THE ESCAPE OF LIGHT FROM WITHIN A MASSIVE OBJECT

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SUMMARY

An analysis is made of the escape of light from within spherically symmetric gravitational masses. It is found that there exist models where a portion of the light emitted inside a star does not escape past the surface; rather, only light within a certain, variable ‘escape cone’ leaves the star. This ‘escape cone’ does not steadily shrink with decreasing radius of emission, unlike the somewhat analogous cone effect for light in an external spherically symmetric field.

1. INTRODUCTION

A few years ago, Synge (1966) reviewed the concept of the so-called ‘escape cone’ of light emitted in the vicinity of gravitationally-intense objects (i.e. where \( r/r_{\text{Schwarzschild}} \geq 1 \)). In a region outside but close to the surface of such objects, not all of the light emitted escapes to a distant observer; rather, light only escapes if it is emitted within a cone which has the property that it monotonically shrinks in angle as its vertex approaches the surface of the object. Zel’dovich & Novikov (1965), and others have also discussed this interesting and unusual effect. These investigations have all dealt with the emission of light from a region exterior to a spherically symmetric mass distribution.

Because of the possible importance of general relativistic effects on the characteristics of stellar objects, we wish to discuss here the emission and escape of light from *inside* spherically symmetric mass distributions. (Recently, for example, interest has arisen in such problems as interior models for quasi-stellar objects (Hoyle & Fowler (1967)), and the question of light escaping from the inside of an assembly of separate heavy bodies. In particular, we will explicitly analyse the interior Schwarzschild field. This model, although idealized, can be analysed analytically; it should give useful insight into the relativistic effects and possible severe restrictions that can exist for the motion and escape of light from within a mass distribution.

We will show that there also exists a ‘cone effect’ for light escaping from within the object. However, the ‘interior escape cone’ does not monotonically shrink with decreasing radius of emission, as does the ‘exterior escape cone’ discussed by Synge.

2. THE INTERIOR FIELD

For constant density, \( \rho(r \leq r_b) = \rho_0 \), \( \rho(r > r_b) = 0 \), we have the Schwarzschild interior field (Schwarzschild 1916), given by

\[
ds^2 = \left[ \frac{3}{2} \sqrt{\left(1 - \frac{r_b^2}{R^2}\right)} - \frac{1}{2} \sqrt{\left(1 - \frac{r^2}{R^2}\right)^2} \right] c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} - r^2(d\phi^2 + \sin^2 \phi \, d\theta^2) \quad 0 \leq r \leq r_b, \quad (1)
\]

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where

\[ R^2 = \frac{3c^2}{8\pi G P_0} \quad \rho_0 = \frac{3M}{4\pi r_b^3} \]

The field equations lead to an immediate restriction (Schwarzschild 1916): in order that the pressure never becomes infinite anywhere, \( r_b \) must be greater than \((9/4)m\). (\( G \) and \( c \) have been set equal to unity; the Schwarzschild radius of the sphere in these units is \( r_s = 2 \ m \).)

Solving the geodesic equations of motion for light \((ds^2 = 0)\), we find \( \phi = 0 \), or motion is restricted to a plane, chosen as \( \phi = \pi/2 \), and

\[ r^2 \dot{\theta} = \alpha = \text{constant} \quad (2) \]

\[ \left[ \frac{3}{2} \frac{1}{\sqrt{\left(1 - \frac{r_b^2}{R^2}\right)}} - \frac{1}{2} \right] \left(1 - \frac{r^2}{R^2}\right)^{1/2} = t = \beta = \text{constant}, \quad (3) \]

where a dot indicates differentiation with respect to an affine parameter. Since the interior and exterior solutions match at \( r_b \), an examination of the exterior equations of motion for light (Appendix A) readily yields

\[ \alpha/\beta = l, \quad (4) \]

the classical impact parameter as measured at infinity.

Combining equations (2), (3) and (4) with \( ds^2 = 0 \), we find

\[ \frac{dr}{d\theta} = r \sqrt{(1 - r^2/R^2)} \left\{ \frac{4r^2}{l^2[B - \sqrt{(1 - r^2/R^2)^2 - 1}]} \right\}^{1/2}, \quad (5) \]

where

\[ B \equiv 3(1 - r_b^2/R^2). \]

Thus the motion of light is characterized by a single parameter, \( l \).

