidently unless it happens that \(G\) has zeros when \(\alpha = ni\) there will be poles in the integrand of \(I_1^*\). Since
\[
-\frac{i}{2\Omega} (p + ik_z w) = ni \Rightarrow w = \frac{ip}{k_z} - \frac{2\Omega n}{k_z}, *
\]

these poles are above the real \(w\) axis when \(Re(p) > 0\). The analytic continuation of \(w\) for all \(p\) is obtained by replacing the integral along the real axis by any contour integral \(\int_{-\infty}^{\infty}\) under the poles. This gives the same result (24) for all \(Re(p) > 0\) but for \(Re(p) \leq 0\) it contains contributions from the pole residues (Fig. 2). With this analytic continuation \(I_f^*\) has no poles in the finite \(p\) plane. Thus from (22)

* Note it is here that we leave out the waves with \(k_z = 0\). Thus, without loss of generuality, we may take \(k_z > 0\).
the poles of $\psi_k$ come only from the zeros of $k^2/4\pi\gamma - I_2^*$. The analytic properties of $I_2^*$ are similar to those of $I_1^*$:

$$I_2^* = \frac{1}{2\Omega} \int_{-\infty}^{\infty} \frac{1}{e^{2\pi x} - 1} F_0 \, dw, \quad \Re(p) > 0, \quad (26)$$

where

$$F_0 = \int_0^\infty \int_0^{2\pi} \left( ik z \frac{\partial f_0}{\partial w} + ik \sin \phi + \phi \frac{\partial f_0}{\partial \phi} \right) e^{z \phi'} 2\Omega w^{2\pi} \frac{d \phi'}{dw} \, d\phi \, dc_{\perp} \, dc_{\parallel}. \quad (27)$$

We again restrict ourselves to those $f_0$ that make $F_0$ an analytic function of $w$ and $\alpha$. (26) is again true for all $\Re(p) > 0$ but for $\Re(p) \leq 0$ there are again contributions from the poles $w = ip/k - 2\Omega n/k$, which are required to make $I_2^*$ analytic for all $p$.

To proceed further we must find the pole of $\psi_k$ furthest to the right in the $p$ plane. That is, we must determine the whereabouts of the root $(p = p_k + iq_k)$ with the largest real part, of the equation $k^2/4\pi\gamma = I_2^*$. The form of $I_2^*$ is too complicated to do this generally without a special choice of distribution function. However we may expect that the system will be most unstable to waves propagating along $O_2$ since in this direction the rotation of the system does not hinder the growth of condensations. We shall discuss these waves for an arbitrary distribution function symmetrical in $w$ and we shall then return to discuss waves of any $k$ for a Maxwellian distribution function $f_0$.

Waves in the $O_2$ direction $k = (0, 0, k)$.—In this case (27) reduces to

$$F_0 = \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \frac{d f_0}{dw} \cos \phi' d\phi' \, dc_{\perp} \, dc_{\parallel} = ik \frac{d \mathcal{F}_0}{dw} \frac{e^{2\pi x} - 1}{\alpha}$$

where

$$\mathcal{F}_0(w) = 2\Omega \int_0^\infty f_0 c_{\perp} \, dc_{\parallel} I_2^* = -\frac{ik}{2\Omega} \int_{-\infty}^{\infty} \frac{d \mathcal{F}_0}{dw} \frac{2\Omega}{w + ikw} \, dw, \quad \Re(p) > 0. \quad (28)$$

Evidently there is only one pole left, the others being zeros of $F_0$. By taking the integral under this pole and deforming its path back up to the real axis we find, for $\Re(p) < 0$,

$$I_2^* = -\int_{-\infty}^{\infty} \frac{d \mathcal{F}_0}{dw} \frac{1}{w + \frac{ip}{k}} \, dw - \frac{2m}{w + \frac{ip}{k}} \frac{d \mathcal{F}_0}{dw} \bigg|_{w = \frac{-ip}{k}}. \quad (28)$$

Case 1.—Solutions of $k^2/4\pi\gamma = I_2^*$ for $\Re(p) > 0$ (unstable). Writing $p = p_k + iq_k$ we find

$$\frac{k^2}{4\pi\gamma} = \left\{ \int_{-\infty}^{\infty} \frac{d \mathcal{F}_0}{dw} \left( w + \frac{q_k}{k} \right) \frac{d \mathcal{F}_0}{dw} \, dw + i \int_{-\infty}^{\infty} \frac{d \mathcal{F}_0}{dw} \left( w + \frac{q_k}{k} \right) \frac{d \mathcal{F}_0}{dw} \, dw \right\}. \quad (29)$$

