RANDOM WAVE-FRONT PERTURBATIONS AND
TELESCOPIC STAR IMAGES

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SUMMARY

A computational model of random perturbations induced in wavefronts in
the entry pupil of telescopes of moderate aperture has been constructed and
used to predict the instantaneous distribution of intensity in the perturbed
stellar images formed by such telescopes.

Parameters of the model have been chosen to give a reasonably life-like
imitation of perturbations due to imperfect 'seeing'; and in order to gain
some idea of the adequacy of the model, instantaneous intensity distributions
in images of α-Aur have been obtained by computer reduction of observa-
tions made at the Cambridge 36-inch telescope using a spectrocon electronno-
graphic image tube. The model and its predictions are described and com-
pared with the observations.

I. INTRODUCTION

A star image in a small telescope of 3 or 4 inches aperture often shows the well
known diffraction disc quite well under conditions of good seeing, although the
associated ring system (predicted by theory and many times photographed in the
laboratory) is seldom seen. The appearances are different in telescopes of larger
aperture. In a 30-inch reflector, for example, the image of a 1st magnitude star
under fairly good seeing conditions bears no resemblance whatever to the Airy
disc and ring pattern; it suggests rather an erupting nucleus of light, intensely
bright and continuously changing. The same is true in any telescope of aperture
greater than about 25 inches.

Not only is the appearance of star images quite different in telescopes of small
and of large aperture; they behave quite differently when (as sometimes happens)
the 'seeing' slowly improves from fairly good to nearly perfect. In a small telescope
(say one of aperture 10 cm or less) the diffraction disc hardly changes in appear-
ce, but its random movements caused by atmospheric tremor gradually settle
down until it appears stationary. In a large telescope, for which the ideal Airy disc
is small compared with the so-called seeing disc under ordinary observing condi-
tions, the structure of the erupting image changes. As the 'seeing' slowly improves
a single intensity-spike or bright core gradually becomes dominant among the
fluctuations in the central part of the image, which continues to fluctuate internally
but gradually loses in brightness as the central spike grows more prominent.
R. F. Griffin (1968, private communication) has observed these changes at the
Coudé focus of the 100-inch Mt Wilson telescope. There is no perceptible fluctua-
tion in the position of this spike. Finally, the central spike becomes completely
dominant and in the end (when perfect seeing is reached) we have the Airy pattern,
of which only the Airy disc is usually visible to the eye.
In the present paper, a beginning is made towards understanding these phenomena by examining the effects on the star image of random perturbations of the incoming wave-fronts. Even an over-simplified model of the perturbations of the wave-fronts entering the aperture of a large telescope may, provided it leads to a reasonably good imitation of the stellar images, be of use in estimating the efficiency of future astronomical instruments. In Sections 2 and 3 we introduce an over-simplified model of this kind and examine computationally the effect of 14 sample perturbations on monochromatic images. In order to compare the results with actual star images, observations of the instantaneous distribution in the images of bright stars, re-imaged from the prime focus of the Cambridge 36-inch telescope, were made using an electromagnetic image tube. The procedures are described in Sections 4 and 5. Comparisons between the general structural characteristics of the computed model images and the observed images are made in Section 6.

The observed images were obtained, with an exposure of 0.0028 s, by means of the greatly improved electronographic method developed by Professor J. D. McGee and co-workers (1966), whose 10 years' work has produced a spectrocon of signal-to-noise ratios of the order of 50 or 100 times those of ordinary photography under the conditions of our observations.

2. Model Perturbations and Model Images

If we had a satisfactory theory of atmospheric turbulence, the most obvious way to apply it to develop a theory of the image fluctuations which determine astronomical 'seeing' would be to proceed in two stages. In the first stage we should calculate the effect of the turbulence on the Rayleigh complex displacement produced in the entry pupil of a telescope by the axial beam of light from a point source (star) at infinity. This effect appears as a statistical set of complex-valued random perturbations of the Rayleigh displacement, which in the absence of perturbation can be taken as equal to 1 throughout the clear aperture.

In the second stage we should apply Huyghens's principle to investigate the effects of these statistically prescribed random perturbations on the telescopic star image. This might be expected to yield a theory of 'seeing'.

Both parts of the above programme seem to be quite out of reach at present. On the one hand, far too little is known about the turbulence structure of the atmosphere, or even (what would suffice for present purposes) about the statistics of the perturbations caused by this turbulence in the phase-amplitude distribution in the entry pupil. On the other hand, almost nothing seems to have been worked out numerically about the relation between individual 'random' perturbations of the phase-amplitude distribution in the entry pupil, even in light from a monochromatic source at infinity on the optic axis of the telescope, and the resulting fluctuations in the form of the star image.

We shall try to gain some insight into the quantitative relation between 'random' perturbations and the resulting fluctuations by examining a number of particular cases computationally. These correspond to image-exposures at arbitrarily selected instants, each exposure being of negligible duration and the instants being too far apart for any detailed correlation between perturbations to be expected. In these circumstances the perturbations at the different instants can be treated as members of a statistical set of random functions.
By making assumptions about the statistical mean (s.m.) spatial power spectrum of this set, we can impose on its members a statistical tendency to show the kind of structural size-distribution which would justify us in taking them as trial models of perturbations resulting, at arbitrary, non-contiguous instants of time, from atmospheric turbulence. Of course, this naive approach can only take us a small part of the way towards an adequate model of the effects of turbulence, because it avoids discussing the manner in which a particular sample perturbation changes with time. (This also would be statistically specified.)

We proceed as follows:

In the plane of the circular entry pupil A (see Fig. 1) we draw rectangular coordinate axes \((\xi, \eta)\), scale-normalized so that the edge of the pupil is represented by the circle \(\xi^2 + \eta^2 = 1\). Let \(S\) denote the square

\[ -M \leq \xi \leq M, -M \leq \eta \leq M, \]

where \(M\) is an integer (taken as 2 in Fig. 1). We need only define the phase-amplitude distribution (Rayleigh complex displacement) \(\Phi(\xi, \eta)\) over the entry pupil A, and not over the whole \((\xi, \eta)\)-plane, since by Huyghen’s principle the normalized image function

\[ D(x, y) = \frac{1}{\pi} \int \int_A \Phi(\xi, \eta) \exp(2\pi i(\xi x + \eta y)) \, d\xi \, d\eta \]

depends only on the values taken by \(\Phi(\xi, \eta)\) in A. Here \(x, y\) are scale-normalized coordinates in the ideal image; the radius of the Airy disc is \(\alpha 6\alpha\) \((x, y)\)-units. However, the analysis is formally simpler if we begin by considering \(\Phi(\xi, \eta)\) over the square \(S\) (see Fig. 1) and use the double Fourier expansion

\[ \Phi(\xi, \eta) = 1 + \sum_{pq} (a_{pq} + ib_{pq}) \exp(2\pi i(\xi u_p + \eta v_q)) \]

\((p, q = 0, \pm 1, \pm 2, \ldots)\).
where \( a_{pq}, b_{pq} \) are real and

\[
(u_p, v_q) = \frac{1}{2M} (p, q) \tag{4}
\]

\[(p, q = 0, \pm 1, \pm 2, \ldots).\]

If we take the numbers \( a_{pq}, b_{pq} \) to be sample values of random variables of the same name, then \( \Phi(\xi, \eta) \) becomes a sample function from a statistical set \( \{\Phi(\xi, \eta)\} \) whose statistical structure depends on the statistical distributions, single and joint, of the \( a_{pq} \) and \( b_{pq} \).