The quantity inside the braces in equation (5) must always be non-negative; therefore, at each \( r \leq r_b \), the allowable range of \( l \) is

\[ 0 \leq l \leq l_t = \frac{2r}{B - \sqrt{(1 - r^2/R^2)}}, \quad (6) \]

\( l_t(dr/d\phi |_{r=r_b} = 0) \) is always greater than \( r \) (classically, \( l_t = r \)). This is an illustration of the curvature of space-time, for \( l \) cannot be given a real interpretation near the source, but only at \( r = \infty \). We must therefore be content to consider \( l \) only as a mathematical constant of the motion.

Kuchowicz (1965) has very neatly solved equation (5) and found

\[ r = R \frac{B \sqrt{[B^2 - 1 + (A + 1 - B^2) \sin^2 \theta] + \sqrt{[A + 1 - B^2] \sin \theta}}}{B^2 + (A + 1 - B^2) \sin^2 \theta} \]

where

\[ A \equiv 4R^2/l^2. \]

Examining this equation shows that, for a sphere in the range

\[ 9/4 \ m < r_b < 3 \ m \]

(recall that \( r_b \) is proportional to \( \rho_0 \)), light emitted at a radius \( r \), with

\[ I \geq I_c = r_b \sqrt{(1 - 2m/r_b)} \]
does not escape the sphere (i.e. there exists a turning point \( \leq r_b \)). \( l_c \) is less than or equal to \( l_t \) in the region

\[
r \geq r_b \left[ \frac{r_b - 0/4 m}{r_b - 3/2 m} \right] \equiv r_c.
\]

In this region, some of the light emitted in the range \( l = 0 \) to \( l_t \) never escapes past \( r_b \). All of the light emitted from \( r \leq r_c \) escapes past the boundary of the sphere. For \( r_b \gtrsim 3 \) m, all of the light emitted in the allowable range of \( l \) at any \( r \) escapes.

These results are illustrated in Fig. 1. For \( r_b \) large, \( l_t \) versus \( r \) is practically a straight line of slope one, for \( g_{00} \) and \( g_{11} \rightarrow 1 \) as \( \rho_0 \) decreases. For \( r_b \gtrsim 3 \) m (as in Fig. 1(a)), \( l_t \) is monotonically increasing with \( r \). For \( r_b \lesssim 3 \) m (as in Fig. 1(b)), \( l_t \) reaches a maximum at

\[
l_{\text{max}} = \frac{r_b}{3} \left[ \frac{4r_b - 9 m}{r_b - 2 m} \right]^{1/2}
\]

and then decreases to \( l_t = l_c \) at \( r = r_b \).

3. CONE OF ESCAPE

We consider the cone of emission of light escaping from the sphere at any \( r \leq r_b \). From the metric, equation (1), the \( r \) and \( \theta \) spatial displacements are

\[
dl_{\theta} = r \, d\theta
\]
\[
dl_r = \sqrt{g_{11}} \, dr.
\]

The half-angle of the cone of emission as measured from the radial direction is then

\[
\tan \theta = \frac{dl_{\theta}}{dl_r} = \frac{r}{\sqrt{g_{11}}} \left( \frac{dl}{dr} \right)_{l_c} = \left( \frac{4r^2}{l_c^2 [B - \sqrt{(1 - r^2/\bar{R}^2}]^2 - 1} \right)^{-1/2}.
\]

Fig. 2 illustrates \( \theta \) for various values of \( r_b \). For \( r_b \gtrsim 3 \) m, \( \theta = \pi/2 \) for all \( r \). (It should be clear that, while the spread from \( \theta \) to \( l_t \) may be decreasing with decreasing \( r \), if all the light emitted at \( r \), in the allowable range \( 0 \) to \( l_t \), escapes the object, the escape cone is still \( \pi/2 \).) For \( r_b \lesssim 3 \) m, we have the interesting result that, with decreasing \( r \), the full cone angle of the envelope of escaping light initially decreases from \( \pi \), reaches a minimum, and then increases to \( \pi \) at \( r_c \). This is in contrast to the result for the exterior Schwarzschild field, where for \( r < 3 \) m (with an emitter exterior to the mass distribution), the corresponding cone envelope is monotonically decreasing.