Equating the imaginary part to zero and assuming as mentioned above that $f_0$ is symmetrical in $w$, we find after the substitution $w' = -w$

$$\int_{0}^{\infty} \left( w + \frac{q_k}{k} \right)^2 \frac{d \mathcal{F}_0}{dw} \, dw = \int_{0}^{\infty} \left( w' - \frac{q_k}{k} \right)^2 \frac{d \mathcal{F}_0}{dw} \, dw'.$$
Hence \( q_k = 0 \) since otherwise one denominator is less than the other throughout the range. Returning to (29)

\[
\frac{k^2}{4\pi\gamma} = - \int_{-\infty}^{\infty} \frac{w}{w^2 + \left(\frac{p_k}{k}\right)^2} \frac{dF_0}{dw} \, dw < - \int_{-\infty}^{\infty} \frac{2}{w^2} \frac{dF_0}{dw^2} \, dw
\]

(30)

provided \( F_0 \) is monotonic decreasing for \( w > 0 \). Thus for such \( F_0 \) there are only solutions for

\[ k^2 < 8\pi\gamma \int_{-\infty}^{\infty} \left( - \frac{dF_0}{dw^2} \right) \, dw. \]

That is, the system can only be unstable to small enough wave numbers (long enough wave-lengths). For a Maxwellian \( F_0 \) this condition is

\[ k^2 < \frac{4\pi\gamma p_0}{c_s^2} \left( = \frac{2\Omega^2}{c_s^2} \text{ from (13)} \right) \]

where \( c_s \) is the "sound" velocity. The condition \( k^2 c_s^2 < 4\pi\gamma p \) is precisely Jeans' criterion for instability in an isothermal gas in which collisions are dominant; we have derived it in the complete absence of collisions.

Returning to the more general (non-Maxwellian) case we see that as \( p_k \) traverses the range \( 0 \to \infty \) the R.H.S. of (30) takes all values between \( 0 \) and

\[ \int_{-\infty}^{\infty} \frac{2}{w^2} \frac{dF_0}{dw^2} \, dw, \]

therefore for any given \( k^2/4\pi\gamma \) in this range there is a solution of (30) for \( p_k \). Thus the above condition is both necessary and sufficient for instability.

Case 2.—Solutions of \( k^2/4\pi\gamma = I_2^* \) for \( Re(p) < 0 \) (stability). We are now in a position to prove that Landau damping occurs for the stable wave-lengths. It is convenient to consider first the possibility of solutions with \( Re(p) = 0 \). The value of the integral \( I_2^* \) under the pole is given correctly if we take the principal part of the integral through the pole and add half the residue. Thus writing \( p = i\eta \)

\[
\frac{k^2}{4\pi\gamma} = P \left[ \int_{-\infty}^{\infty} - \frac{dF_0}{dw} \frac{1}{w + q_k} \, dw \right] - \pi i \left. \frac{dF_0}{dw} \right|_{w = -q_k/k}.
\]

The first two terms are real while the third is imaginary; thus there can only be solutions where \( \left( \frac{dF_0}{dw} \right)_{-q_k/k} = 0 \) and only then if we may find a \( k \) such that

\[
\frac{k^2}{4\pi\gamma} = P \left[ \int_{-\infty}^{\infty} - \frac{dF_0}{dw} \frac{1}{w + q_k} \, dw \right],
\]

i.e. if

\[
P \left[ \int_{-\infty}^{\infty} - \frac{dF_0}{dw} \frac{1}{w + q_k} \, dw \right] > 0.
\]

† This result is actually more general and holds as a necessary and sufficient condition for instability for all distributions \( F_0 \) which have no real roots for \( \lambda \) of the equation

\[
\int_{-\infty}^{\infty} \frac{dF_0}{dw} \frac{1}{w^2 + \lambda^2} \, dw = 0.
\]
If $\mathcal{F}_0$ decreases monotonically from its maximum then the only solution of this type is $q_k = \alpha$. Thus (for such $\mathcal{F}_0$) all save the critical wave-length have $p_k \neq \alpha$. Since for large $t$ the waves vary like $\exp \left[ (p_k + iq_k)t \right]$ this implies that all stable wave-lengths damp.