From (2) and (3) we obtain, after calculation

\[
D(x, y) = \frac{J_1(2\pi z)}{\pi z} + \sum_{pq} (a_{pq} + ib_{pq}) \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}}, \tag{5}
\]

where

\[z = \sqrt{x^2 + y^2}, \quad Z_{pq} = \sqrt{A^2 + B^2}, \quad A = u_p + x, \quad B = v_q + y \tag{6}\]

and \( J_1 \) is Bessel's function of order 1. From (4) and (6) we see that, for each value of \( (p, q) \), \( Z_{pq} \) is the distance (measured in \( (x, y) \)-units) from \( (x, y) \) to the lattice point \((-p/2M, -q/2M)\) in the \((x, y)\)-plane. Thus (5) express the image function \( D(x, y) \) as a sum of Airy complex displacement patterns centred respectively on the lattice points \((-p/2M, -q/2M)\) in the \((x, y)\)-plane.

The function \( I(x, y) = |D(x, y)|^2 \) gives, on multiplication by a suitable normalizing constant, the intensity at the points \((x, y)\) of the focal plane which lie within a moderate distance (say 10 or 20 Airy disc radii) of the \((x, y)\)-origin. We shall refer to it as the image-intensity function.

As is well known, every function \( \Phi(\xi, \eta) \) which is continuous and bounded in \( S \) has an expansion (3) which converges to \( \Phi(\xi, \eta) \) in mean square over \( S \) in the sense that, as \( N \to \infty \),

\[
\iint_S |\Phi(\xi, \eta) - \Phi_N(\xi, \eta)|^2 \, d\xi \, d\eta \to 0, \tag{7}
\]

where \( \Phi_N(\xi, \eta) \) denotes the partial sum

\[
1 + \sum_{p^2 + q^2 \leq N^2} (a_{pq} + ib_{pq}) \exp(2\pi i(\xi u_p + \eta v_q)) \tag{8}
\]

of the series (3). It then follows that, as \( N \to \infty \), the value of the expression

\[
D_N(x, y) = \frac{1}{\pi} \int_A \Phi_N(\xi, \eta) \exp(2\pi i(\xi x + \eta y)) \, d\xi \, d\eta
\]

\[
= \frac{J_1(2\pi z)}{\pi z} + \sum_{p^2 + q^2 \leq N^2} (a_{pq} + ib_{pq}) \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}}, \tag{9}
\]

which is itself a partial sum of the series (5), tends to \( D(x, y) \) uniformly in \((x, y)\) over the whole \((x, y)\)-plane. For by Schwarz's inequality

\[
|D(x, y) - D_N(x, y)|^2 = \left| \frac{1}{\pi} \int_A [\Phi(\xi, \eta) - \Phi_N(\xi, \eta)] \exp(2\pi i(\xi x + \eta y)) \, d\xi \, d\eta \right|^2
\]

\[
\leq \frac{1}{\pi^2} \int_A |\Phi(\xi, \eta) - \Phi_N(\xi, \eta)|^2 \, d\xi \, d\eta \int_A |\Phi(\xi, \eta) - \Phi_N(\xi, \eta)|^2 \, d\xi \, d\eta
\]

\[
= \frac{1}{\pi} \int_S |\Phi(\xi, \eta) - \Phi_N(\xi, \eta)|^2 \, d\xi \, d\eta
\]

and, by (7), the last expression tends to zero as \( N \to \infty \).
This means that the partial sums (9) provide us with uniform approximations $D_N(x, y)$ to the image function $D(x, y)$ even when no assumption is made about "periodic boundary values" of $\Phi(\xi, \eta)$ round the edge of $S$. Although the $\Phi_N(\xi, \eta)$ are each periodic on the boundary of $S$ (in the sense that

$$\Phi_N(-M+\alpha, \eta) = \Phi_N(M-\alpha, \eta)$$

for $-M \leq \eta \leq M$ and

$$\Phi_N(\xi, -M+\alpha) = \Phi_N(\xi, M-\alpha)$$

for $-M \leq \xi \leq M$) their mean square limit over $S$ is not subject to any such condition, even when $\Phi$ is continuous and bounded in $S$. As an illustration, we might take $\Phi(\xi, \eta) = \exp \left(2\pi i(\alpha \xi + \beta \eta)\right)$, where $\alpha, \beta$ are real constants. This gives, directly from (2),

$$D(x, y) = \frac{J_1(2\pi Z(\alpha, \beta))}{\pi Z(\alpha, \beta)},$$

(10)

where $Z(\alpha, \beta) = \sqrt{(\alpha + x)^2 + (\beta + y)^2}$; that is, a displaced Airy pattern centred on the point $(x, y) = (-\alpha, -\beta)$. At the same time, we can still treat this example by the general procedure, expanding the function $\Phi(\xi, \eta) = \exp \left(2\pi i(\alpha \xi + \beta \eta)\right)$ as in (3) and proceeding by way of (9) to (5). Comparing results, we see that the series (5) has the sum (10) and the series (3) converges in mean square over $S$ to the function $\exp \left(2\pi i(\alpha \xi + \beta \eta)\right)$ from which we started. But if $(\alpha, \beta)$ happens to coincide with $(u_p, v_q)$ for some integers $p, q$ then the series on the right of (3) has just one non-zero term under the $\sum$ and therefore converges uniformly, while its sum has "periodic boundary values" round the edge of $S$.

We return to the set $\{\Phi(\xi, \eta)\}$ and try to assign to it a statistical structure which will make it suitable to present purposes. First we take the $a_{pq}$ and $b_{pq}$ in (3) to be independent gaussian random variables, each centred on the value zero, with r.m.s. values satisfying the conditions

$$\langle a_{pq}^2 \rangle = \langle b_{pq}^2 \rangle = \text{const.} \times \phi(u_p, v_q)$$

and

$$\sum_{pq} \left(\langle a_{pq}^2 \rangle + \langle b_{pq}^2 \rangle\right) = 2\sigma^2,$$

where the angle brackets denote statistical means and where the constant $\sigma$ and the function $\phi$ are to be chosen later. To do this we must take

$$\langle a_{pq}^2 \rangle = \langle b_{pq}^2 \rangle = \sigma^2 \phi(u_p, v_q)\phi(u_p, v_q)/P,$$

(11)

where $P$ stands for

$$\sum_{pq} \phi(u_p, v_q).$$

By Parseval's theorem we see from (3) that

$$\left\langle \frac{1}{(2M)^2} \int \int_S \left| 1 - \Phi(\xi, \eta) \right|^2 d\xi d\eta \right\rangle = \sum_{pq} \langle a_{pq}^2 + b_{pq}^2 \rangle = 2\sigma^2,$$

so that $\sigma$ is a statistical parameter which measures the strength of the perturbations, while the set of values $\phi(u_p, v_q)$ $(p, q = 0, \pm 1, \pm 2, \ldots)$ is the discrete analogue, for a set $\{\Phi(\xi, \eta)\}$ defined over the finite square $S$, of the usual s.m. (statistical mean) spectral power density $\phi(u, v)$.

