GRAVITHERMODYNAMICS—I

PHENOMENOLOGICAL EQUILIBRIUM THEORY AND ZERO TIME FLUCTUATIONS

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Summary
We inquire into the physics of a self-gravitating medium in quasi-static equilibrium, using the phenomenological approach of thermodynamics. Long range gravitational forces modify the thermodynamic functions and equations of state. The theory is discussed at three levels of increasing generality. First, we use an analogy with an imperfect van der Waals gas to illustrate the basic physical problems, including that of collective gravitational ‘shielding’. Second, we consider a quasi-particle theory based on the plasma-like aspects of a gravitating gas. Thirdly, we formulate a self-consistent, non-linear field theory which incorporates the effects of gravity into the thermodynamics of the medium.

The theory is applied to investigate the most probable spectrum of fluctuations consistent with equilibrium. All three levels support the result that the spectrum has at least one peak at a wavelength of about the Jeans length. Furthermore, for systems having about the Jeans volume, fluctuations are of order unity, rather than $N^{-1/2}$ as in a perfect gas. Macrofluctuations, resulting from the long range forces, promote gravitational turbulence and instability. In several respects, this resembles the first-order phase transition of an imperfect gas. These results have interesting implications for the formation of stellar systems and galaxies. They support the view that the ‘equilibrium’ state of a self-gravitating system is hierarchical rather than smooth and that the initial conditions for galaxy formation may be chaotic rather than quiescent. The most probable state of a sufficiently large system which is not yet collapsing is to be on the verge of non-linear collapse.

1. Introduction. This report initiates a series whose aim is a deeper understanding of the physics of self-gravitating media. The present paper discusses phenomenological equilibrium theory. Later discussions will consider phenomenological non-equilibrium and statistical approaches.

The modification of thermodynamics by gravity is an interesting physical problem in itself. Additional motives for this work are recent developments in cosmology and in stellar dynamics. A fundamental problem of cosmology is to explain the rise of structure in the Universe. To do this, two hypotheses have been proposed. The first is that the initial conditions of the Universe are themselves structured and portions of this structure distort and propagate through time and space as the Universe expands. What forms remain, we see as galaxies and larger clusters. The second hypothesis supposes that, for a while, the expansion is essentially homogeneous. But eventually an instability occurs, grows, and organizes the Universe.

Harrison (1967) has reviewed the problems of each hypothesis. An unhappy feature of the primordial structure idea is that known physics is unlikely to apply in

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the dense phase when initial conditions are imposed. These initial conditions also induce the unsatisfying feeling that the problem is solved by fiat, although this feeling may be mollified by assuming an irregular spectrum of fluctuations in which all modes can be excited (Misner 1967). With the instability hypothesis, the main difficulty is that growth rates for gravitational (e.g. Lifshitz 1946; Bonnor 1957) and non-gravitational (Saslaw 1967) instabilities are too slow to be important if these instabilities arise in the linear regime suggested by the usual assumption of $N^{-1/2}$ statistical fluctuations.

The attractiveness of the instability picture lies in the usual view that a nearly uniform expansion seems to provide the most simple and 'natural' starting point. There is an appeal in the attempt to obtain structure from an amorphous perfect gas using known physical instabilities. But perhaps the most 'natural' state of a self-gravitating gas is not a uniform one. For such a gas is not really perfect; its thermodynamics is modified by long range force. Thus its 'natural' fluctuations may be much larger than of order $N^{-1/2}$. If this were the case, then instabilities could begin in the non-linear regime and grow rapidly enough to produce large scale structure. It would not be necessary to postulate initial irregularities. It would be sufficient to apply classical physics to the 'natural' state of the Universe after radiation and matter decouple. In other words, we investigate the physical basis for the assumption of initial turbulence.

In stellar dynamics we seek to understand similar problems involving fluctuations and the nature of typical equilibrium. Again the equilibrium study is a necessary preliminary to the non-equilibrium analysis which will provide some insight into the relaxation (Lynden-Bell 1967) and evolution (Spitzer & Saslaw 1966; Spitzer & Stone 1967) of dense stellar systems. Here a great stimulus has been the observations of violent events in galactic nuclei.

Six sections comprise the rest of this paper. Section 2 briefly discusses the relation between gravithermodynamics and the more usual approaches to the physics of self-gravitating media and the problem of equilibrium. To illustrate the physics, a van der Waals analogy is used in Section 3 and the resultant problem of 'shielding' forms Section 4. Two generalized levels of the theory are described in Sections 5 and 6, and Section 7 briefly considers the relation of the equilibrium theory to cosmology and galaxy formation.

2. Comparison with other approaches, and the problem of equilibrium. Before proceeding, it may be helpful to place the techniques we use in perspective. The most approximation-free method of examining the physics of self-gravitating media is by numerical $N$-body computations (e.g. Aarseth 1963, 1965), though even here very close encounters are not treated exactly. Total information about the system can be obtained in principle. But for $N \gtrsim 1000$ the task of computing is still prohibitive and one resorts to less informative statistical models. Complicated non-linear integro-differential equations for a distribution function $f(p, q, t)$ usually result. When most realistic, these are non-Markovian (Prigogine & Severne (1966) have discussed the uniform case). To solve exactly even a semi-realistic model is usually prohibitive and one must be content with averaged moments of $f(p, q, t)$. These moments are obtained by assuming an $a$ priori form for $f(p, q, t)$ such as a Maxwellian. This is the hydrodynamic approximation in which all detailed information of the microstructure of the system is lost. Fortunately the essentially phenomenological equations which result are often solvable in interesting cases.
We are looking for an approximation with the solvability of hydrodynamics, but with a closer tie to the microscopic level. In particular, the method must be powerful enough to describe fluctuations. A thermodynamic approach seems likely to meet these requirements and is intrinsically well worth investigating. Our thermodynamics, however, will be changed by gravity. Obviously for a system with time dependent parameters we want to use a non-equilibrium theory. A reasonable first approximation is to derive an equilibrium theory which describes zero-time fluctuations and can also apply to non-equilibrium systems which evolve quasi-statically. Thus equilibrium gravithermodynamics (GTD) is useful in the same way as ordinary equilibrium thermodynamics for describing time dependent systems.

The rigorous meaning of equilibrium remains a fundamental problem for all thermodynamics, but especially for systems with long range forces. Realizing the possible dangers in not having a rigorous basis, we will proceed conservatively. This means following tested analogies when possible, requiring that results reduce to classical ones in the relevant limits, and extending the theory in a straightforward way.

First let us try to clarify the problem of equilibrium and indicate why it need not pose a vital objection. Laboratory systems in which long range forces and chemical reactions are negligible can be forced to remain in equilibrium for arbitrary time by controlling their boundary conditions. Not so for large gravitating systems which ultimately tend to collapse, unless prevented by internal forces which do not dissipate. Formally, the free energy of such a system, when internal forces are neglected, has no minimum. There is no equilibrium and this leads to gravitational collapse. But if there are, for example, pressure gradients, there can be a local or relative minimum to the total free energy as a function of, say, average density. A star is in such an equilibrium state, albeit in a thermally metastable one since if suddenly cooled, it will collapse. Observed astronomical systems are usually in local equilibrium.

Thus the new fundamental thermodynamic equations must generally involve gradients of state variables as well as the variables themselves. Therefore they may no longer be homogeneous to the first order. Physically, this problem arises because the energy and entropy need no longer be extensive quantities. The major consequence of this is that we will assume, as in the theories of imperfect gases, surface tension, etc. that the most probable local equilibrium state is found by maximizing the modified entropy. We shall see that for the range of interest in equilibrium GTD, this is still equivalent to finding a local minimum of the resultant free energy.

The forming, fading fluctuations themselves provide a crude criterion for equilibrium. If their decay time scale

$$\tau_d \approx \frac{\lambda}{c}$$

(where $\lambda$ is the wavelength of the fluctuation and $c$ is the sound speed) is less than the time scale, $\tau_b$, for a substantial change in background quantities, then the system is quasi-static and the equilibrium theory provides a reasonable approximation. However, if they do not decay, but grow on a time scale less than $\tau_b$, the equilibrium approximation is not useful. We shall first formulate the theory for modes for which $\tau_d < \tau_b$ and only then extrapolate the results from this safe region. An important finding will be the manner in which the equilibrium theory breaks down for $\tau_d \approx \tau_b$. Not only does it imply that the most probable state of the system contains...
growing modes, but it also strongly suggests the presence of macrofluctuations as well.

3. An illustrative model—the van der Waals gas

3.1 Introduction. Since the mathematics of the van der Waals model is quite simple, the basic physical principles of an imperfect self-gravitating gas can be introduced more transparently in this section. Fortunately, for astronomical purposes we do not require an imperfect gas model to give high numerical accuracy.

A detailed and important previous discussion of the modification of an equation of state by gravity is by Bonnor (1956). He did not, however, consider—as we will—a local thermodynamics dependent on both local and global properties of the system. Rather, he formally added the equilibrium pressure needed to truncate an isothermal sphere to the perfect gas equation of state. This modification is valid only on the boundary of the sphere, and only globally. Unfortunately it does not yield an equation of state which is meaningful in the usual thermodynamic sense. A recent thorough discussion of the isothermal sphere problem has been given by Lynden-Bell & Wood (1968).