Further investigation requires some specialized form of distribution; we therefore return to a discussion of wave-lengths of all $k$ for a Maxwellian distribution.

Maxwellian distribution.

$$f_0 = Ae^{-\beta \varphi}, \quad \beta = \frac{1}{2c_s^2}$$

from (14)

$$\frac{\Omega^2}{2\pi \gamma} = \rho_0 = \int_0^\infty f_0 \, d^3c = A \left( \frac{\pi}{\beta} \right)^{3/2},$$

therefore

$$A = \frac{\Omega^2 \beta^{3/2}}{2\pi \rho_0 \gamma}.$$  

Our first aim is to simplify $I_2^*$ so that we can discuss the solutions for $p$ of $I_2^* = k^2/4\pi \gamma$. Substituting (31) into (27) the $\phi$ and $c_\perp$ integrations may be performed to obtain $\dagger$:

$$F_0 = -4\pi \Omega (1 - e^{-2\alpha}) Ae^{-\beta w^2}$$

$$- 2\pi \int_0^{2\pi} \left( ik w + 2\Omega x \right) A e^{e^2 - \beta w^2} \exp \left[ - \frac{k^2 \sin^2 \phi'}{4\Omega^2 \beta} \right] d\phi'.$$

$$- F_0/(1 - e^{2\alpha})$$ appears in formula (26) for $I_2^*$. Note that for $Re(p) < 0$

$$\frac{1}{1 - e^{2\alpha}} \int_0^{2\pi} e^{e^2} \exp \left( B \sin^2 \frac{\phi'}{2} \right) d\phi' = \sum_{n=-\infty}^{\infty} e^{2\alpha n} \int_0^{2\pi} e^{e^2 + B \sin^2 \phi'/2} d\phi'$$

$$= \int_0^{2\pi} e^{e^2 + B \sin^2 \phi'/2} d\phi',$$

since $\sin^2 \phi'/2$ has period $2\pi$ in $\phi'$.

Using (33), (34) in (26) we obtain for $Re(x) < 0$, i.e. for $Re(p) > 0$,

$$I_2^* = 2\pi \sqrt{\frac{\pi}{\beta}} A + \int_{-\infty}^{\infty} \int_0^{2\pi} \left( ik w + 2\Omega x \right) A e^{e^2 - \beta w^2} \exp \left[ - \frac{k^2 \sin^2 \phi'}{4\Omega^2 \beta} \right] d\phi' \, dw.$$  

$\dagger$ We remember $\kappa = \frac{k_1 c_\perp}{2\Omega}$ and first integrate the second term by parts $w.r.t.$ $\phi'$; we expand the $\cos$-$\cos$ in the exponent as a product of sins and use the formula

$$\int_0^{2\pi + \alpha} \exp (2i\kappa \sin \phi \sin \phi'/2) d\phi' = 2\pi J_0 (2\kappa \sin \phi'/2).$$

Finally we perform the $c_\perp$ integration using

$$\int_0^{\infty} c_\perp e^{-\beta c_\perp^2} J_0 (c_\perp s) \, dc_\perp = \frac{1}{2\beta} \exp \left( - \frac{s^2}{4\beta} \right).$$

With (5)

$$s = \frac{k_1 c_\perp \sin \frac{\phi'}{2}}{2\Omega}.$$
For \( \text{Re}(\alpha) < 0 \) the integrals are absolutely convergent so we may reverse their order; remembering \( \alpha = -1/2 \Omega (p + ik \omega) \) we perform the \( \omega \) integration to obtain

\[
I_2^* = 2\pi A \sqrt{\frac{\pi}{\beta}} \left( I - \int_0^\infty \frac{p}{2\Omega} \exp \left( \frac{\kappa^2}{2} \phi' \right) \frac{\partial}{\partial \phi} \left( \frac{p}{2\Omega} \phi' + 2\kappa \sin^2 \frac{\phi'}{2} \right) d\phi' \right)
\]

where \( \text{Re}(p) > 0 \) and

\[
\bar{x} = \frac{k}{(8\Omega^2 \beta)^{1/2}}.
\]