These conditions evidently do not suffice to make every sample set of coefficients in (3) define a sample function. But it can be shown (Kolmogoroff 1950...
that if in (11) \( \phi(u, v) \) is chosen to satisfy the physically natural condition

\[
\sum_{pq} \phi(u_p, v_q) < \infty,
\]

then the series

\[
\sum_{pq} (a_{pq}^2 + b_{pq}^2)^{1/2}
\]

will converge except in a subset of cases with total probability zero. It follows that, outside this zero-probability subset, the series (3) converges absolutely and uniformly in \( S \), and is the Fourier expansion in \( S \) of its sum \( \Phi(\xi, \eta) \), which is itself continuous in \( S \) and has equal boundary values at corresponding boundary points at opposite sides of \( S \). In other words, almost all the sample functions \( \Phi(\xi, \eta) \) are now continuous and bounded in \( S \), and have continuous boundary values satisfying the two equations

\[
\begin{align*}
\Phi(-M+\alpha, \eta) & = \Phi(M-\alpha, \eta) \quad (-M \leq \eta \leq M), \\
\Phi(\xi, -M+\alpha) & = \Phi(\xi, M-\alpha) \quad (-M \leq \xi \leq M).
\end{align*}
\]

Let \( \{\Phi(\xi, \eta)\}_C \) denote the set of members of \( \{\Phi(\xi, \eta)\} \) outside the above zero-probability subset, so that all members of \( \{\Phi(\xi, \eta)\}_C \) are continuous. An arbitrary choice of the \( a_{pq}, b_{pq} \) under the constraints (11), (12) leads, with probability 1, to a member of \( \{\Phi(\xi, \eta)\}_C \).

However, even if the function \( \phi(u_p, v_q) \) is chosen so that it falls off rapidly to zero as \( w_{pq} = \sqrt{u_p^2 + v_q^2} \to \infty \), the set \( \{\Phi(\xi, \eta)\}_C \) will usually include functions \( \Phi(\xi, \eta) \) for which the Fourier expansions (3) each contain terms

\[
(a_{pq} + ib_{pq}) \exp\left(2\pi i (\xi u_p + \eta v_q)\right)
\]

with non-negligible absolute value \( \sqrt{a_{pq}^2 + b_{pq}^2} \) and with large spatial frequency \( w_{pq} \). Each such term contributes, by (5), a term

\[
(a_{pq} + ib_{pq}) \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}}
\]

to the image function \( D(x, y) \); when \( w_{pq} \) is large enough, (15) represents mainly light scattered right out of the bright central part of the image.

On the other hand we have the following result:

Corresponding to any positive numbers \( \epsilon \) and \( \delta \), we can choose a constant \( w_1 \) so large that, for all \( (x, y) \) and all \( N \geq 2MW_1 \)

\[
|D(x, y) - D_N(x, y)|^2 < \epsilon
\]

except in a subset of sample functions of total probability \( < \delta \). (This follows from (11) and (12).)

Thus we lose nothing of importance for present purposes (and avoid some difficulties connected with the notion of randomness) by fixing a finite upper limit \( w_1 \) to the spatial frequencies \( w_{pq} = \sqrt{u_p^2 + v_q^2} \) which are to be embodied in the random perturbations. In physical terms, we use a statistical set of band-limited functions (3) as the model which is to represent the perturbed complex displacements in \( S \). This is statistically equivalent to setting \( \phi(u_p, v_q) = 0 \) whenever \( u_p^2 + v_q^2 > w_1^2 \); more exactly, it is equivalent to fixing \( a_{pq} = b_{pq} = 0 \) whenever \( p^2 + q^2 > (2M)^2 w_1^2 \). Henceforth we use \( \{\Phi(\xi, \eta)\} \) to denote this last set; its members are all exponential polynomials.
It is easy to show that the new set \( \{ \Phi(\xi, \eta) \} \) is random in \( S \) (entropy-maximized in \( S \)) under the constraint (11) together with the band-limitedness. (For the method of proof see Goldman (1953), Sections 4.3, 4.6 and Appendix 8.)

Now let us take a function \( \Psi(\xi, \eta) \), continuous and bounded in \( S \), and expand it in a Fourier series over \( S \) of the form

\[
\Psi(\xi, \eta) = \sum_{pq} (\alpha_{pq} + i\beta_{pq}) \exp(2\pi i(\xi u_p + \eta v_q)),
\]

where the coefficients \( \alpha_{pq} + i\beta_{pq} \) satisfy the condition

\[
\sum_{pq} (\alpha_{pq}^2 + \beta_{pq}^2) < \infty.
\]

Then all those functions

\[
\Phi(\xi, \eta) + \Psi(\xi, \eta) = \sum_{pq} [(a_{pq} + \alpha_{pq}) + i(b_{pq} + \beta_{pq})] \exp(2\pi i(\xi u_p + \eta v_q))
\]

in which \( \Phi(\xi, \eta) \) belongs to \( \{ \Phi(\xi, \eta) \} \), and lies outside the exceptional set, provide series \( \Psi(\xi, \eta) \) which tend to a limit \( D_{n+y}(x, y) \) uniformly in \( (x, y) \) as \( N \to \infty \).

This means that we can apply the same methods to an image with aberrations, by superposing computed \( D \)-functions or, what is equivalent, by using (9) with \( a_{pq} + ib_{pq} \) replaced by \( (a_{pq} + \alpha_{pq}) + i(b_{pq} + \beta_{pq}) \), where the \( \alpha_{pq}, \beta_{pq} \) are the same for every sample function. However, we shall not pursue this generalization further in the present paper, but only mention without proof that it can be used to explain how quite small aberrations may perceptibly reduce the photographic performance of a large telescope working in a turbulent atmosphere.

If we consider a larger square \( S_2 \) in which the perturbations form an ergodic set with regard to spatial displacements belonging to \( S \) (by which we mean here that, for \( -M \leq \xi_0, \eta_0 \leq M \), the perturbation functions (sample functions) \( \Phi(\xi, \eta) \) are defined in \( S_2 \) together with a border of width \( M \) surrounding it, and that the statistics of the set \( \{ \Phi(\xi + \xi_0, \eta + \eta_0) \} \) are the same as those of \( \{ \Phi(\xi, \eta) \} \) if we subdivide \( S_2 \) into a large number of squares \( S \), each of side \( 2M \), and suppose that in the Fourier coefficients \( A_{PQ} + iB_{PQ} \) of \( \Phi(\xi, \eta) \) over \( S_2 \) all the numbers \( A_{PQ}, B_{PQ} \) are statistically independent random variables of zero mean and prescribed r.m.s. values, then we can calculate the statistical mean r.m.s. values of the \( a_{pq}, b_{pq} \) for each of the squares \( S \). For, each \( S \)-coefficient provides a Fourier term whose \( S \)-cutoff can be expanded in \( S \); thus each \( a_{pq} + ib_{pq} \) is a linear function in \( A_{PQ} + iB_{PQ} \) for any of these \( S \). As the \( A_{PQ}, B_{PQ} \) are statistically independent we can therefore determine \( \langle a_{pq}^2 \rangle, \langle b_{pq}^2 \rangle \) for the given \( S \). These \( a_{pq}, b_{pq} \) are, of course, no longer statistically independent. Thus we can find the power spectrum over \( S \) from that for \( S_2 \).

Because the variables \( a_{pq}, b_{pq} \) are here no longer statistically independent, it follows that the set \( \{ \Phi(\xi, \eta) \} \) of page 204 cannot itself represent a set of \( \Phi \) cut out by \( S \) from an ergodic set in \( S_2 \), but has a less complicated statistical structure. In particular, it has the simplifying property that

\[
\langle a_{pq}a_{pq}' \rangle = 0, \quad \langle b_{pq}b_{pq}' \rangle = 0
\]

whenever \( (p, q) \neq (p', q') \). The set of page 204 is more random than would be one with the same discrete power spectrum \( \phi(u_p, v_q) \) cut out by \( S \) from an ergodic set in \( S_2 \). For the ergodicity requires that the probability distributions of the coefficients \( A_{PQ} + iB_{PQ} \) shall each be radially symmetrical about zero in the Argand
plane. (A shift \((\xi', \eta')\) in the \((\xi, \eta)\)-plane multiplies \(A_{PQ} + iB_{PQ}\) by
\[
\exp(2\pi i(\xi' u_P + \eta' v_Q)),
\]
i.e. rotates it through an angle \(2\pi(\xi' u_P + \eta' v_Q)\) in the Argand plane.) The derived
\(a_{pq} + ib_{pq}\) then have the same symmetry, and the rest follows from the fact that the
set of page 203 is entropy-maximized in \(S\) under the constraint of given \(\phi(u_p, v_q)\).