3.2 Equation of state. Most thermodynamics texts discuss the van der Waals equation which is frequently written

\[(P + N^2 a/V^2)(1 - Nb/V) = NkT/V\]  

where \(P\) and \(T\) are the local pressure and temperature, \(N\) and \(V\) are the total number of particles and volume of the system, \(k\) is Boltzman's constant, and \(a\) and \(b\) are constants arising from particle interactions. Main advantages of this equation are: (1) simplicity; (2) closed form (unlike a virial expansion); (3) ability to describe first order phase transitions (i.e. allows for two states at the same pressure but different densities); (4) reasonable accuracy when applied to gas–liquid systems, which are those most analogous to gravitational condensation. The outstanding feature of equation (1) is its loops in the \(P - V/N\) diagram for isotherms of less than critical temperature. Such loops are inevitable in a uniform canonical ensemble (Widom 1957). Thus the van der Waals model is representative of a large class of physically similar but more mathematically complicated equations of state for uniform systems. Gradients do not enter, and this is an important deficiency. Our analysis in Section 3.3 explores the physical implications of this deficiency.

To pursue this model, we must evaluate the coefficients \(a\) and \(b\) in equation (1). In general they are given by

\[b = 2\pi \int_0^{2\sigma_0} (1 - e^{-U_{12}/kT})r^2 \, dr\]  

\[a = -2\pi kT \int_{2\sigma_0}^R (1 - e^{-U_{12}/kT})r^2 \, dr\]  

where \(U_{12}(r)\) is the two particle interaction energy, and \(2\sigma_0\) is the radius of the 'hard core' part of the interaction potential. The first integral, in which \(U_{12}\) is essentially infinite, is easily evaluated: \(b = 16\pi\sigma_0^3/3\). Since \(U_{12}\leq kT\) when we consider a gas of gravitating molecules at reasonable temperature, the second integral becomes

\[a = -2\pi \int_{2\sigma_0}^R U_{12}r^2 \, dr\]
Normally the upper limit of equation (4) is taken as infinite, since molecular interactions decrease at least as rapidly as $U_{2r} \sim r^{-5}$. If we did this for the gravitational case the integral would diverge. Therefore, we formally impose an upper limit, $R$, and consider its meaning later (in Sections 3.3 and 4). Here we just remark that $R$ has two possible \textit{a priori} interpretations: as a cut-off resulting from some sort of inherent shielding in $U_{12}$, or as a geometric limit of the system. This latter may be related to a 'collective shielding'. On evaluation

$$a = \pi G \bar{m}^2 (R^2 - 4\bar{v}^2) = (3/4\pi)^{2/3} \pi G \bar{m}^2 V^{2/3}$$

(5)

where $\bar{m}$ is an average particle mass, and $V$ is the 'excess volume' associated with $R$. No longer are the coefficients necessarily constant. Their dependence on thermodynamic quantities will depend on the interpretation of $R$. This will become clearer in Section 3.3.

3.3 Sound speed and the Jeans analysis. Insight into the meaning and limitations of the gravitational van der Waals model can be obtained by calculating the modification of sound speed by gravity using a thermodynamic approach. In general the sound speed over a volume $V$ is

$$c^2 = \left( \frac{\partial P}{\partial \rho} \right)_S = -\frac{V^2}{\dot{m}N} \left( \frac{\partial P}{\partial V} \right)_T \left[ 1 - \left( \frac{\partial T}{\partial S} \right)_V \left( \frac{\partial P}{\partial T} \right)_V \right]$$

(6)

where $S$ is the entropy and $\bar{m}$ is the average mass of an element of the medium. Calculation is easier with the right hand side of equation (6) which is obtained by using thermodynamic identities. Let us write the coefficients in the general form

$$a = a_0 V^\alpha T^\beta$$

(7a)

$$b = b_0 V^\gamma T^\epsilon.$$  (7b)

Furthermore, define a fractional excluded volume

$$x = Nb/V$$

(8)

and define

$$y = Na/kTV.$$  (9)

This last quantity is essentially the ratio of the interaction energy of one particle with all the others to the kinetic energy of that one particle. Standard equations for the thermodynamic functions of a van der Waals gas give, after some algebra

$$c^2 = \frac{kT}{\bar{m}(1-x)^2} \left[ 1 - \gamma x - (1 - \alpha)(1 - x)^2 x + \frac{(1 + x (\epsilon - 1) - \beta (1 - x)^2 x^2)}{\beta (1 - x)^2} \right].$$

(10)

For a perfect gas $x = y = \alpha = \beta = \gamma = \epsilon = 0$ and equation (10) reduces to the usual form.

In astronomical applications to a gravitating molecular gas (though not necessarily to a dense stellar system) $x \ll 1$, and therefore the system is far from its critical point in the usual sense of imperfect gases. This usual critical point occurs when short range molecular forces balance kinetic forces. In our case we soon will find that long range forces can balance kinetic forces and produce a mock critical
point. The essential difference between real and mock critical points is that the former are in stable equilibrium, but the latter are not. To see this consider the case when \( R \) in equation (5) represents the geometric volume of the system \( (R \gg r_0) \). Then equation (10) becomes 
\[
c^2 = kT(5 - 4y)/3\bar{m} \tag{11}
\]
and \( c^2 < 0 \) for \( y > 5/4 \). Note that since the density is constant, \( y \sim R^3 \) and by considering large enough volumes we provoke an imaginary sound speed. This corresponds to gravitational instability and we have, from an imperfect gas viewpoint, reproduced the Jeans instability. In fact it is easy to show that the radius at which this instability sets in is related to the Jeans length by
\[
\frac{R}{\lambda_J} = \frac{1}{\pi} \sqrt{\frac{3}{5}} y_0 \tag{12}
\]
where \( y_0 = y(c=0) \). In this case \( y_0 = 5/4 \). There are two comments. Firstly, we have reproduced Jeans instability without the usual method of perturbing an inconsistent zero order state; secondly, it is not obvious a priori that the van der Waals formalism can reproduce Jeans instability which requires \( y_0 \) to be real, positive, and of order unity. Solutions for \( c^2 = 0 \) in equation (10) do not generally satisfy these conditions. In particular, from the particle viewpoint, a new dimensionless parameter, \( G\bar{m}/v^2 l \), enters in addition to \( R/\lambda_J \). This new parameter (fundamental in stellar dynamics), where \( v^2 \) = mean square velocity and \( l \) = interparticle distance, is essentially \( N^{-2/3} \) and usually very small. It might, a priori, have appeared in equations (10) and (12).

Next consider the view that \( R \) results from gravitational shielding in \( U_{12} \), independent of thermodynamic parameters of the system. This is easily seen to be inconsistent with the requirement that equation (10) reproduces the Jeans instability, since from equation (12) we would have \( R = \text{constant} \approx y_0\lambda_J \) and thus \( R \) would in fact depend on the thermodynamic state of the system. Other interesting systems (e.g. when the mass and volume of the particles are related to thermodynamic parameters of the system) can be discussed on the basis of the general equation (10), but we will not pause to do that here.

From the viewpoint leading to equation (4), the only relevant factor is that \( R \) is finite. That this is a result of an instability of systems of a certain size suggests that there is a 'collective shielding' effect. We will briefly explore the meaning of this from a hydrodynamic viewpoint in Section 4 and from a thermodynamic viewpoint in Section 6.

What happens when we perform the standard Jeans analysis using equation (1) rather than the perfect gas equation of state? This will be interesting because we must now include gravity in the static equation of state, as well as in the momentum equation. In fact since the perfect gas equation of state is incorrect for a static self-gravitating medium, we might expect this procedure to provide a more accurate result. Performing the analysis in the usual way, we are lead again to the Jeans dispersion relation 
\[
\sigma^2 = c^2 k^2 - 4\pi G\rho_0 \tag{13}
\]
but now with equation (11) for \( c^2 \). So for systems of volume slightly less than the critical Jeans volume, \( c^2 = 0 \) and \( \sigma^2 < 0 \) for all wavelengths! What are we to make of this? It implies that even the slightest, smallest perturbations grow.
It is here that the van der Waals model breaks down, for it neglects the effects of pressure gradients and these prevent the small perturbations from growing. There is a related, more important, but more subtle reason why the standard Jeans analysis breaks down when it incorporates the van der Waals equation. Although one may have a local equation of state which depends on global properties of the system, the hydrodynamic equations on which the Jeans analysis is based are purely local. Introduction of global properties into them via the equation of state is inconsistent and this inconsistency becomes important when \( R \approx \lambda \). For this calculation to be consistent, the equation of state must be taken in the local limit \( V \to 0 \), \( N \to 0 \), \( N/V \to \text{constant} \). Hierarchical systems consisting of many gaseous blobs or of stars can, interestingly enough, be described consistently by this Jeans–van der Waals analysis. The parameters of the Jeans analysis apply to the complete system, while the effects of internal gravity of each blob are described by the coefficient \( a \). This coefficient will now depend on the tidally distorted shape of the blob. In turn this is related to the global parameters of the entire system. Of course in this approximation only uniform blobs can be studied, so this applies most realistically to the problem of incipient fragmentation. The application to hierarchies whose fragments have density gradients can be made more usefully with the more general equation of state derived in Section 6.

Nevertheless, this simple model serves us well. It leads the way to considering hierarchies. It indicates interesting new effects concerning fluctuations when \( \epsilon = 0 \). In the next section we shall pursue this problem of fluctuations.