This formula for \( I_2^* \) has only been derived for \( \text{Re}(p) > 0 \); both the formula for the expansion of \( 1/(1 - e^{2\pi \alpha}) \) and the integral taken along only the real axis would be invalid for \( \text{Re}(p) < 0 \). However it is noteworthy that we have arrived at a formula for \( I_2^*(p) \) which is analytic over the whole \( p \) plane\(^*\). Since the analytic extension of a function is unique the formula (35) is valid for any complex values of \( p \). Let us now recapitulate what we are doing and why we are doing it. We showed earlier that for large \( \psi_k(t) \) varied like exp \([e(\xi_k + i\eta_k)]t\) where \( p = \xi_k + i\eta_k \) is the pole of \( \psi_k(p) \) furthest to the right in the complex \( p \) plane. We also showed that the poles of \( \psi_k(p) \) came only from the zeros of \( k^2/4\pi \gamma = I_2^* \) (see (22)). To find these zeros \( I_2^* \) had to be analytically continued into the part of the \( p \) plane where they might occur. We have found the continuation in the form (35) so we have only to discuss solutions of \( k^2/4\pi \gamma = I_2^* \) to determine the behaviour of \( \psi_k(t) \) and thereby the stability of the system to disturbances of wave number \( k \). Writing down this equation and using (35), (36) and (32) we have

\[
2\bar{x}^2 = I - \int_0^\infty \frac{p}{2\Omega} \exp \left( \frac{\kappa^2}{2} \phi' \right) \frac{\partial}{\partial \phi} \left( \frac{p}{2\Omega} \phi' + 2\kappa \sin^2 \frac{\phi'}{2} \right) d\phi'
\]

\[
= I - I_3^*, \quad \text{say.}
\]

Consider solutions with \( p \) real and greater than zero:

as \( p \to \infty \), \( I_3^* \to 1 \)

as \( p \to 0 \), \( I_3^* \to 0 \)

as \( I_3^* \) is a continuous function of \( p \) in the range \( 0 \to \infty \). Thus whenever\(^\dagger\)

\( 2\bar{x}^2 - 1 < 0 \) there is a real positive solution for \( p \) so that the system is unstable to wave numbers with \( k^2/4\Omega^2 \beta < 1 \), i.e. \( k^2c_s^2 < 4\pi \omega \rho_0 \) (by (13) and (36)). This is again Jeans’ criterion for instability. This is actually necessary and sufficient for instability for we show in the Appendix that where \( 2\bar{x}^2 - 1 > 0 \) there are no solutions for \( p \) with \( \text{Re}(p) > 0 \). We also show there that the only pure imaginary (i.e. ‘‘oscillatory’’) solutions are those with \( p = 0 \). Thus again all stable disturbances damp.

\(^*\) \( \text{Re}(p) \) should be bounded below but the value of the lower bound is immaterial. We merely require some bound to obtain uniformity of convergence for the integral in (35).

\(^\dagger\) Excepting those at limiting stability (and those with \( k \cdot \Omega = 0 \) already omitted from discussion).

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Estimates of the damping rate.—We can only be interested in the stable range \( \kappa^2 > \frac{1}{4} \). We shall estimate the damping rates for large \( \kappa^2 \), i.e. small wave-lengths.

Case 1.—\( \kappa_z^2 \) large. The predominant contributions to \( I_3^* \) arise from \( \phi' \) small. Thus putting \( \sin \phi'/z = \phi'/z \) we obtain

\[
2 \kappa^2 = 1 - \int_0^\infty \frac{p}{2\Omega} \exp\left(-\frac{\kappa^2}{2} \phi^2 + \frac{p}{2\Omega z} \phi \right) d\phi.
\]

Evidently our approximation is only good when \( -Re(p)/(2\Omega \kappa^2) \ll 1 \). Thus if \( -Re(p)/(2\Omega \kappa) \ll 1 \), also

\[
2 \kappa^2 - 1 = -\frac{p}{2\Omega} \int_0^\infty \exp\left(-\frac{\kappa^2}{2} \left(\phi + \frac{p}{2\Omega \kappa^2}\right)^2\right) d\phi e^{\phi/(8\Omega^2 \kappa^2)}
\]

while if \( -Re(p)/(2\Omega \kappa) \gg 1 \) we may extend the integration without inserting the ‘‘(1/2)’’:

\[
2 \kappa^2 - 1 = -\frac{p}{2\Omega} \sqrt{\frac{2\pi}{\kappa^2}} \exp\left(\frac{p^2}{8\Omega^2 \kappa^2}\right) \bigg(\phi \bigg) \bigg(\frac{1}{2}\bigg) \tag{38}
\]

provided

\[
-\frac{Re(p)}{2\Omega \kappa^2} \ll 1.
\]