By using, in place of (3), the double Fourier expansion
\[
\Phi(\xi, \eta) = 1 + \sum_{PQ} (A_{PQ} + iB_{PQ}) \exp(2\pi i(\xi u_P + \eta v_Q))
\]
\((P, Q = 0, \pm 1, \ldots)\)
of \(\Phi(\xi, \eta)\) in \(S_2\), where now
\[
(u_P, v_Q) = \frac{i}{2KM} (P, Q), \quad K \geq 1,
\]
we obtain in place of (5) a new expansion
\[
D(x, y) = \frac{J_1(2\pi x)}{\pi x} + \sum_{PQ} (A_{PQ}^{(2)} + iB_{PQ}^{(2)}) \frac{J_1(2\pi Z_{PQ}^{(2)})}{\pi Z_{PQ}^{(2)}},
\]
uniformly convergent in \((x, y)\), in which
\[
Z_{PQ}^{(2)} = \sqrt{(A_2)^2 + (B_2)^2}, \quad A_2 = u_P + x, \quad B_2 = v_Q + y.
\]
This is true whether \(K\) is an integer or not; we restrict \(K\) to integer values, however,
so that \(S_2\) shall be divisible into squares \(S\) of side \(2M\).

By starting from a Fourier expansion (20) over the square \(S_2\) of side \(2KM\),
and then taking the new Fourier coefficients \(A_{PQ}, B_{PQ}\) to be statistically independent
real gaussian random variables, restricted by the conditions
\[
\langle A_{PQ}^2 \rangle = \langle B_{PQ}^2 \rangle; \quad \langle A_{PQ}^2 + B_{PQ}^2 \rangle = \phi(u_P, v_Q)/(2KM)^2,
\]
we can thus provide the set \(\{\Phi(\xi, \eta)\}\) in \(A\) with a more complex statistical structure.
The new structure is the same as would have resulted from replacing \(M\) by \(KM\)
in the original analysis (equations (2)–(19)); that is to say by increasing the size of
\(S\), while keeping the aperture \(A\) unchanged.

From (20)–(22) we see that the effect of replacing \(M\) by \(KM\) in equations
(2)–(19) has been to replace each term
\[
(a_{pq} + ib_{pq}) \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}}
\]
in (5) by a cluster of \(K^2\) terms
\[
(A_{PQ}^{(2)} + iB_{PQ}^{(2)}) \frac{J_1(2\pi Z_{PQ}^{(2)})}{\pi Z_{PQ}^{(2)}}
\]
corresponding to the \(K^2\) points of a square lattice, of mesh \(1/2KM\), occupying a
square of side \(1/2M\) centred on the point \((x, y) = -(u_P, v_Q)\). The larger \(KM\)
is taken, the more natural appears the assumption that the \(A_{PQ}\) and \(B_{PQ}\) are statistically independent. Or, what is equivalent, the larger \(M\) is taken in the original
analysis (equations (2)–(19)), the more plausible the model becomes. Computation
time sets a practical upper limit to the values of \(M\) which can be used in numerical
work, however; and in practice it is a question of finding the smallest value of \(M\).
which produces an acceptable model when the results of computation are compared with actual observed images.

Returning to the model with independent, zero-centred gaussian $a_{pq}, b_{pq}$ satisfying (11), we see from equation (5) that the intensity function

$$I(x, y) = |D(x, y)|^2 = \left( \frac{J_1(2\pi z)}{\pi z} + \sum_{pq} a_{pq} \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}} \right)^2 + \left( \sum_{pq} b_{pq} \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}} \right)^2$$

(23)

and hence that the statistical mean $\langle I(x, y) \rangle$ satisfies the equation

$$\langle I(x, y) \rangle = \left( \frac{J_1(2\pi z)}{\pi z} \right)^2 + \sum_{pq} \langle a_{pq}^2 + b_{pq}^2 \rangle \left( \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}} \right)^2$$

$$= \left( \frac{J_1(2\pi z)}{\pi z} \right)^2 + \frac{2}{P} \sum_{pq} \phi(u_p, v_q) \left( \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}} \right)^2,$$

(24)

by (11).

In terms of the statistical set $\{\Phi(\xi, \eta)\}$, the statistical mean (s.m.) intensity function in the image is given by (24) and the statistical fluctuations in the relative intensity at any point $(x, y)$ by the value distribution of $I(x, y) - \langle I(x, y) \rangle$ at this point. We are mainly concerned with the cases where the function $\phi(u, v)$ is radially symmetrical about the $(u, v)$ origin, where it has a gently rounded central peak, falling away smoothly on all sides as $\sqrt{u^2 + v^2}$ increases, and flattening out as its

**Fig. 2.** Relative intensities in the model image with $M = 1, N = 10, \tau = 3, \sigma = 0.125.$
value approaches zero. In such a case, the sum

$$\sum_{pq}$$

on the right of (24) is, for smooth, slowly changing $\phi$ and for $M \geq 1$, everywhere roughly equal to $\phi(-x, -y)$, while the first term $(J_1(2\pi x)/\pi x)^2$ contributes a narrow central intensity spike, comparable in width with the Airy disc, which dominates the image when $2\sigma^2$ is small. Thus we can obtain an image which reproduces the striking features observed visually on a 100-inch telescope by R. F. Griffin, as already mentioned, and measured photoelectrically by Meinel (1960) on an 82-inch telescope.

It now seems reasonable to specialize in (11) and (12) by taking

$$\phi(u_p, v_q) = \exp \left(-\frac{(u_p^2 + v_q^2)}{\tau^2}\right) \text{ for } p^2 + q^2 \leq N^2$$

$$= 0 \quad p^2 + q^2 > N^2,$$

where the values of the parameters $N, \tau$ can be chosen later.

The cases computed may be specified by giving the values of $\tau, N, M$ and the parameter $\sigma$ which measures the strength of the perturbations. They fall into three sets

1. $(\tau, N, M) = (3, 10, 1); \sigma = 0.125, 0.5, 1.0, 2.0, 10.0$
2. $(\tau, N, M) = (3, 20, 2); \sigma = 0.25, 1.0$
3. $(\tau, N, M) = (6, 20, 2); \sigma = 0.25, 0.5, 1.0, 2.0, 5.0, 10.0, 15.0$.