3.4 Zero-time fluctuations. Several important physical ideas set the stage for the application of classical fluctuation theory to the van der Waals model. Consider a gravitating cloud of large extent. Carve out from this cloud a spherical region concentric with the centre. This region forms our basic system. It is to a Gibbs ensemble of such systems that our thermodynamic theory applies, rather than to a Maxwell–Boltzmann ensemble. For in the latter, long range forces would prevent the isolation of energy into separate systems. Nor would it be possible to carve a single cloud into more than one spherically symmetric system.

Each system in the Gibbs ensemble is assumed to interact weakly with an energy reservoir consisting of all the other systems. Their temperature is therefore constant. Also we assume that the fraction of systems in a particular state equals the probability of finding one particular system in that state. This probability is given by the Boltzmann postulate.

\[
w = w_0 e^{\Delta S/k}
\]  

where \( \Delta S = S - S_0 \) and \( S_0 \) is the entropy of the state with maximum probability \( w_0 \)—often called the equilibrium state. (This meaning of equilibrium must be distinguished from its mechanical connotations.)

Classical theory (e.g. Fowler 1936; Landau & Lifshitz 1958) proceeds by expanding \( \Delta S \) (resulting from a change in free energy in the isothermal case) in equation (14) in a Taylor series. The zeroth term vanishes in equilibrium, the first gives the gaussian distribution (second moments) and the others involve higher moments. The \( n \)th term of the expansion is proportional to

\[
\frac{\partial^n P_0}{\partial V_0^n} \left( \frac{V - V_0}{V_0} \right)^{n+1} V_0^{n+1}
\]
where the subscript denotes the equilibrium state. When the sound speed tends to \( c \), \( \frac{\partial P_0}{\partial V_0} \) tends to \( c \) and the second moment of the density (which normally is \( \sim N^{-1/2} \)) becomes infinite. Since normally for some value of \( n \),

\[
\frac{\partial^n P_0}{\partial V_0^n}
\]

remains finite, this is used to calculate the \( n+1 \)th moments and the root mean \( n+1 \)th fluctuation. Such fluctuations are usually much greater than \( N^{-1/2} \), though not of order unity. It would seem possible to settle the problem simply by using our van der Waals equation for the moments derived from equation (14). But it is not so easy.

Some years ago Greene \& Callen (1951) noticed that the expansion procedure for equation (14) gives correct results only for the second moments derived from the gaussian distribution. All moments however can be calculated exactly. Applying their formalism to density fluctuations and writing \( \delta N = N - N_0 \), we find for the first three exact moments:

\[
\langle \delta N^2 \rangle = k \left( \frac{\partial N}{\partial (\mu/T)} \right)_{1/T, V}^{1/T, V} \tag{15}
\]

\[
\langle \delta N^3 \rangle = k^2 \left( \frac{\partial^2 N}{\partial (\mu/T)^2} \right)_{1/T, V}^{1/T, V} \tag{16}
\]

\[
\langle \delta N^4 \rangle = k^3 \left( \frac{\partial^3 N}{\partial (\mu/T)^3} \right)_{1/T, V}^{1/T, V} + 3k^2 \left( \frac{\partial N}{\partial (\mu/T)} \right)_{1/T, V}^{1/T, V} \tag{17}
\]

where \( \mu \) is the chemical potential. By manipulations involving thermodynamic identities these can be written more usefully:

\[
\langle \delta N^2 \rangle = \frac{kTN}{V} \left( \frac{\partial N}{\partial P} \right)_{T, V}^{T, V} \tag{18}
\]

\[
\langle \delta N^3 \rangle = k^2 T^2 \frac{N^2}{V^2} \left[ 2 \frac{\partial N}{\partial P} \left( \frac{\partial N}{\partial P} \right)^2 + \frac{\partial^2 N}{\partial P^2} \right]_{T, V} \tag{19}
\]

\[
\langle \delta N^4 \rangle = k^3 T^3 \frac{N^3}{V^3} \left[ 4 \frac{\partial N}{\partial P} \left( \frac{\partial N}{\partial P} \right)^3 + 7 \frac{\partial N}{\partial P} \frac{\partial^2 N}{\partial P^2} + \frac{\partial^3 N}{\partial P^3} \right]_{T, V} + 3k^2 T^2 \frac{N^2}{V^2} \left( \frac{\partial N}{\partial P} \right)^2_{T, V}. \tag{20}
\]

Higher order moments can be found by an iterative procedure, but these are sufficient to demonstrate our point. For the van der Waals equation

\[
\left( \frac{\partial N}{\partial P} \right)_{T, V} = \frac{V - Nb}{\gamma} \tag{21}
\]

where

\[
\gamma = kT + b \left( P + \frac{N^2}{V^2} a \right) - \frac{2aN}{V} \left( 1 - \frac{N}{V} b \right). \tag{22}
\]

For \( V \gg Nb \), \( \gamma = 0 \) when the thermal energy of a particle approximately equals its gravitational energy. But now we notice that the higher moments also become infinite when \( \gamma = 0 \). Indeed the higher the moment the more strongly it diverges as
\( \gamma \) tends to 0, since its divergent terms do not generally cancel. This invalidates any Taylor expansion for \( \Delta S \) in equation (14) and implies that the equilibrium moments are infinite. Hence we cannot use the standard thermodynamic fluctuation formulae, but must revert to the basic equation (14) and the van der Waals equation for free energy. This approach will be used in Section 6 with a more general formula for the free energy.

It is important to distinguish between these fluctuations and the more usual critical fluctuations. First, a critical point results from short range forces. Second, at a critical point

\[
\left( \frac{\partial^2 P}{\partial (V/N)^2} \right)_T = 0
\]

and

\[
\left( \frac{\partial^3 P}{\partial (V/N)^3} \right)_T < 0
\]

as well as

\[
\left( \frac{\partial P}{\partial (V/N)} \right)_T = 0.
\]

(Tisza 1951) so long as the thermodynamic functions have no mathematical singularities at this point. Thus a critical fluctuation is in neutrally stable equilibrium while the gravitational fluctuations are unstable. The formal theory thus agrees with our physical expectation, and it remains to clarify the meaning of collective shielding.

4. Collective shielding. Not only is collective shielding important for the cut-off in the van der Waals coefficient, but also for the conceptual foundation of our Gibbs ensemble. If a basic system is larger than the shielding length, it may be regarded as essentially infinite in extent. Particle orbits within the shielded region are unaffected by the outer parts of the system which simulate an infinite uniform force-free background. In this way we can reconcile the Jeans analysis with finite physical systems.

Shielding may be examined from three viewpoints, each providing a different insight. In this section we do a simple macroscopic hydrodynamic analysis based on the Jeans equation (23). In Sections 5 and 6 the pseudoplasma and microscopic approaches are used.

By perturbing the hydrodynamic equations for an isothermal gravitating gas, one obtains the familiar Jeans equation for density perturbation:

\[
(\Box^2 + k_J^2)\rho_1 = 0 \tag{23}
\]

where \( \Box^2 \) is the d’Alambertian operator and \( k_J \) is the Jeans wave number. Usually this equation is Fourier analysed in time and space to obtain the dispersion relation, equation (13). In doing this, much information contained in equation (23) is lost since one mode, unless dominant, tells very little about the structure of the perturbation. For a time dependent background, the usual procedure is to Fourier analyse in space and solve the time dependent equation. But since we desire the shape of the perturbation we take the opposite procedure. In fact, consider a growing perturbation

\[
\rho_1 = \rho_1(r)e^{\alpha t} \tag{24}
\]
for which equation (23) becomes the Helmholtz equation

\[(\nabla^2 + \chi)\rho_1(r) = 0\]  

(25)

where

\[\chi = 4\pi G \rho_0 - \frac{c^2}{c_0^2} \frac{\sigma^2}{c_0^2} \]

(26)

and \(c_0\) is the sound speed in a perfect gas.

The relevant case is when \(\chi > 0\) and \(\rho_1(r = 0)\) is finite (if \(\chi\) were less than \(0\) the perturbation would collapse faster than free fall, if \(\chi\) were equal to \(0\) then \(\rho_1(r = 0)\) would be infinite). Then we find

\[\rho_1(r) = \rho_1(r=0) \frac{\sin \sqrt{\chi} r}{\sqrt{\chi} r} \]

(27)

i.e. damped sinusoidal behaviour, arising essentially from mass conservation. With equation (27) we solve Poisson’s equation for the perturbed gravitational potential:

\[\phi_1 = c_1 + \frac{c_0}{\sqrt{\chi} r} + \frac{4\pi G \rho_1(r=0) \sin \sqrt{\chi} r}{\chi \sqrt{\chi} r} \]

(28)

Thus \(\phi_1\) is a superposition of a constant term (which can be chosen to be zero), a term representing a point charge at the origin, and a damped shielding term which becomes negative for \(\sqrt{\chi} r > \pi\). A background which changes with time may be crudely represented by a time dependent shielding length \(\chi(t)\) implied by \(\rho_0(t)\) and \(c_0(t)\) in equation (26).

The solution (28), with \(c_1 = c_2 = 0\), indicates that collective shielding causes the effects of a perturbation of gravitational potential to die rapidly away in distances greater than about the Jeans length. This shape is reminiscent of the Yukawa potential postulated by Layzer (1964) for \(\phi_0\) rather than \(\phi_1\). Indeed if the sign of equation (23) were negative, it would become the Klein–Gordon equation with the Yukawa solution, but it would not then conserve mass.