The damping rate is \(-Re(p)\) where \( p \) is the solution of (38) with the largest real part. This turns out to be the solution with real \( p \), given approximately by \( p = -2\Omega \kappa^2 \), where

\[
\kappa^2 = \sqrt{\frac{2}{\tau}} \xi e^{\phi/2} \bigg(\frac{1}{2}\bigg).
\]

Since \( \xi \) is large like \( \sqrt{\log \kappa} \) the ‘‘(1/2)’’ should not be there.

i.e.

\[
p \approx -2\Omega \kappa \sqrt{2 \log \left(\frac{\kappa^2}{\sqrt{2}}\right)^{-1}}; \quad \kappa = \frac{k}{(8\Omega^2 \rho)^{1/2}} = \frac{kc_g}{2\Omega} \tag{39}
\]

Note this result and (38) are independent of the direction of \( \kappa \) provided \( \kappa_z^2 \) is large. It is also interesting that the order of magnitude of \(-p\) is \( kc_g \), thus the damping time \( 1/(-p) \sim \lambda/2\pi c_g \) which is the time taken for diffusion with the sound velocity to smear out the disturbance.

Case 2.—\( \kappa_z^2 \) small; \( \kappa_{l}^2 \) large (waves propagating almost perpendicular to the axis of rotation). We use the expansion

\[
e^{\kappa_z^2 \cos \phi} = \sum_{n=-\infty}^{\infty} I_n(\kappa_z^2) e^{in\phi}
\]

to express (37) as

\[
2 \kappa^2 - 1 = -\frac{p}{2\Omega} \sum_{n=-\infty}^{\infty} I_n(\kappa_z^2) e^{-2i} \int_0^\infty \exp\left(-\frac{\kappa_z^2}{2} \phi^2 + \frac{p}{2\Omega} \phi - in\phi\right) d\phi,
\]
We now write $q = \frac{p}{2\Omega} - in$ and integrate by parts

$$\int_0^\infty \exp \left( -\frac{\kappa_z^2}{2} \phi^2 + q\phi \right) d\phi = \left[ -\frac{e^{-\kappa_z^2\phi^2/q^2} - q\phi}{q^2} \right]_{-\infty}^\infty \int_0^\infty \phi \exp \left( -\frac{\kappa_z^2}{2} \phi^2 + q\phi \right) d\phi$$

$$= \frac{1}{q^2} + \kappa_z^2 \left[ \phi \exp \left( -\frac{\kappa_z^2}{2} \phi^2 + q\phi \right) \right]_{-\infty}^\infty \int_0^\infty \left( 1 - \kappa_z^2 \phi^2 \right) \exp \left( -\frac{\kappa_z^2}{2} \phi^2 + q\phi \right) d\phi$$

$$= \frac{1}{q} + \kappa_z^2 + O(\kappa_z^4).$$

Put

$$p' = \frac{p}{2\Omega}.$$

Thus

$$2\kappa^2 - 1 = p' \sum_{-\infty}^\infty I_n(\kappa_z^2) e^{-\kappa_z^2} \left( \frac{1}{p' - in} + \frac{\kappa_z^2}{(p' - in)^2} + O(\kappa_z^4) \right)$$

$$= -\sum_{-\infty}^\infty \frac{e^{-\kappa_z^2} I_n(\kappa_z^2) 2p'^2}{p'^2 + n^2} \left( 1 + \kappa_z^2 \frac{p'^2 - 3n^2}{(p'^2 + n^2)^2} + O(\kappa_z^4) \right).$$