![Airy disc size](image-url)

**Fig. 3.** Image with $\sigma = 0.5$; all else as in Fig. 2.
In set (3), the notation \( N = (20) \) means that in the sum (5) the terms with \( p^2 + q^2 \leq (40)^2 \) are taken into account, i.e. \( \phi(w) \) is ‘cut-off’ at \( w = 10 \), not at \( w = 5 \). But only the terms of (5) with \( p^2 + q^2 \leq (20)^2 \) are actually evaluated; the remainder, \( \text{viz.} \) those for which \( (20)^2 < p^2 + q^2 \leq (40)^2 \), appeared, from auxiliary computations of one sample case along a grid of lines \( y = 0, \pm 1, \pm 2, \ldots \) in the circle \( x^2 + y^2 < 25 \) and of other sample cases along the line \( y = 0 \) in this circle, to have no significant effect on the image intensities inside the circle \( x^2 + y^2 < (4.5)^2 \). (Some effect appeared as \( x^2 + y^2 \) approached the value 25.) Inside the circle these outer terms only show their effect by making an appreciable contribution to the mean square fluctuation \( \langle |\Phi(\xi, \eta) - 1|^2 \rangle \), given at each point \((\xi, \eta)\) in the aperture \( A \) by the equations

\[
\langle |\Phi(\xi, \eta) - 1|^2 \rangle = \sum_{p^2 + q^2 < 1600} \langle a_{pq}^2 + b_{pq}^2 \rangle = 2\sigma^2, \tag{27}
\]

in which, as usual, the angle brackets denote statistical means. This approximate treatment reduces by a factor 4 the amount of computation needed for Figs 12, 13, 14, 16, 17, 18 and 19.

3. DISCUSSION OF THE COMPUTED CASES AND THE CORRESPONDING DIAGRAMS

Briefly enumerated, the computed cases are displayed in Figs 2–19 as follows.

(1) ‘Coarse mesh’ \((M = 1)\), \( N = 10 \), \( \phi(u, v) = \exp \left( - (u^2 + v^2)/9 \right) (\tau = 3) \). Intensity distributions in the circle \( x^2 + y^2 < 25 \): \( \sigma = 0.125, 0.5, 1.0, 2.0, 10.0 \)
Phase-intensity distributions in the plane of the aperture: 
\( \sigma = 0.125 \) (Fig. 5(a) and (b)); \( \sigma = 0.5 \) (Fig. 6(a) and (b)).

All the cases (1) are derived from the same sequence of gaussian random numbers, normalized to make the statistical m.s. deviation \( \langle |\Phi(\xi, \eta) - 1|^2 \rangle \) equal to \( 2\sigma^2 \) at each point \((\xi, \eta)\) of \( S \).

(2) 'Fine mesh' \( (M = 2) \), \( N = 20 \), \( \phi(u, v) = \exp \left( -\left( u^2 + v^2 \right) / 9 \right) \) \( (\tau = 3) \).

Intensity distributions in the circle \( x^2 + y^2 < 25 \): \( \sigma = 0.25 \), \( \tau = 1 \) (Figs 10 and 11).

The terms 'coarse mesh' and 'fine mesh' refer to the spacing of the lattice points \((x, y) = \left( -u_p, -v_q \right) = \left( -p/2M, -q/2M \right)\), at which are located the peaks of the individual terms on the right of (5).

(3) 'Fine mesh, double aperture' \( (M = 2) \), \( N = 20 \),

\[ \phi(u, v) = \exp \left( -\left( u^2 + v^2 \right) / 36 \right) \] \( (\tau = 6) \).

**Fig. 5(a)** Phase distribution (radians) in the plane of the aperture for the image of Fig. 2.

\( N = 20 \) has the meaning explained on page 209. Intensity distributions in the circle \( x^2 + y^2 < 25 \): \( \sigma = 0.25 \), \( 0.5 \), \( 1 \), \( 2 \), \( 5 \), \( 10 \), \( 15 \) (Figs 12, 13, 14, 16, 17, 18 and 19).

The sets (2) and (3) are all derived from a second sequence of gaussian random numbers, normalized as before to make the m.s. deviation of the complex displacement equal to \( 2\sigma^2 \) at each point \((\xi, \eta)\) of \( S \). The curves in the intensity diagrams show relative intensities in each case. In the phase diagrams, (Figs 5(a) and 6(a)) the phases are measured in radians.
We proceed to consider some features of interest in these diagrams. Figs 2–8 refer to the first set of cases, in which a computer-generated sequence of gaussian random numbers is normalized to an adjustable r.m.s. value $\sigma$ and used to define the coefficients $a_{pq} + ib_{pq}$ in the Fourier expansion (3) of a random complex disturbance in the square $S$ circumscribing the aperture $A$ (case $M = 1$). The corresponding diffraction image (intensity distribution) was computed in the case $\sigma = 0.125, 0.5, 1.0, 2.0, 10.0$ (Figs 2, 3, 4, 7 and 8 respectively) and also the phase distribution and the intensity distribution over the aperture for $\sigma = 0.125$ (Fig. 5(a) and (b)) and for $\sigma = 0.5$ (Fig. 6(a) and (b)).

The value of $N$ in these figures was taken as 10, so as to include those terms of the series (5) whose peaks (at the points $(x, y) = (-u_p, -v_q)$) lie in the circle $x^2 + y^2 < 5^2$ filled by the distributions to be computed. The remaining terms were considered small enough to be disregarded, in examining image structure inside the circle, except very near to its edge. (A more thorough treatment would follow the method used in the set of cases with $(\tau, N, M) = (6, 20, 2)$; see page 209. This supported a similar conclusion, though without providing a rigorous proof.)

It is interesting to notice, in Fig. 2, the disintegration of the outer Airy bright rings under the small amount $\sigma = 0.125$ of perturbation, while the innermost bright ring is not very much distorted and the central disc is hardly disturbed. In
Fig. 3, where the perturbations are four times as large \((\sigma = 0.5)\), the first bright ring has broken up; the disc is still hardly affected.

The phase distributions and intensity distributions over the aperture in these two cases also show features of interest. In the first case (Fig. 5(a) and (b)) the phase distribution, shown in radians in Fig. 5(a), appears at first sight a purely random structure. The fact that the phase values along the top edge of the square \(S\) must repeat those along the bottom edge, and those along the right-hand edge must repeat those along the left, does not appear to lead to any recognizable resonance effects in the form of the contour lines within the aperture. It does of course impose

![Diagram](https://example.com/diagram.png)

**Fig. 6(a).** Phase distribution (radians) in the plane of the aperture for the image of Fig. 3. Curves of differing phase-value modulo \(2\pi\) intersect only at points of zero intensity.

the restriction that the phases at the two points \((x, y) = (\pm r, 0)\) are equal, and likewise those at the points \((x, y) = (0, \pm r)\), on the edge of the aperture. Similar remarks apply to the relative intensities, shown in Fig. 5(b). The variation of these intensities from point to point of the aperture constitutes the phenomenon of scintillation.

The choice of power-spectrum function \(\phi(u, v) = \exp \left(-\frac{u^2 + v^2}{\tau^2}\right)\) corresponds to a certain clustering of the component spatial frequencies \(\sqrt{u^2 + v^2}\)
in the expansion (3) around the value $r/\sqrt{2}$, which is the maximum of the function $r \exp (-r^2/\tau^2)$. Thus the constraints (11), where $\phi$ is given by (25) and where $\tau = 3$, may be expected to produce an impression of 'blobs' of diameter about 0.5 $(x, y)$-units in the phase-intensity distributions. Fig. 5(a) and (b) are consistent with this; they give a visual impression of atmospheric 'blobs' of about this size. With an aperture diameter of 40 cm, the value $\tau = 3$ thus corresponds to disturbances which give an impression of atmospheric 'blobs' of about 10-cm diameter.