Macroscopically, the physical reason for this shielding is that in choosing a certain time dependence for a growing mode, we force the spatial part of the perturbation to be coherent. This coherence cannot be enforced over distances greater than those over which information can propagate in a growth time scale, namely about \(c/(G \rho_0)^{1/2}\). As \(\chi \to 0\), the shielding length becomes infinite because gravity dominates and its information propagates instantaneously (in this Newtonian approximation) rather than with \(c\).

5. The gravitational pseudoplasma

5.1 Introduction. To deepen our understanding, it is useful to develop a quasi-particle approach to both shielding and modified thermodynamics. Consider an infinite homogeneous gravitating medium containing localized regions whose density differs from the mean, \(\rho_0\). If this mean background is subtracted from the local density, we are left with perturbed areas of positive and negative gravitational mass. These blobs or quasi-particles move more or less coherently through the background. We shall refer to them as ‘excitations’. They have positive inertial mass—modified by the induced mass (a factor of about \(3/2\) times a shape factor, for incompressible cases, less for a compressible medium, see Prandtl (1952)) which is essentially a result of the drag of the background (and which we will neglect since

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the mass can be renormalized.) The force between two excitations is

$$F \sim (m_1 + \delta m_1)(m_2 + \delta m_2) = m_1 m_2 + m_1 \delta m_2 + m_2 \delta m_1 + \delta m_1 \delta m_2$$  \hspace{1cm} (29)$$

where $m = \rho_0 V$, $\delta m = V \delta p$. The $m_1 m_2$ term results from the background and is subtracted out. Terms of the form $m \delta m$ represent the excess force on the excitation from the uniform background. But since this excess force is, on average, symmetric it does not—when integrated over the background—lead to a net acceleration. So the only effective force is $\delta m_1 \delta m_2$, the force of one excitation on another. Excitations of positive mass attract other positives and repel negatives. Negatives attract negatives. This suggests a tendency both cluster and to shield. Furthermore, excitations accrete the background.

Many people have noted the analogy with a plasma, except for the course of the repulsion of like plasma particles. This sign difference prompts the name pseudo-plasma. Gravitational plasma is pseudo in other respects as well. For example the shape and mass of the excitations which form its particles may be strongly time dependent. Drag and accretion are present. There is no reason why ‘particles’ of one sign should be much more massive than those of the other. Despite these differences we can exploit the plasma analogy to consider collective shielding, polarization, and modifications of thermodynamics. Again we adopt the viewpoint of a Gibbs ensemble of virtual states in equilibrium. It should be noted that this procedure differs considerably from the usual way in which a gravitational $N$-body system is treated as a plasma (e.g. Lynden-Bell 1962; Sweet 1963).

5.2 Collective shielding. Positive excitations tend to coalesce and expel negative ones from their midst. The negatives provide a region of lower density which shields against nearby positive agglomerates. This can be illustrated heuristically by comparison with a Debye sphere. Since there are differences from the Debye analogy, we derive the phenomenological theory in some detail.

Let the average number of excitations per unit volume having mass $\delta m_i$ be $n_{10}$. By mass conservation

$$\sum_i n_{10} \delta m_i = 0.$$  \hspace{1cm} (30)$$

In a cloud of excitations we assume there is a balance between kinetic and potential energy such that the number distribution is given by the Boltzmann factor

$$n_i = n_{10} e^{-\phi \delta m_i / kT}$$  \hspace{1cm} (31)$$

where $\phi = \phi_0 + \phi_1$. Poisson’s equation is

$$\nabla^2 \phi = -4 \pi G \sum_i n_0 \delta m_i.$$  \hspace{1cm} (32)$$

Next the usual step of putting $\phi_0 = 0$ is taken. There are two cases to consider.

Case 1: $\phi \delta m_i / kT \ll 1$. This is the normal case for real plasmas. The procedure is to expand the exponential in equation (31) and substitute into equation (32) to first order. The zero order term is zero by mass conservation (30). The solution of the resulting differential equation with radial symmetry is

$$\phi_1 = -G \overline{\delta m_i} e^{-\mathcal{H} r}$$  \hspace{1cm} (33)$$

where

$$\mathcal{H}^2 = \frac{4 \pi G}{kT} \sum_i (\delta m)^2 n_{10}$$  \hspace{1cm} (34)$$
and the sum is over the different mass excitations which have an average mass $\overline{m}_l$
Thus there is a shielding length, $\mathcal{H}^{-1}$, closely related to the Jeans length. In fact if
excitations of only one mass are present, then $\mathcal{H}^{-1} = \lambda_{J (\rho)}$.

Case 2: $\delta m/kT < t < 1$. When the excitations themselves have little random
motion, a more general situation arises. Poisson's equation becomes

$$\nabla^2 \phi_1 = -4\pi G \sum_i n_i \delta m_i e^{-\phi_1 \delta m_i / kT}.$$  \hspace{1cm} (35)

Let us consider the case

$$\delta m = \pm \delta m$$  \hspace{1cm} (36)

and furthermore

$$n_{+0} = n_{-0} \equiv n_0.$$  \hspace{1cm} (37)

Equation (35) becomes

$$\nabla^2 \phi_1 = 8\pi G n_0 \delta m \sinh (\phi_1 \delta m / kT).$$  \hspace{1cm} (38)

When $|\phi_1| \delta m / kT \gg 1$, equation (38) can be reduced to Abel's equation of the first
kind. For that case an examination of the form of the derivatives shows that $-\phi_1 \delta m / kT$ decreases with radius and again has a characteristic shielding length
closely related to $\lambda_J$. In general, though, equation (35) must be solved numerically.

Equation (38) differs from the Lane-Emden equation for two fundamental reasons. First, $\sinh \phi_1$ arises since there are both positive and negative particles.
Second, the gas of excitations is collisionless so there is no polytropic or perfect gas
equation of state. Thus these viewpoints are complementary rather than equivalent.
Unlike a normal plasma, mass inhomogeneity tends to grow, thereby superimposing the Jeans mode along with phase mixing (Landau damping), two stream instability,
and accretion (trapping). The shortest time scale for growth of these instabilities is
apparently $\sim (G \delta \rho)^{-1/2}$.

5.3 Polarization. A useful way to investigate a system is to stimulate it and study
its response. A gravitational pseudoplasma provides its own internal stimuli and we
must find its self-consistent response. This leads to polarization of the medium,
Somewhat analogous to the electrostatic case. In Section 5.4 a thermodynamic
description will be based on this polarization effect.

Let the positive and negative excitations be described by normalized distribution
functions $f^\pm (r, v, t)$ such that the number density and average velocity are given by

$$n = \int f (r, v, t) \, d^3v$$  \hspace{1cm} (39)

$$\mathbf{V} = n^{-1} \int \mathbf{v} f (r, v, t) \, d^3v.$$  \hspace{1cm} (40)

Next consider the motion of an excitation, taking the random velocities into account.
Our method will be to obtain the mean radial separation of excitations for small perturbations, $G(r, t)$, in the gravitational field. This gives the induced dipole
density, thence the 'gravitational susceptibility' and the 'digravnic constant'
(along the dielectric constant).

Excitations move according to

$$m^i \mathbf{r} = -m \mathbf{E}(r, t)$$  \hspace{1cm} (41)
where \( m^i(g) \) are their inertial (gravitational) masses and \( m^\sigma = g\mathbf{V}\phi \geq 0 \), as long as effects of drag and accretion are neglected. (Drag can be approximated by mass renormalization and accretion is usually very slow.) Now

\[
m^i \sim \rho_0 + \rho_1
\]

\[
m^\sigma \sim \rho_1
\]

and hence

\[
|m^i| \gg |m^\sigma|.
\]

We Fourier analyse the perturbed field:

\[
G(\mathbf{r}, t) = G_1e^{i(kr - \omega t)}
\]

where, to lowest order

\[
\mathbf{r} = \mathbf{r}_0 + \mathbf{V}_0t
\]

and the 0 subscript denotes equilibrium values. Solving equation (41) with (45):

\[
\mathbf{r} = \mathbf{r}_0 + \mathbf{V}_0t + \frac{m^\sigma}{m^i} \frac{G(\mathbf{r}, t)}{(\omega - \mathbf{k} \cdot \mathbf{V}_0)^2}.
\]

The dipole moment density is

\[
\mathcal{P} = m^\sigma\langle n\mathbf{r} \rangle = m^\sigma \int_0^\infty rf(\mathbf{r}, v_0) d^3v_0
\]

where the separation is averaged over the equilibrium distribution (assumed to be even in \( v_0 \)) and \( n \) is the number density of excitations. Since \( \langle r \rangle = 0 \) in equilibrium, the dipole moment density becomes

\[
\mathcal{P} = \frac{m^\sigma}{m^i} G(\mathbf{r}, t) \int_0^\infty \frac{f(v_0)}{(\omega - \mathbf{k} \cdot \mathbf{V}_0)^2} d^3v_0.
\]

As an illustration, suppose a finite number of excitation streams, each with velocity \( V_q \) in the x-direction:

\[
f(v_0) = \sum_q n_q \delta(v_0 - V_q).
\]

Then

\[
\mathcal{P} = \mathcal{P}_x = \frac{m^\sigma}{m^i} G_x(r, t) \sum_q \frac{n_q}{(\omega - \mathbf{k}V_q)^2}.
\]

Resonances occur for excitations moving with the phase velocity \( \omega/k \), where \( \omega \) is the frequency of the variation in the perturbed gravitational field. These are directly analogous to the van Kampen modes for a plasma. A less artificial example is the simplest distribution function—constant in a prescribed region of velocity space. This leads again to a resonance.