Now by hypothesis the L.H.S. is large, real and positive while we recall

$$e^{-\mu I_n(\mu)} \sim \frac{1}{(2\pi\mu)^{1/2}}$$
as $\mu \to \infty$. Hence the R.H.S. is quite small unless $p'^2$ is near an imaginary integer. In that case only one term contributes. Writing $p'^2 = -n^2 + \delta$ we have approximately

$$2\kappa^2 - 1 = -\frac{e^{-\kappa_z^2} I_n(\kappa_z^2)}{\delta} \left( -n^2 + \delta \right) \left( 1 + \kappa_z^2 (4n^2) + O(\kappa_z^4) \right)$$

$$\delta = \frac{n^2 e^{-\kappa_z^2} I_n(\kappa_z^2) 2\kappa^2 - 1}{2\kappa^2 - 1} \left( 1 - \frac{4\kappa_z^2 [2\kappa^2 - 1]}{n^2 [e^{-\kappa_z^2} I_n(\kappa_z^2)]^2} + O(\kappa_z^4) \right) < 0.$$

Thus to within our approximation there is no damping in these directions since $p'$ is imaginary.

*Oscillations of the distribution function.*—For stable oscillations the potential damps out, but we now show that the distribution function oscillates finitely. The large time behaviour of $f_k$ depends on the position of the pole of (21) with greatest real part. Since the poles of $\psi_k$ are all to the left of the real axis we see that the dominant poles of (21) are those with $\alpha = ni$, i.e. $p + ik_z v = -2\Omega ni$. These are all on the imaginary axis and they therefore correspond to undamped oscillations. This result is not unexpected for in the non-gravitating oscillations of the spherical cluster the distribution function only converged in the mean. Once $\psi_k$ has died out the disturbances in the distribution function merely propagate along the unperturbed orbits. In a finite system this normally leads to mean convergence but for our infinite system the orbits are not closed in the $z$ direction and the disturbances propagate out of sight in the $\pm\kappa$ directions.
Conclusion.—A stellar system may reach an equilibrium without the aid of any mechanism normally described as dissipative. The "encounterless" "diffusion" as stars travel along their orbits is sufficient provided that no considerable fraction of the orbits is iso-periodic. Jeans' condensational instability can occur in a collisionless star-gas but in disturbances of considerably smaller scale the density reverts to its equilibrium value with a time constant of the order of

$$\tau = \frac{\lambda}{2\pi c_s} \left( \sqrt{\frac{2}{\gamma}} \frac{(\pi/2)^{3/2} c_s^2}{\rho_0 \lambda^2} \right)^{-1}$$

$$= \frac{1}{2\pi} \left( \text{"diffusion time"} \right) \left[ 4 \log \left( \frac{2}{\pi} \left( \frac{\text{Radial period}}{\text{"diffusion time"}} \right)^{1/4} \right) \right]^{-1/2}$$

where $\gamma$ = gravitational constant, $c_s$ = "sound velocity", $\rho_0$ = equilibrium density, $\lambda$ = wave-length of disturbance. The distribution function at best converges in the mean but may oscillate finitely in systems with isoperiodicity trouble. Thus a system whose density has achieved a steady state will have information about its birth still stored in the peculiar velocities of its stars.

None of these results has been generally established but they are inferences from those special systems in which it has been possible to perform the calculations.

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Department of Astrophysics, California Institute of Technology, Pasadena, Cal., U.S.A.

On leave of absence from: Clare College, Cambridge.

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APPENDIX

We have to show that if $2\bar{\kappa}^2 - I > 0$ there are no solutions, $p$, with $Re(p) \geq 0$ of equation (37) rewritten below:

$$2\bar{\kappa}^2 - I = -\int_0^\infty \frac{p}{2\Omega} \exp \left\{ -\left( \frac{\bar{\kappa}^2}{2} \phi'^2 + \frac{p}{2\Omega} \phi' + 2\bar{\kappa}^2 \sin^2 \frac{\phi'}{2} \right) \right\} d\phi' = -I_4.$$  

Write $p/2\Omega = p_1 + ip_2$. We require solutions with $p_1 \geq 0$ of

$$2\bar{\kappa}^2 - I = -\int_0^\infty (p_1 \cos p_2 \phi' + p_2 \sin p_2 \phi') \exp \left\{ -\left( \frac{\bar{\kappa}^2}{2} \phi'^2 + p_1 \phi + 2\bar{\kappa}^2 \sin^2 \frac{\phi'}{2} \right) \right\} d\phi'$$

with

$$o = \int_0^\infty (p_1 \sin p_2 \phi' - p_2 \cos p_2 \phi') \exp - (\ldots) d\phi'. \quad (40)$$