**Fig. 6(b). Relative intensity distribution in the plane of the aperture for the image of Fig. 3.**

The choice $N = 10$ then allows us to compute, from (9), the value of the approximation:

$$I_N(x, y) = |D_N(x, y)|^2$$

(28)

to the image intensity function in the circle $x^2 + y^2 < 25$ in the focal plane, where for light of wavelength $\lambda$ the linear size of the $(x, y)$-unit is equal to $F\lambda$ in a system of local ratio $F$. The angular equivalent on the sky of the linear $(x, y)$-unit $F$ in the focal plane is evidently $F\lambda/f$ radian $= \lambda/2a$ radian $= 10.313/2a$ arc sec when $\lambda = 500$ m$\mu$, where in the last expression the aperture diameter $2a$ is measured.
in centimetres. Thus when the aperture diameter is 40 cm, the \((x, y)\)-unit is \(0.258\) arc sec, the Airy disc radius (to the first dark ring) is
\[
0.61 \ (x, y)\text{-unit} = 0.63 \text{ arc sec}
\]
and the circle \(x^2 + y^2 < 3^2\) has angular radius \(3.145\), which is between three and six times that of the astronomer’s “seeing disc” under good seeing conditions. From Fig. 5(b) we also see that, even for the rather small r.m.s. perturbation

\[\text{Fig. 7. Image with } \sigma = 2; \text{ all else as in Fig. 2.}\]

represented by Fig. 2, the light-intensity varies by a factor of more than 6 between different points of the aperture. The stellar scintillations seen when stars are observed with the naked eye can likewise show intensity variations of this order in conditions of good seeing.

Fig. 6(a) and (b) show in the same way the phase and intensity distributions corresponding to Fig. 3, where the perturbations in the entry pupil are four times as great \((\sigma = 0.5)\) and, as will be seen from the coming together of the equiphase curves in Fig. 6(a), the intensity in the aperture falls to zero at a number of isolated points. The size of the ‘blobs’, as estimated by visual inspection, is seen to be unchanged when we notice that the intensity-maxima in Fig. 6(b) occur at very nearly the same places as those in Fig. 5(b).

The behaviour of the intensity-distribution in the focal plane as the size \(\sigma\) of the perturbations is further increased (their form remaining otherwise unchanged)
may be followed in Figs 7 and 8, in which the parameter \( \sigma \) has the respective values 2 and 10. In Fig. 7, no visible sign of the ring structure remains and subsidiary peaks are beginning to appear, comparable in size with the Airy disc; but the Airy disc still remains dominant. In Fig. 8 (\( \sigma = 10 \)) the central peak (Airy disc) no longer dominates the instantaneous image, being surpassed in brightness by one of the subsidiaries. In all the intensity-diagrams, the curves show relative intensities, i.e. the computed intensity-functions are left unnormalized. In Figs 2, 3, 4 and 7 they have been multiplied by 100 for convenience in labelling the isophotes.

**FIG. 8. Image with \( \sigma = 10 \); all else as in Fig. 2.**

In these figures, the relative intensity distribution can be interpreted as an instantaneous picture of the rapidly forming and reforming subsidiary peaks which result from the perturbations. Each lasts only a few hundredths of a second in an actual star image, and together they give the central part of the image its sparkling or erupting appearance. In Fig. 8, one of these ‘sparkles’ is momentarily as bright as the central Airy bright spot, which is itself considerably distorted.

According to equation (5), the ‘sparkles’ arise from the joint fluctuations of the coefficients \( a_{pq} + ib_{pq} \) of Airy displacement-patterns centred on the respective points \( (x, y) = (-p/2M, -q/2M) \), where in the present case \( M = 1 \). Generally speaking, each ‘sparkle’ arises from an unusually large fluctuation of a small group of one or more contiguous displacement patterns. Fig. 9(a) shows such a
small group; each circle represents the boundary of the central peak of one of the complex-valued terms \( (a_{pq} + ib_{pq}) J_1(2\pi Z_{pq})/\pi Z_{pq} \) on the right of (9), while the corner points of the squares are lattice points \((-\frac{1}{2}p, -\frac{1}{2}q)\). Because of the statistical independence of the \( a_{pq} \) and \( b_{pq} \), the statistical mean square fluctuation of \( D_N(x, y) \) at the point \((x, y)\) is

\[
\sum_{p^2 + q^2 \leq N^2} \langle a_{pq}^2 + b_{pq}^2 \rangle \left( \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}} \right)^2,
\]

and it is a simple matter to verify that when \( M = 1, \tau = 3 \) the interpolation function

\[
\sum_{p^2 + q^2 \leq 100} \exp\left( -(p^2 + q^2)/(2M\tau)^2 \right) \left( \frac{J_1(2\pi Z_{pq})}{\pi Z_{pq}} \right)^2
\]

gives a reasonably good approximation to the function \( \exp(-x^2+y^2/\tau^2) \) in the circle \( x^2 + y^2 < (4.5)^2 \). This helps to explain why, in spite of the form of the terms on the right of equation (5), the ‘sparkles’ in the computed image (Fig. 8) show no discernible tendency to locate themselves at the lattice points

\[(x, y) = (-u_p, -v_q) \quad (p, q = 0, \pm 1, \pm 2, \ldots),\]

even in the crude model with \( M = 1 \).

In the remaining computed cases, shown in Figs 10–19, \( M \) is given the value 2 and (see Fig. 9(b)) the lattice points \((x, y) = (-u_p, -v_q)\) form a mesh twice as fine as before. To cover the same circle \( x^2 + y^2 \leq 25 \) in the focal plane we now have to set \( N = 20 \) in (9); that is to sum over the 1257 terms of (5) for which \( p^2 + q^2 \leq 400 \). The approximation (30), in which the sum is now over values of \( p, q \) satisfying \( p^2 + q^2 \leq 400 \), is also improved by the change. The use of these ‘fine-mesh’ computations, with their greater demands on computing time, has two main objects. First, it enables us to verify that the estimated size of the ‘sparkles’, which was roughly comparable with that of the Airy disc in the model constructed with \( M = 1 \), remains pretty much the same in the fine-mesh model with \( M = 2 \), while their distribution shows the same absence of underlying lattice

![Mesh size: ![](a)](b)

**Fig. 9. Anatomy of a ‘sparkle’** (a) in the model with \( M = 1 \), (b) in the model with \( M = 2 \) (‘fine mesh’). The circles, each equal in size to the Airy disc of the system, indicate major contributions from individual terms of the series (9). (See text.)
structure and the same overall brightness trends as the 'sparkles' approach the centre.

Secondly, in the case $M = 1$ the restriction on the perturbations $\Phi(\xi, \eta)$ which we have called 'periodicity on the boundary of $S$', and which applies to all the sample functions of the model set, imposes some constraint on the statistics of the images through the fact that, in the plane of the aperture, $\Phi(-1, 0) = \Phi(1, 0)$ and $\Phi(0, -1) = \Phi(0, 1)$. It is not easy to be sure that this constraint is of negligible importance for present purposes in the case $M = 1$, though we may suspect it.

![Fig. 10. Relative intensities in the model image with $M = 2$ ('fine mesh'), $N = 20$, $r = 3$, $\sigma = 0.25$.](https://academic.oup.com/mnras/article-abstract/158/2/199/2603079)

But it is easier in the case $M = 2$, which has at least 900 more free parameters and in which the boundary of $S$ is well outside the rim of the aperture. Thus the similarity in appearance of the computed results for $M = 2$ and $M = 1$ reassures us about two doubtful assumptions which we should like to make for the case $M = 1$. Nevertheless, the remainder of the computations described in this paper were carried out with $M = 2$; that is, using the fine-mesh model.