Next consider a general expression for polarization. We expand

\[
\frac{1}{(\omega - \mathbf{k} \cdot \mathbf{V}_0)^2} = \frac{1}{\omega^2} \left( 1 + 2 \frac{\mathbf{k} \cdot \mathbf{V}_0}{\omega} + \frac{3}{\omega} \left( \frac{\mathbf{k} \cdot \mathbf{V}_0}{\omega} \right)^2 + \cdots \right)
\]

and substitute this into equation (49). Recalling that \( f(v_0) \) is even and assuming as is
usual in plasma physics (Bohm & Gross 1949) that $f \to 0$ as $v_0 \to \omega/k$, we find from equation (49) to second order

$$\mathcal{P}_\pm = \frac{m_{\pm}^{\ell}}{m_{\pm}^t} \frac{n_{\pm}^{\ell}}{\omega^2} \left(1 + \frac{k^2 \langle v_0^2 \rangle_{\pm}}{\omega^2}\right) G_{(r, t)} \equiv \beta_{\pm} G$$

(53)

The + and − signs refer to collections of positive and negative excitations, and brackets denote averages over the distribution function. Equation (53) relates the general response (polarization) of the system to its internal stimulus $G$. Thus $\beta$ is the susceptibility. Note that $\mathcal{P}$ does not depend on the sign of the excitation.

Analogous to the dielectric constant $\epsilon$, we define the digravic constant $\gamma$ by

$$\gamma \equiv 1 + 4\pi G(\beta_+ + \beta_-) = 1 + 8\pi \frac{m^{\ell} n^{\ell} G}{m^t \omega^2} \left(1 + \frac{k^2 \langle v_0^2 \rangle}{\omega^2}\right)$$

(54)

where the second equality assumes that the masses, number densities, and distribution functions are the same for both types of excitations. It is easy to extend the theory if this is not the case, but it would become unnecessarily complicated for our present purposes. Note that $\gamma > 1$, compared with $\epsilon > 1$ for the dielectric case at low frequencies. But $\epsilon < 1$ for normal plasmas. Thus polarization decreases the effect of the gravitational field, and produces a shielding or screening of distant matter. A similar result was found qualitatively, though in a less physical way, by Miller (1966) for stellar systems. However, we also see that for perturbations which grow or decay, $\omega$ is imaginary and the gravitational susceptibility $\beta$ changes sign, making $\gamma < 1$. The qualitative difference between growing and oscillating modes will play an important part in the thermodynamics.

In order that this analysis be consistent, we must show that the induced density is small compared to the original density perturbation $\rho_1 = m^t n$. Induced density is given by

$$-\nabla \cdot \mathcal{P} = -\beta \nabla \cdot G = 4\pi G \rho_1 \beta = \frac{m^{\ell} \omega_j^2}{m^t \omega^2} \rho_1 \left(1 + \frac{k^2 \langle v_0^2 \rangle}{\omega^2}\right)$$

(55)

where

$$\omega_j^2 = 4\pi G \rho_1.$$  

(55a)

From equation (44) we see that $|\nabla \cdot \mathcal{P}| \ll \rho_1$.

Rewriting the digravic constant in a more transparent form:

$$\gamma = 1 + 2 \frac{m^{\ell} \omega_j^2}{m^t \omega^2} \left(1 + \frac{k^2 \langle v_0^2 \rangle}{\omega^2}\right)$$

(56)

shows that for high frequencies $\gamma \to 1$. In a plasma, when $\epsilon = 0$ only longitudinal oscillations occur. For the pseudoplasma, these modes propagate only when $\langle v_0^2 \rangle \neq 0$ and their dispersion relation—very similar to that of Jeans—is given by equation (56) with $\gamma = 0$. In this case $\omega^2 \approx -\omega_j^2 \leq 0$, showing that equations of the same sign tend to form unstable clusters. When modes do not propagate, their frequency is the Jeans frequency modified by the ratio $2m^{\ell}/m^t$. This modification factor is yet another difference between the plasma and the pseudoplasma.

5.4 Thermodynamics. Work: the stage has been set for calculating the thermodynamic state variables of a pseudoplasma. Consider a macroscopic spherical
volume containing many excitations. The induced surface mass density of the sphere is

$$m_s = \frac{1}{4\pi G} \int \mathbf{\Gamma} \cdot d\mathbf{S}$$  \hspace{1cm} (57)$$

where \( \mathbf{\Gamma} \) is the analogue of the electric induction,

$$\mathbf{\Gamma} = \mathbf{G} + 4\pi G \mathbf{P} = \gamma \mathbf{G}$$  \hspace{1cm} (58)$$

and

$$\nabla \cdot \mathbf{\Gamma} = 0.$$  \hspace{1cm} (59)$$

since no excitations pass through the boundary of the sphere. By doing an amount of work

$$\delta W = \phi \delta m_s = \frac{1}{4\pi G} \int \nabla \cdot (\phi \delta \mathbf{\Gamma}) \, dV$$  \hspace{1cm} (60)$$

one can bring a spherical shell of mass \( \delta m_s \) from infinity and add it to the surface. Using equations (59) and (60), \( \nabla \phi = \mathbf{G} \), and the vector identity for \( \nabla \cdot (\phi \delta \mathbf{\Gamma}) \), find

$$\delta W = \frac{1}{4\pi G} \int (\mathbf{G} \cdot \delta \mathbf{\Gamma}) \, dV$$  \hspace{1cm} (61)$$

analogous to the electrostatic result. Although the derivation is restricted to spherical symmetry, this restriction does not appear in the result which seems quite general. Equation (61) is not yet complete; it lacks a contribution from the kinetic energy of collective wave motions. To add this we may replace \( \mathbf{\Gamma} \) by

$$\mathbf{\Gamma}^I = \mathbf{G} \frac{\partial (\omega \gamma)}{\partial \omega}$$  \hspace{1cm} (62)$$

which includes the kinetic gravitational induction, in analogy with normal plasma oscillations. This work must be added to the thermodynamic variation in total energy.

Free energy: the change in internal energy per unit volume, \( u^*(s, \rho, \mathbf{\Gamma}) \), is

$$du^* = T \, ds + \mu d\rho + \mathbf{G} \cdot d\mathbf{\Gamma}^I/4\pi G$$  \hspace{1cm} (63)$$

where \( \mu \) is the chemical potential per unit mass and \( s \) is the entropy density. Free energy, \( F^*(s, \rho, \mathbf{\Gamma}^I) \) is

$$F^* = U^* - TS$$  \hspace{1cm} (64)$$

and

$$dF^* = -S \, dT + \mu d\rho + \mathbf{G} \cdot d\mathbf{\Gamma}^I/4\pi G.$$  \hspace{1cm} (65)$$

Thus \( \mathbf{G} \) has also the meaning

$$\mathbf{G} = 4\pi G \left( \frac{\partial u^*}{\partial \mathbf{\Gamma}^I} \right)_s = 4\pi G \left( \frac{\partial F^*}{\partial \mathbf{\Gamma}^I} \right)_T, \rho.$$  \hspace{1cm} (66)$$

Since \( \mathbf{G} \) is a more natural physical parameter than \( \mathbf{\Gamma}^I \), we use it as the independent variable by making the Legendre transformations

$$u(s, \rho, \mathbf{G}) = u^* - \mathbf{G} \cdot \mathbf{\Gamma}^I/4\pi G$$  \hspace{1cm} (67)$$

and

$$F(T, \rho, \mathbf{G}) = F^* - \mathbf{G} \cdot \mathbf{\Gamma}^I/4\pi G.$$  \hspace{1cm} (68)$$
Thence

\[ du = T \, ds + \mu \, d\rho - \mathbf{\Gamma} \cdot d\mathbf{\Phi}/4\pi G \]  
(69)

\[ dF = -s \, dT + \mu \, d\rho - \mathbf{\Gamma} \cdot d\mathbf{\Phi}/4\pi G \]  
(70)

and

\[ \mathbf{\Gamma} = -4\pi G \left( \frac{\partial u}{\partial \mathbf{\Phi}} \right)_{S, \rho} = -4\pi G \left( \frac{\partial F}{\partial \mathbf{\Phi}} \right)_{T, \rho}. \]  
(71)

Using equation (62) and integrating equation (70), gives for the change in free energy (at constant \(T\) and \(\rho\)) due to 'turning on' the gravitational interaction

\[ \Delta F(T, \rho, \mathcal{G}) = F - F_0 = \frac{\mathcal{G}^2}{8\pi G} \frac{\partial(\omega \gamma)}{\partial \omega}. \]  
(72)

Having obtained the energy and free energy, it is now possible to calculate all other thermodynamic properties of the pseudoplasma. Here we just discuss the entropy briefly.