Multiplying the second of these by $p_2/p_1$ and adding (assuming $p_1 \neq 0$)

$$2\bar{\kappa}^2 - I = -\int_0^\infty \left( p_1 + \frac{p_2^2}{p_1} \right) \cos p_2 \phi \exp \left\{ -\left( \frac{\bar{\kappa}^2}{2} \phi'^2 + p_1 \phi + 2\bar{\kappa}^2 \sin^2 \frac{\phi'}{2} \right) \right\} d\phi'$$

$$= -\left( p_1 + \frac{p_2^2}{p_1} \right) I_4, \quad \text{say.}$$

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We shall now show that \( I_4^* > 0 \). This proves the required result for \( 2p_1 = Re(p) > 0 \). We shall return to the case \( p_1 = Re(p) = 0 \) later. We have just defined

\[
I_4^* = \int_0^\infty \cos p \phi \exp \left\{ - \left( \frac{\kappa_2^2}{2} \phi^2 + p_1 \phi + 2\kappa_2^2 \sin^2 \frac{\phi'}{2} \right) \right\} d\phi'.
\]

Using

\[
2 \sin^2 \frac{\phi'}{2} = 1 - \cos \phi'
\]

and

\[
e^{k \cos \phi} = \sum_{-\infty}^{\infty} I_n(k) e^{in\phi},
\]

\[
I_4^* = \sum_{-\infty}^{\infty} e^{-\kappa_2^2} I_n(\kappa_2^2) Re \int_0^\infty \exp \left\{ - \left( \frac{\kappa_2^2}{2} \phi^2 + \left( p' + in \right) \phi \right) \right\} d\phi ; \quad p' = \frac{p}{2\Omega}.
\]

Since \( e^{-\kappa_2^2} I_n(\kappa_2^2) > 0 \), \( I_4^* > 0 \) provided

\[
I_5^* = Re \left( \int_0^\infty \exp \left\{ - a\phi^2 - b\phi \right\} d\phi \right) > 0
\]

for all real \( a > 0 \) and all complex \( b \), \( Re(b) > 0 \). I am grateful to Dr G. Münch for the following ingenious proof of this:

\[
I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-a(y+iy)^2} dy,
\]

therefore

\[
e^{-a\phi} = \int_{-\infty}^{\infty} e^{-ay - 2aiy \phi} dy
\]

thus

\[
I_5^* = Re \int_{-\infty}^{\infty} e^{-ay} \int_{-\infty}^{\infty} e^{-2aiy + b\phi} d\phi dy
\]

\[
= Re \int_{-\infty}^{\infty} \frac{b e^{-ay}}{2aiy + b} dy = \int_{-\infty}^{\infty} \frac{b e^{-ay}}{b^2 + (2ay + b)^2} dy > 0
\]

where \( b = b_1 + ib_2 \) and \( b_1 > 0 \) by hypothesis. This completes the proof for \( Re(p) > 0 \).

We now consider \( p_1 = Re(p) = 0 \). From (37) we see \( p = 0 \) gives \( 2\kappa^2 - 1 = 0 \) whereas \( 2\kappa^2 - 1 > 0 \) by hypothesis. Hence \( p_2 \neq 0 \) (40) reads

\[
o = p_2 \int_0^\infty \cos p_2 \phi \exp \left\{ - \left( \frac{\kappa_2^2}{2} \phi^2 + 2\kappa_2^2 \sin^2 \frac{\phi'}{2} \right) \right\} d\phi',
\]

therefore

\[
o = \sum_{-\infty}^{\infty} e^{-\kappa_2^2} I_n(\kappa_2^2) Re \int_{-\infty}^{\infty} \exp \left\{ - \left( \frac{\kappa_2^2}{2} \phi^2 + i(p_2 - n) \phi' \right) \right\} d\phi'.
\]

But the integral is merely

\[
\frac{1}{b} \int_{-\infty}^{\infty} \exp \left\{ - \left( \frac{\kappa_2^2}{2} \phi^2 \cos b\phi' \right) \right\} d\phi', \quad b = p_2 + n
\]

which is the Fourier transform of a Gaussian. Thus every term in (41) is positive so there are no solutions. Taking our two results together there are no solutions, \( p \) of (37) with \( Re(p) \geq 0 \) and \( 2\kappa^2 - 1 > 0 \).

† The Bessel functions \( I_n \) should not be confused with the integrals \( I_n^* \).

References