In all the new computed cases the coefficients $a_{pq} + ib_{pq}$ were again derived from a single sequence of gaussian random numbers, multiplied in each case by a suitable normalizing constant to give the prescribed value $\sigma\sqrt{2}$ to the r.m.s. fluctuation of $\Phi(\xi, \eta)$. 
First, the two relative intensity diagrams shown in Figs 10 and 11 were computed. Comparing Fig. 11 with Fig. 4, we see that the size and general distribution of the sparkles appears to be much the same in these two cases, one computed from the fine-mesh model \((M = 2, N = 20)\), the other from the coarse-mesh model with \(M = 1, N = 10\); both for \(\sigma = 1\). We also see, comparing Fig. 10 with Fig. 11, that neither the spatial separation of the sparkles nor their brightness relative to each other shows any significant variation as \(\sigma\) changes from 0.25 to 1.0; only their brightness relative to the central disc is increased. The sparkles in the coarse-mesh diagrams (Figs 3, 4, 7 and 8) showed similar behaviour, at least for \(\sigma \geq 0.5\).

Next, the effect of doubling the linear aperture of the telescope was examined, using the fine-mesh model. Because of the scale-normalizations adopted, this doubles the linear \((\xi, \eta)\)-scale and halves the \((x, y)\)-scale in the focal plane. Consequently the only change needed in the previous analysis is a doubling of the parameter \(\tau\), giving \(\tau = 6\) in the specification (25), where we now have, when \(p^2 + q^2 \leq N^2\),

\[
\phi(u_p, v_q) = \exp \left(-\frac{(u_p^2 + v_q^2)}{36}\right).
\]

(31)

The 'seeing disc', actually unaltered, appears twice as large as before in the new \((x, y)\)-units; the 'blobs' in the plane of the aperture appear half-size in the doubled \((\xi, \eta)\)-units. The Airy disc, still of radius 0.61 \((x, y)\)-unit, is half its previous angular diameter.
To compute the expression (9) over the circle \(x^2 + y^2 \leq 25\) (now half its previous angular diameter) we cut-off the power function (31) to zero outside the circle \(p^2 + q^2 \leq (40)^2\) (20 becomes 40 because \(\tau\) is now doubled) and treat the terms for which \((20) < p^2 + q^2 \leq (40)^2\) in the way described on p. 209.

The computed relative intensity diagrams for \(\sigma = 0.25, 0.5, 1.0, 2.0, 5.0, 10.0, 15.0\) are shown in Figs 12, 13, 14, 16, 17, 18 and 19 respectively. The ratio of apparent 'blob' size to aperture diameter in the \((\xi, \eta)\)-plane is about the same, in the present model where \(\tau = 6\), as for a telescope of 80-cm aperture working in the Earth's atmosphere. Fig. 15, in which is shown the square root of the relative intensity along the line \(y = 0\) in the case \(\sigma = 1\), is included here to assist visualization in Fig. 14. All the figures were constructed with the help of sets of computer-plotted curves giving \(\sqrt{I}\) along lines \(y = constant\). For technical reasons, two-dimensional plotting routines, which would have been much faster, were rejected as too insensitive.

Some interesting inferences can be drawn from Figs 12 to 19, though of course none of them receives a rigorous proof. By comparing Fig. 12 with Fig. 10 and Fig. 14 with Fig. 11 we see that, when \(\sigma = 0.25\) or \(\sigma = 1.0\), the change from \(\tau = 3\) to \(\tau = 6\) in (25), giving a more slowly drooping power spectrum, leaves the fine detail of the instantaneous image almost unchanged; only the general brightness
level, with the fluctuations smoothed out, now falls off more slowly as $x^2 + y^2$ increases. This fits in with the fact that the ‘seeing disc’ is now twice as large as before in comparison with the Airy disc, though its actual angular size is unaltered.

The ‘sparkles’, which when plotted in $(x, y)$-coordinates appear very nearly the same in Figs 12 and 14 as in Figs 10 and 11 respectively, are evidently reduced in scale by a factor 2 when the aperture is doubled, and remain comparable in size with the Airy disc.

![Fig. 13. Image with $\sigma = 0.5$; all else as in Fig. 12.](image)

In the cases $\sigma = 0.25$ (Figs 10 and 12) and $\sigma = 0.5$ (Fig. 13) a curious distortion of the first Airy ring, suggestive of a mixture of coma and astigmatism, is a rather striking feature. When $\sigma = 1$ (Figs 11 and 14) it is still visible, though less striking, and as $\sigma$ increases through the remaining values $\sigma = 2$ (Fig. 16), 5 (Fig. 17), 10 (Fig. 18) and 15 (Fig. 19) it becomes clear that the effect is due to a chance configuration resulting (when $\sigma$ is in the same range of size as 0.5 and 1.0) from interaction between the random fluctuations and the first Airy bright ring. A chance effect of a different shape, but arising in the same way, can be seen when $\sigma = 0.5$ in the first bright ring of Fig. 3, which was computed from the ‘coarse mesh’ model ($M = 1$).

We can draw some further inferences by observing that Figs 12 and 14 may be given a second, equally valid interpretation. Namely, we can regard the change
from $\tau = 3$ to $\tau = 6$ simply as a change in the power spectrum of the perturbations, the optical system being left unchanged. Then, comparing Fig. 12 with Fig. 10 and Fig. 14 with Fig. 11, we see that a change in the 'seeing' (or, more correctly, in what corresponds to the 'seeing' in our models) has left the 'sparkles' substantially unchanged in size; but the general run-up of sparkle-brightness near the Airy disc is less pronounced when the seeing disc is larger.

Finally, returning to the double-aperture interpretation, we see from Figs 16 to 19 that the Airy disc becomes 'lost in the noise' (indistinguishable from the subsidiary peaks) by the time $\sigma$ has reached the value $15$.

We should expect, on the basis of the foregoing, that sparkle size will remain comparable with Airy disc size for still larger apertures, for example in telescopes of 100-inch or 200-inch aperture, provided always that the mirror imperfections are negligibly small. For, in the above models with $M = 1$, $M = 2$, the transition to larger apertures requires only an increase in the value of the 'seeing' parameter $\tau$, or more generally, the replacement of the discrete power spectrum $\phi(u_p, v_q)$ by the more slowly falling spectrum $\phi(u_p/m, v_q/m)$, where the constant $m$ is the ratio of the new aperture diameter to the old and where $(u_p, v_q) = (p/2M, q/2M)$ as before. But now the computations become impractically heavy, so that we are unable to obtain computed images by the simple methods of the present paper.

**Fig. 14. Image with $\sigma = 1$; all else as in Fig. 12.**
However, two short-exposure photographs (0.01 s), made by Texereau at Haute Provence with a 76-inch telescope, and referred to again below, support the expectation.

The question has a possible bearing on astronomical photography, since the formation of a latent image is most favoured by frequent bright sparkles of size somewhat less than the photographic grain size (say 3 λ), such as would be formed by an F/3 telescope of very good figure and of large aperture. For this would favour the arrival of several photons on the same photographic grain within a time interval of the order of 0.01 s. Errors of figure on the mirror would reduce the frequency of the brighter sparkles; large aperture would, at a given focal ratio, increase it. According to the above, the short focal ratio (F/3:3) of the 200-inch telescope may possibly be one of the factors in its high photographic performance.

![Image of a graph](https://example.com/graph.png)

**Fig. 13.** Square root of relative intensities along the line y = 0 in the image shown in Fig. 14.

4. OBSERVATION OF TELESOMIC STAR IMAGES

To form some idea about the usefulness of the models described in Section 2 we need to observe several image-intensity distributions, members of the statistical set \{\|D(x, y)\|^2\} belonging to an actual star image, over a period of time during which the corresponding set of wavefront perturbations may be regarded as statistically stationary. This means that observations with the shortest practicable exposure time should be obtained in quick succession on a night when the 'seeing' conditions appear to be changing very slowly. Prime-focus photographs taken by J. Texereau and made public by G. Courtès (1962; Figs 39 and 40) suggest that an exposure time of a few milliseconds may be regarded as 'instantaneous', and a nominal exposure time of 2.5 ms was accordingly chosen for the present work. So short an exposure time precluded any attempt to observe in monochromatic light.