Entropy:

\[ S = -\left( \frac{\partial F}{\partial T} \right)_{\rho, \mathcal{G}} = S_0(T, \rho) + \frac{\mathcal{G}^2}{8\pi G} \left[ \left( \frac{\partial \gamma}{\partial T} \right)_{\rho} + \omega \left( \frac{\partial^2 \gamma}{\partial T \partial \omega} \right)_{\rho} \right], \]  
(73)

where

\[ \left( \frac{\partial \gamma}{\partial T} \right)_{\rho} = \frac{2k}{n \, m^4 \, \omega^2} \left[ \int \frac{d^3v}{\omega^2} \left( v_0 \right)^2 \frac{\partial}{\partial T} f(v_0, T) d^3v_0 + \frac{2}{} \left( \frac{\partial k}{\partial T} \right)_{\omega, \rho} \right]. \]  
(74)

\(T\) is the temperature related to the random velocities of the excitations. In a simple case \(f(v_0, t)\) is Maxwellian and \(\langle v_0^2 \rangle = 3R' T\), where \(R' = \text{Boltzman constant/m}^4\). If furthermore the wave number \(k\) is temperature independent, then

\[ S = S_0(T, \rho) - \frac{\mathcal{G}^2 R'}{4\pi G} \frac{m^9 \omega_j^2}{m^4 \omega^4} k^2 \]  
(75)

which is independent of the sign of \(m^9\). In general, a statistical evolution equation must be adopted for \(f(v_0, T, t)\), giving \(S(t)\).

Most probable state: the most probable system in our Gibbs ensemble is that with minimum free energy consistent with imposed constraints. In normal thermodynamics the only constraint is that the system be in equilibrium at constant temperature with an energy reservoir. In gravithermodynamics we must also exclude from the ensemble all condensed collapsing systems whose free energy decreases indefinitely, i.e. systems without a relative minimum in \(F\). We retain only metastable systems.

The free energy is

\[ F(T, \rho, \mathcal{G}) = F_0(T, \rho) - \frac{\mathcal{G}^2}{8\pi G} + \frac{\mathcal{G}^2 m^9 \omega_j^2}{4\pi G} \left( 1 + \frac{3k^2 \langle v_0^2 \rangle}{\omega^2} \right). \]  
(76)

Generally the second term on the r.h.s. dominates the third. Thus the free energy of an hierarchial system (one with \(\mathcal{G} \neq 0\)) is less than that of a uniform system. Furthermore, a system with growing or decaying modes \(\omega_0^2 < 0\) has a still lower free energy compared to systems with oscillating modes. Thus the last term in
equation (76) acts, in a sense, like a hyperfine splitting of free energy among systems in the Gibbs ensemble!

If $\omega < 0$, we write

$$\omega_*^2 = - \omega^2$$  \hspace{1cm} \text{(77)}

To minimize the free energy, we must maximize

$$\frac{1}{\omega_*^2} \left( 1 - \frac{3k^2 \langle v_0^2 \rangle}{\omega_*^2} \right).$$  \hspace{1cm} \text{(78)}

If thermal motion is neglected, then relation (78) has a maximum as $k \to \infty$ and long wavelengths are the most unstable, as expected from Jeans analysis. (This assumes that derivatives $dG/d\omega_*$ are small compared to other terms. If this is not the case then they must be included self-consistently via Poisson's equation.) But when the random motions of the modes (excitations) themselves are taken into account, this result is modified in a manner which is quite unexpected from the Jeans analysis. Of course for a detailed study of relation (78) we need more higher order terms. However, the most important conclusions follow without these extra terms. It is easy to show that frequencies which minimize relation (78) satisfy the equation

$$\frac{1}{3} = 2 \frac{\langle v_0^2 \rangle}{v_p^2} \frac{\langle v_0^2 \rangle}{v_p v_g}$$  \hspace{1cm} \text{(79)}

where $v_p$ and $v_g$ are their phase and group velocities. For the special case when $v_g = 3v_0/5$, the free energy is minimum when $v_p = v_0$. Thus the resonance found from the microscopic kinetic viewpoint of Section 5.3 survives in the macroscopic theory as an entropy maximum, consistent with the constraints discussed previously.

If $\omega$ and $k$ are related by the Jeans dispersion relation, (79) becomes

$$\lambda^2 = \frac{1}{k^2} \frac{\lambda_j^2}{\left( 1 - \frac{6 \langle v_0^2 \rangle}{3 \langle v_0^2 \rangle + c^2} \right)}.$$  \hspace{1cm} \text{(80)}

Hence $\lambda \to \lambda_j$ as $\langle v_0^2 \rangle \to 0$ or $c^2 \to \infty$. In general, the wavelength which minimizes free energy is slightly longer than $\lambda_j$, but it becomes infinite if $c^2 = 3 \langle v_0^2 \rangle$. Whether this occurs depends on the form of the distribution function. The distribution function also determines whether it is reasonable to ignore higher order terms in relation (78) and frequently it is.

A pseudoplasma approach is a considerable improvement on a van der Waals type theory in that it inherently incorporates non-uniformities. Already it has provided interesting insight into the thermodynamics and most probable state of inhomogeneous media. Discussion of this model could be carried further, but we truncate it here in favour of Section 6 where the theory is developed in a more general way, independent of any a priori model or equation of state.

6. **Self-consistent field theory**

6.1 **Introduction.** Without imposing any model on the system, we want to determine the modification of thermodynamics by gravity, when inhomogeneities and fluctuations are included. To do this we extend the Landau theory of second order phase transitions (e.g. Landau & Lifshitz 1958; Brout 1965). Normally this theory is applied to phase transitions in an external field (e.g. ferromagnetism) and

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the basic equations are linear. But for gravitating systems the field is produced by the fluctuations themselves. Self consistency requirements now lead to more difficult non-linear basic equations.

6.2 Basic equations. The free energy has the form

\[ F = \int f(r) \, d^3r \]  

(81)

in which the local free energy density is expanded as

\[ f(r) = f_0(T) + m\langle n \rangle \langle \phi \rangle + a(n - \langle n \rangle)^2/2 + b(\nabla(n - \langle n \rangle))^2/2. \]  

(82)

The free energy density for a homogeneous system of non-interacting particles, of average mass \( m \), is \( f_0(T) \); \( \langle n \rangle \) is the average number density for an average system in our Gibbs ensemble (note the different meaning of brackets than in Section 5); \( n(r) \) is the local number density in a particular system. Furthermore, the phenomenological coefficients, which incorporate effects both of long and short range forces, are

\[ a = \left( \frac{\partial^2 f}{\partial n^2} \right)_{T, \langle n \rangle} = \frac{1}{n} \left( \frac{\partial P}{\partial n} \right)_{T, \langle n \rangle} \geq 0; \]  

(83)

\[ b = -\left( \frac{\partial f}{\partial \nabla n} \right)_{T, \langle n \rangle} + \left( \frac{\partial^2 f}{\partial (\nabla n)^2} \right)_{T, \langle n \rangle} \geq 0. \]  

(84)

Alternatively, a somewhat more general form of the theory results if \( b \) is taken as an empirically determined constant. If \( a \) were less than \( 0 \), the average system would be infinitely dense, and if \( b \) were less than \( 0 \) it would have enormous density gradients as well. That would describe the absolutely unstable state, rather than the local equilibrium in which astronomical systems are found. Thus the inequalities in equations (83) and (84) incorporate the new constraint discussed in Section 5.4. The second equality in equation (83) follows from the Gibbs–Duhem relation and is only approximately correct when long range forces become important. It provides a useful intuition for the meaning of \( a \) and \( b \), since for a nearly perfect gas \( a \approx kT/\langle n \rangle \) and \( b \approx (\text{scale length of inhomogeneity})^2 \cdot (a) \). Equilibrium, gravity, departures from the average density and inhomogeneities are described by the terms on the r.h.s. of equation (82). Higher order terms could be included, but we try to understand simpler things first. Similarly, the theory can be generalized for a non-uniform background \( \langle n \rangle (r) \).

The gravitational term in equation (82) gives the energy perturbation from a potential resulting from a density fluctuation. The potential acts on all the unshielded particles. Thus

\[ \langle \phi \rangle = -Gm \int \int \frac{\eta(r')}{|r - r'|} e^{\frac{F(r) - E(\phi, r')}{kT}} \, d^3r' \, d\Gamma_{\phi, r'} \]  

(85)

where

\[ \eta(r) = n(r) - \langle n \rangle. \]  

(86)

The \( d^3r' \) integral is from Poisson's equation and the second integral is an average over an unperturbed Gibbs distribution (\( d\Gamma \) is normalized phase space volume). By interchanging the order of integration

\[ \langle \phi \rangle = -Gm \int \left( \frac{1}{|r - r'|} \right) \eta(r') \, d^3r'. \]  

(87)
This may be thought of as the difference in gravitational potential between the most probable system in our Gibbs ensemble and a fiducial system (which may very rarely occur) having a uniform density which is the average density of the most probable system. Even if the integration limits of equation (85) are finite, $\langle \phi \rangle$ is still a well defined quantity for the ensemble. Hence the background can be treated consistently, in contrast with the hydrodynamic treatment used by Jeans.