The redshift of the light emitted of course increases as the point of emission moves closer to the centre. For a photon emitted radially \( (l = 0) \) from a point at rest in the object, the redshift relative to an observer at infinity is simply

\[
z = 1/\sqrt{g_{00}} - 1
\]
\[
= \left[ \frac{3}{2} \sqrt{\left(1 - \frac{r_b^2}{\bar{R}^2}\right) - 1/2} \right] \left(1 - \frac{r_b^2}{\bar{R}^2}\right)^{-1} - 1,
\]

which increases with decreasing \( r \).
Fig. 1. Range of light emitted ($l_o$ to $l_1$) as a function of $r/r_b$ for various values of $r_b$. For $r_b < 3m$, not all the light emitted in the region $r_o < r < r_b$ escapes past $r_b$ (shaded section).
**Fig. 2.** The half-angle of the cone of escaping light as a function of $r/r_b$, for various values of $r_b$. Curve (a), $\theta = \pi/2$, is valid for $r_b \geq 3m$. Curves (b), (c), and (d) correspond respectively to $r_b = 2.86m$, 2.35m, and 2.27m.

**Fig. 3.** The transit time, in units of $m$, from $r = 0$ to $r = r_b$ for a radial photon ($l = 0$), as a function of $r_b$. The transit time is longer than the classical case, especially in the limit as $r_b$ approaches its minimum allowable value, $(9/4)m$.

**4. Transit Time of Light**

It is interesting too to examine the transit time of light travelling within the sphere, for it emphasizes the importance of general relativity for large $\rho_0$. From the geodesic equations, we find

$$\frac{dr}{dt} = \frac{l_{g00}}{r\sqrt{-g_{11}}} \sqrt{\left(\frac{r^2}{g_{00}l^2} - 1\right)}.$$
This equation can be integrated analytically, yielding

\[ t - t_0 = \frac{R}{\sqrt{(B^2 - 1)}} \sin^{-1} 2\frac{[A(A - \sigma)(\sigma + 1 - B^2)(B^2 - 1)]^{1/2}}{\sigma(A + 1 - B^2)} \]

where

\[ A \equiv 4R^2/l^2 \quad \text{and} \quad \sigma \equiv R^2 \left[ \frac{B - \sqrt{(1 - r^2/R^2)}}{r} \right]^2. \]

In the limit of vanishing \( \rho_0 \), we have

\[ t - t_0 \approx \sqrt{(r^2 - l^2)} + 4/3 \frac{(r^2 - l^2)^{3/2}}{R^2} + o \left( \frac{1}{R^{2n}} \right), \]

the classical result, plus terms of order \( 1/R^{2n} \). (See Fig. 3.)

5. CONCLUSION

Gravity places unusual restrictions on the emission and escape of light from within a spherically symmetric mass distribution. There exist regions where the escape cone is less than \( \pi \). While the interesting range of \( r_b \) here (or equivalently, \( \rho_0 \)) is not very great, recent work by Kuchowicz (1968) indicates that other, perhaps more realistic configurations have a much greater latitude of dimensions within which these capture effects may exist.

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REFERENCES


APPENDIX A

The exterior Schwarzschild solution is given as

\[ ds^2 = (1 - 2 m/r) \, dt^2 - \frac{dr^2}{(1 - 2 m/r)} - r^2 (d\phi^2 + r^2 \sin^2 \phi \, d\theta^2). \]

The relevant geodesic equations of motion for light (\( ds^2 = 0 \)) are readily found to be

\[ r^2 \ddot{\theta} = p = \text{constant} \]

\[ (1 - 2 m/r) \dot{t} = c = \text{constant}. \]
If we consider a test particle at infinity, of mass $m$ and with velocity $v$ as measured at infinity, equations (A1) and (A2) can be written as

\begin{align*}
\sqrt{r^2} = p &= \frac{lv}{\sqrt{1 - v^2}} \\
(1 - 2m/r)t = t &= \frac{v}{\sqrt{1 - v^2}},
\end{align*}

where $l$ is the classical impact parameter of the particle as measured at infinity. Therefore, for light, where $v = 1$, we have

\begin{equation}
p/c = l.
\end{equation}

Since the interior and exterior solutions match at $r_b$, we have from equations (2), (3), (A3), and (A4),

\begin{align*}
\alpha &= p \\
\beta &= c.
\end{align*}

Therefore,

\begin{equation}
\frac{\alpha}{\beta} = \frac{p}{c} = l.
\end{equation}