The most probable system is supposed to be the one with the maximum entropy allowed by its constraints. In normal thermodynamics this is also the system with minimum free energy for given temperature (and volume) when it is in diathermal contact with a heat reservoir (which may be the Gibbs ensemble itself). The conditions that these hold in GTD are that $\partial u/\partial s \geq 0$ and $u$ is a single valued continuous function of $s$. These must be verified a posteriori, but equation (82) leads us to believe that they will hold at least until $m\langle n \rangle \langle \phi \rangle \gtrsim f_0(T)$, and we will therefore adopt the energy minimum criterion. Thus using equations (81), (82) and (87) we find

$$\delta F = 0 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[ -G\langle n \rangle m^2 \frac{1}{|r-r'|} + a \delta(r-r') \eta(r') \right. $$

$$ - \delta(r-r') b \nabla^2 \eta(r') \left. \right] \delta \eta(r') d^3r' d^3r$$

(88)

in which $\delta(r-r')$ is the Dirac delta function. Since $\delta \eta(r')$ is arbitrary, the quantity in brackets must be zero:

$$\delta(r-r') [a-b \nabla^2] \eta(r') = Gm^2 \langle n \rangle \frac{1}{|r-r'|}.$$

(89)

Multiplying equation (89) by $\eta(r')$ and integrating over $r'$:

$$\eta(r) [a-b \nabla^2] \eta(r) = -m < n > < \phi >.$$

(90)

This is the basic equation for the spatial dependence of the perturbed density. In its full form it is a non-linear differential-inegral equation for $\eta(r)$. When $\langle \phi \rangle = 0$, there are two solutions: $\eta = 0$ and $(a-b \nabla^2) \eta = 0$. The latter gives the Yukawa solution. However from equation (82) we see that the first of these solutions actually minimizes the free energy, so it is the one to use. Equation (90) gives uniform density as the self-consistent solution for zero gravitational perturbation, as it should.

From this thermodynamic approach we can extract more basic information—the two point correlation function

$$g(r, r') = \langle \eta(r) \eta(r') \rangle$$

(91)

which relates density perturbations at different places. To derive an equation for $g$ we first perturb equation (90) by letting $\eta \rightarrow \eta + \delta \eta$, $\phi \rightarrow \phi + \delta \phi$ then subtract the zero order equation (90), then substitute $b \nabla^2 \eta$ from equation (90). The result is

$$\left[ a \eta - \frac{m \langle \phi \rangle}{\eta} - b \nabla^2 \right] \delta \eta = -m \langle n \rangle \langle \delta \phi \rangle.$$

(92)

Using in equation (92) the well known result (e.g. Kadanoff et al. 1967)

$$\delta \eta = \frac{1}{kT} \int g(r, r') \langle \delta \mu \rangle (r) d^3r'$$

(93)
where \( \mu \) is the chemical potential, gives

\[
\int \left\{ \left[ a_\eta - \frac{m\langle n \rangle}{\eta} \right] \langle \phi \rangle - \eta b \nabla^2 \right\} g(r, r') + 2kT \langle n \rangle \delta(r - r') \langle \delta \phi \rangle \, d^3r' = 0. \tag{94}
\]

Since \( \langle \delta \phi \rangle \) is arbitrary, the quantity in braces vanishes, leaving the correlation function to satisfy the basic equation

\[
\left[ a_\eta - \frac{m\langle n \rangle}{\eta} \right] \langle \phi \rangle - \eta b \nabla^2 \right\} g(r, r') = -2kT \langle n \rangle \delta(r - r') \tag{95}
\]

where \( \eta(r) \) is determined from equation (90). As \( \eta \to \infty \), \( g(r, r') \) becomes a delta function, as would be expected.

From \( g(r, r') \) it is possible to derive thermodynamic functions in the standard manner. However, it is unlikely that properties of long range systems are well described by low order correlation functions, or possibly even by a finite number of them. Therefore we do not pursue \( g(r, r') \) any further, except for this remark: in an hierarchical system, \( g \) taken at face value describes the correlation between members of the \( n-1 \) level from fluctuations on the scale of the \( n \) level (from equation (90)). Thus it may be useful for investigating the formation of binaries in stellar systems.

### 6.3 Modified thermodynamics

Now we examine the solution and major thermodynamic consequences of equation (90). For spherical symmetry, this basic equation is rewritten as

\[
\frac{\partial^2 \eta}{\partial x^2} + \frac{2}{x} \frac{\partial n}{\partial x} - \frac{a^2}{b} \frac{\Phi}{\eta} - \frac{\Phi^2}{b} = 0 \tag{96}
\]

where \( \lambda \) is some scale length (e.g. the Jeans length) of the density, \( x \equiv r/\lambda \), and

\[
\Phi \equiv m\langle n \rangle \langle \phi \rangle \tag{97}
\]

is negative for positive density fluctuations and vice versa. In principle, equation (96) is solved for \( \eta(a, b, \Phi, \lambda) \). Then \( \Phi \), \( b \), and \( a \) are determined by the self consistency requirements of equations (97), (85), (84) and (83) (or \( b \) is prescribed, in lieu of equation (84)). Then \( g(r, r') \) can be found from equation (95). To compute these solutions is a major numerical undertaking beyond our present scope. Here we extract some information by a more modest physical approach.

Formula (96) describes an isothermal density fluctuation. It differs from the isothermal sphere equation for three reasons: there is no hydrostatic equilibrium; the equation of state is neither polytropic nor perfect gas; higher order non-linear terms are ignored in equation (82). Let the central boundary conditions of the fluctuation be

\[
\eta(x=0) = \eta_0 \tag{98}
\]

\[
\frac{d\eta}{dx} \bigg|_{x=0} = 0 \tag{99}
\]

Expanding the solution of equation (96) from the centre leads to

\[
\eta(x) = \eta_0 + \frac{\lambda^2}{b} \left( a\eta_0 + \Phi \right) \frac{x^2}{3!} + \frac{\lambda^4}{b^4} \left( a^2 \eta_0 - \frac{\Phi^2}{3} \right) \frac{x^4}{5!} + \frac{\lambda^6}{b^6} \left( 3a^2 \eta_0 + \frac{7a^2 \Phi}{\eta_0} + \frac{17a\Phi^2}{\eta_0^2} + \frac{3\Phi^3}{\eta_0^3} \right) \frac{x^6}{3 \cdot 7!} + \cdots \tag{100}
\]
From this we see that there are solutions in which the density increases outward. For certain values of the parameters, these solutions—if they are self-consistent—may have lower total free energy than the more usual solutions with negative gradients. This is another interesting possible effect of the long range forces. The qualitative form of solutions to equation (96) may be surveyed by letting $\eta = \eta_0 \xi$ (which eliminates the first derivative) and considering the slope of $\eta'(\eta)$. Again, there are seen to be solutions indicating shielding. Here we note that at the mock critical point, when $a = \lambda$, there is a special solution to (96):

$$\eta = \pm \sqrt{\frac{\Phi}{2b}} r$$  \hspace{1cm} (101)

for which $\Phi$ must be positive. From equation (85) we see that since $\eta$ can have either sign, correlations may be so strong that fluctuations increase radially. In this case

$$f(r) = f_0(T) - 4\Phi$$  \hspace{1cm} (102)

so $\Phi$ will have its maximum value consistent with equation (85).

Free energy: this is our basic thermodynamic quantity. From equations (100), (82) and (81), it is to fourth order:

$$F = F_0(T) + a\eta_0^2 \left( \frac{1}{2} + \zeta + \frac{1}{10} \frac{\lambda^2}{b} (1 + \zeta)(4 + \zeta) V_* \right)$$

$$+ \frac{1}{280} \frac{a^2 \lambda^4}{b^2} (1 + \zeta) \left( \frac{9}{2} + \frac{7}{3} \zeta - \frac{1}{6} \zeta^2 \right) V_*^4$$

$$+ \frac{1}{37800} \frac{a^3 \lambda^6}{b^3} (1 + \zeta) \left( 32 + 8 \zeta + \frac{44}{3} \zeta^2 + \frac{86}{3} \zeta^3 \right) V_*^3 + \cdots$$  \hspace{1cm} (103)

where

$$V_* \equiv 3V/4\pi \lambda^3 \equiv V/V_\lambda$$  \hspace{1cm} (104)

and

$$\zeta \equiv \frac{\Phi}{a\eta_0^2}.$$  \hspace{1cm} (105)

There are several comments. First, $V$ is the volume of the basic system we consider in our Gibbs ensemble. Second, the terms are actually independent of the scale length $\lambda$. Third, the coefficients of powers of $V_*$ are exact, and it can be shown that in every order they contain the factor $(1 + \zeta)$. Fourth, $a\lambda^2/b$ is dimensionless and by suitably choosing $\lambda$ (reflected of course in $V_*$) we can have

$$a\lambda^2/b = 1.$$  \hspace{1cm} (106)

A reasonable choice of $\lambda$ could be $\lambda_0$. Fifth, $\zeta$ roughly measures the ratio of gravitational energy to a combination of gravitational and kinetic energy. To lowest order

$$\zeta = aV^{2/3}$$  \hspace{1cm} (107)

but its actual magnitude, given by the proportionality factor $a$, can only be found from the numerical consistency analysis which relates $a$ and the most probable value of $\eta_0$. This and the relation of $\zeta$ to fluctuations are briefly considered in Section 6.4.
Pressure: to third order, using equations (107), (106) and (103)

\[
P = -\left(\frac{\partial F}{\partial V}\right)_T = P_0(T) - \frac{\alpha \eta_0^2}{2} \left[ 1 + \frac{\alpha}{10} \left( \alpha + \frac{2}{5} V_\lambda^{-2/3} \right) V_\lambda^{2/3} \right. \\
\left. + \frac{7}{15} \left( 5 \alpha + \frac{9}{56} V_\lambda^{-2/3} \right) V_\lambda^{2/3} V_\lambda^{2/3} + \cdots \right]
\]

(108)

where \(P_0(T)\) is the perfect gas pressure. Gradients are contained implicitly in the added terms. It is interesting that the effects of gravity (which can reduce \(P\) considerably) enter first through \(\alpha\) and only in higher order via \(\Phi\).

Any other modified thermodynamic quantity of interest can now be computed from equation (103). Some, such as the entropy, will depend on the temperature directly from the Gibbs averaging. The important thing to note is that as \(V \to 0\) (i.e. small systems or small regions in a large system) our results reduce to ordinary thermodynamics. This limit also applies as \(\eta_0 \to 0\) for a uniform medium.

6.4 Fluctuation spectrum. With this background it is possible to compute fluctuation moments from equations (18)–(20). However for non-numerical analysis, it is more convenient to use the Boltzmann postulate (14) in the form

\[
w = w_0 e^{-(F - F_0)/kT}.
\]

(109)

The fluctuation spectrum thus depends on all four quantities, \(a, \eta_0, \xi,\) and \(V\), which are related by self-consistency. From the explicit dependence of \(F\) on \(\xi\) in equation (103) and using equation (106) we note immediately that the probability of a fluctuation increases greatly for \(\xi < -\frac{1}{2}\) (for \(V_\lambda \leq 1\)). However, as \(\xi\) becomes even more negative (\(\xi < -15/2 - \frac{9}{2} \sqrt{21}\)) for \(V_\lambda = 1\) the probability of a fluctuation again decreases. A more detailed examination may indeed show a spectrum with generally low amplitude, except for occasional sharp peaks. This result is also suggested by the pseudoplasm in equation (75). It is closely related to hierarchies of random motions.

To second order in the free energy, the probability of a fluctuation is

\[
w = w_0 \exp \left\{ \frac{V}{kT} \left[ \alpha \eta_0^2 \left( 1 + \alpha + \frac{1}{10} \frac{2 \eta_0^2 \xi^2}{b} (1 + \xi)(1 + 4 \xi) V_\lambda^{2/3} + \cdots \right) \right] \right\}
\]

(110)

Since the total probability must be unity, it is necessary to recalculate the normalization constant of equation (110) each time the expansion is taken to a higher order. For example if gravity is absent, \(\xi = 0\) and for a perfect gas

\[a = kT/\langle n \rangle.
\]

Thus equation (110) reduces in lowest order to

\[w = w_0 e^{-N/2\langle n \rangle} \]

(111)

from which \(w_0 \approx N^{-1/2}\), the usual result. The higher order terms in equation (111) are neglected in standard thermodynamic fluctuation theory since they arise from the back reaction of the fluctuation on the thermodynamic state functions.

Qualitatively from equation (110) we see that, roughly, as long as \(E_{\text{grav}} < E_{\text{thermal}}\) the exponent is negative and the fluctuations small. As soon as this inequality is reversed, there is an abrupt change in the probability of a fluctuation, reminiscent of a phase transition, but differing from a normal phase transition for reasons mentioned in Section 3. This change can be illustrated quantitatively for an exact
case, $\zeta = -1$, representing a constant fluctuation (in which $g(\mathbf{r}, \mathbf{r}')$ has the Yukawa form and particle orbits are very highly correlated inside the shielded region.) Now we must normalize

$$w = w_0 e^{y^2}$$

with

$$y^2 = \frac{N}{4} \left( \frac{\eta_0}{\langle n \rangle} \right)^2$$

This requires

$$1 = \frac{4w_0}{\sqrt{N}} \int_0^\infty e^{y^2} dy$$

where the limit is, for the moment, left unspecified, but we assume $\kappa(N/4)^{1/2} \gg 1$. Using

$$\lim_{x \to \infty} \int_0^x e^{t^2} dt = e^{x^2/2}$$

(differentiate both sides and let $x \to \infty$) shows that

$$w = \frac{\kappa N}{4} e^{N(x^2 - \kappa^2)/4}$$

$$x^2 \leq \kappa^2$$

where $x = (4/N)^{1/2} y$. This is essentially a ‘one sided’ delta function at $x \leq \kappa$. The probability of a fluctuation is enormous over a very small part of the spectrum. Next we must find $\kappa$, the maximum value of $|\eta/\langle n \rangle|$. Within our theoretical framework, we want the value of $\kappa$ to be one which minimizes the free energy $F = F_0 - \Phi/2$. Evidently this is the maximum value of $\kappa$ consistent with the theory, i.e. $\kappa = 1$. So the most probable fluctuation is here of order unity rather than $N^{-1/2}$ as in a perfect gas, and the spectrum is peaked at about the Jeans length. A more realistic case would have a broader peak. Although this example is somewhat special, it seems to illustrate the main physical points.

Apparently there is a profound connection between the desire of a self-gravitating system to maximize its entropy and collapse and its desire to fluctuate wildly. Under these conditions it is no longer possible to separate hydrodynamic, statistical, and thermodynamic descriptions of the system.

As a result, this theory suffers from the same insurmountable difficulty as all fluctuation theories involving short range forces: they invalidate themselves when $\eta \gg \langle n \rangle$. Nevertheless, it is perhaps encouraging that short range theories give surprisingly accurate quantitative results until the fluctuations actually become macroscopic. We may hope that long range theory will provide at least reasonable qualitative and semi-quantitative information until this regime is reached. All indications are that when the long range theory breaks down, it breaks down violently. Whether this breakdown may herald the formation of galaxies is considered in Section 7.
To know how well an equilibrium theory applies in the actual Universe, we must as a very minimum know the scale factor \( R(t) \). Not only is this fundamental quantity unknown, but there is no observationally tested theory by which we understand the physics of \( R(t) \) sufficiently to derive it with confidence.

It is possible to consider a view from which the problem of the origin of perturbations is fairly naturally resolved, though at the expense of modifying the familiar —though untested—cosmological models. We might try, for example, a new approach based on the following hypothesis: regions of the Universe which are not in rapid gravitational collapse evolve quasi-statically in the sense of GTD. Here rapid means in times short compared to the Hubble time \( H^{-1} = R/\dot{R} \).

An immediate consequence of the hypothesis is that non-collapsing regions are sufficiently close to GTD equilibrium to have fluctuation spectra similar to those discussed in Sections 5 and 6. By considering regions which collapse, and knowing the time when they began collapsing, we can constrain \( R(t) \) and the possible cosmological models. It is sufficient to consider regions small enough to be Newtonian. (For scales comparable with \( R \), the Newtonian approximation does not apply. If the observed microwave background is a relic of the primordial fireball, then our hypothesis involves a relativistic version of GTD until the matter and radiation decouple. Until decoupling, the sound speed is essentially the speed of light, so it is likely that the particle horizon will be the analogue of the critical (Jeans) length and fluctuations may occur on this larger scale. The self gravity of the photons may cause great fluctuations in photon density and these fluctuations will carry the matter with them by the electron drag discussed by Peebles (1965). This will depend on the role of short range photon–photon interactions (e.g. those mediated by scattered particles) under these conditions. During decoupling the critical length decreases. Since this decrease is rapid compared to the Hubble time, fluctuations should result on a wide range of scales, but their amplitude during decoupling will be only partially developed, probably far below their equilibrium amplitudes. Furthermore, viscosity will be important. After decoupling, however, the hypothesis ensures that the decoupled fluctuations will be in GTD equilibrium. The present application would start at this point.)

On this picture, the first stages in galaxy formation are coherent non-linear fluctuations in regions of about the Jeans length. (To see if larger ones form, detailed numerical calculations of the spectrum are necessary. Collective shielding may prevent their formation.) These fluctuations would not arise if the problem were considered from the simpler hydrodynamic viewpoint. A rough criterion for them to be in equilibrium is that sound can cross them in a time short compared with the time \( (\approx H^{-1}) \) for the background density to change by a factor of, say, two. This requires that

\[
H^2 < 4\pi G \rho_0. \tag{118}
\]

The stronger this inequality is satisfied, the closer the Jeans modes are to equilibrium. Non-equilibrium theory will determine how closely equation (118) must be satisfied for the equilibrium approximation to be useful. If equation (118) were applicable now, it would require a high density universe if a Friedman model were used: \( \sigma_0 = 4\pi G \rho_0 / 3 H_0^2 > 1/3 \). But since the Friedman models may not be appropriate, one could directly adopt (or adapt) the hypothesis as a basis for cosmology, or try to incorporate it into general relativity.

The non-linear fluctuations of about the Jeans mass which are present are un-
stable and contract appreciably in times short compared to $H_0^{-1}$. If the dispersion in
time over which they contract is small, a possibility suggested by the ill-understood
collective properties of the medium, their radiation from molecular hydrogen may
contribute substantially to the present infra-red background radiation (Saslaw &
Zipoy 1967). These clouds are the basic particles for a medium which itself fluctuates strongly. Clouds of clouds may form, and a structure of hierarchies grows.
Because higher levels of the hierarchy depend on the random velocities of lower
levels, further evolution depends upon new relaxation times. Rapidly the problem
becomes a dynamic and chaotic one. But already enough non-linear structure is
present over scales which make it likely that galaxies and clusters can develop.

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