The observing system applied to the telescopic star image must be capable of handling image details considerably smaller than the Airy disc, which is of diameter...
$2.44F\lambda = 6.0 \mu$ for $\lambda = 550 \mu\mu$ at the F/4.5 prime focus of the 36-inch telescope used. Thus some linear magnification of the star image is necessary, and it was clear that with a photographic plate and a 36-inch telescope aperture working in a sky of poor transparency, the reduction in image-intensity caused by this magnification would lead to excessive photographic noise (in other words, to unacceptably small signal-to-noise ratios) in the recorded images. A spectracon electronographic image-tube was accordingly used for the investigation. The spectracon, developed by J. D. McGee et al. (1966) is a single-stage, magnetically focused image-tube, in which electrons are accelerated sufficiently to penetrate a thin mica end-window. The electron image is recorded directly on a strip of melinex-backed nuclear emulsion pressed into contact with the 25 mm x 5 mm mica window. Electronography has big advantages over photography in conditions of low integrated intensity. A spectracon with an S20 photocathode makes effective use of about 10 per cent of the incident photons, as against 1 per cent or less for direct photography. Further, the finer grain structure of nuclear emulsions compared with those of ordinary photography gives greatly reduced emulsion granularity noise. A further advantage of electronography is the strictly linear response of emulsion optical density with exposure, a property discussed further in Section 5.

An S20 spectracon was selected and incorporated in a prime focus camera

---

**FIG. 16.** Relative intensities in the model image with $\sigma = 2$, all else as in Fig. 12.
together with an 8-mm microscope objective lens and a camera shutter giving nominal exposures down to 2.5 ms. The camera is shown diagrammatically in Fig. 20. Retractable finder mirrors for successive location of the image are shown in the in-beam position in the diagram. They were of great value in locating the F/90 final image. All settings of optical components inside the camera were made in the laboratory and the only adjustment needed at the telescope was the movement of the whole camera to bring the primary image into the object plane of the micro-objective. Trial exposures of α-Lyrae were first made with a linear magnification of ten and the results were sufficiently encouraging to suggest that a linear magnification of 20 combined with the use of Ilford G5 emulsion would give recorded images with a satisfactory signal-to-noise ratio. Ilford G5 is the coarser and faster

Fig. 17. Image with σ = 5; all else as in Fig. 12.

of the two nuclear emulsion types commonly used in electronography, Ilford L4 being the slower, finer-grain type. The use of L4 would have resulted in images of lower density though superior signal-to-noise ratio. The choice of G5 was made partly on the grounds that good quality contact prints of the images could be made simply and easily, while the emulsion signal-to-noise ratio was still entirely adequate.

During further trials, made at a time when only 2nd and 3rd magnitude stars were well placed for observation, conventional Hartmann tests were used in lining up the telescope mirrors. Once arrived at, good alignment could be maintained by

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using a much quicker check than the Hartmann test. It consisted in making exposures of several seconds duration on faint stars, with the same recording system, to give time-integrated images, and checking that these images appeared circularly symmetrical. Faint stars were used to avoid risk of damage to the photocathode. Of course this method would not be sufficiently sensitive using ordinary photography at prime focus. Usually little or no primary mirror adjustment was found to be necessary and the check was quickly carried out.

![Diagram](https://i.imgur.com/45X5Q.png)

**Fig. 18. Image with \( \sigma = 10 \); all else as in Fig. 12.**

When \( \alpha \)-Aur was conveniently observable in 1971 January, several satisfactory short-exposure images were obtained. Precautions against local causes of bad seeing within the dome were carried out in accordance with usual Cambridge practice: the dome and mirror were continuously ventilated and the mirror refrigerated during the day to within 2\( ^{\circ} \)C of the anticipated night temperature.

Prints of the electrophotographs of four images of \( \alpha \)-Aur, obtained during 4 min on a night (1971 January 22) when seeing conditions appeared to be changing very slowly, are shown in Plate I. The exposure was 2.8 ms in each case. The construction of isophote diagrams for these recorded images is described in the next section.
5. Processing of the Observations

The linear relation between emulsion density $D$ and exposure $E$ is a well-established property of electrographs. (As in photography, $D$ is connected with the transmission $T$ by the equation

$$D = \log_{10} \left( \frac{T_0}{T} \right),$$

$T_0$ being the transmission corresponding to zero exposure, while if a patch of photocathode is illuminated with intensity $I$ for a time $t$, the exposure $E = It$.)

![Fig. 19. Image with $\sigma = 15$; all else as in Fig. 12.](image)

The linearity has been verified experimentally by Kahan & Cohen (1969) for densities up to $D = 6$ ($T = 10^{-6}T_0$) for the Ilford L4 emulsion and to

$$D = 2.2 \ (T = 6.3 \times 10^{-3}T_0)$$

for Ilford G5, which saturates at $D = 5.5$.

The above-mentioned star images of $\alpha$-Aur. on Ilford G5 were measured with a microphotometer of square scanning aperture ($32 \mu \times 32 \mu$) at sampling point forming a square-mesh grid (mesh size $16 \mu \times 16 \mu$) and covering a square of size 1.6 mm $\times$ 1.6 mm. Since the Airy disc diameter $2.44 F\lambda$ is $110 \mu$ when $F = 90$ and $\lambda = 0.5 \mu$, we can regard convolution with a $32 \mu \times 32 \mu$ aperture as producing
PLATE I. Half-tone enlargements (×10) of four consecutively recorded electrograph images of Capella, observed 1971 January 22 at zenith distance 24°. Calibrated exposure time 2.8 ms in each case.
an acceptably small (though not a negligible) smoothing effect on the recorded image, together with a negligible small-scale distortion of the finest image detail.

The area covered, 1.6-mm square, included a clear border outside the detectable star image, which provided a means of checking for uniform sensitiveness over the part of the photocathode employed and for absence of 'drift' in the electronics. These checks were made four times during each run and, provided the apparatus had been stabilized by warming up for 4 hr beforehand, were always well satisfied. In fact, the instrumental sensitivity was found to vary by less than 1 per cent.

![Diagram of Spectracon Camera with Re-imaging System and Image Location Facilities](https://example.com/diagram.png)

**Fig. 20. Spectracon camera with re-imaging system and image location facilities.**

In each run, 100 scans along consecutive lines gave $100 \times 100$ automatically punched entries on seven-track paper tape, which were later processed by the computer TITAN, using a Fortran inverse interpolation routine (two-dimensional) in conjunction with a Calcomp plotter, to obtain preliminary versions of Fig. 21(a)–(d). Each figure required between 4 and 8 min of plotter time. The numbered curves in the figures represent curves of equal microphotometer reading, expressed in terms of photographic density (convoluted of course with the $32 \mu \times 32 \mu$ microphotometer aperture; that is to say somewhat smoothed, but not so much as
FIG. 21. Relative intensity curves (see text) in image of \( \alpha \)-Aur (Capella) recorded electronographically in Plate I. The numbers on the curves stand for relative image intensities above background.
to obscure their features appreciably). In each of the drawings produced by
the plotter, two or three small patches of unsatisfactory interpolation, resulting from
a certain unavoidable crudity in the fast-running Fortran program, needed to be
improved by human intervention; and a general crinkliness along large contours,
which it seemed safe to interpret as an effect of photographic granularity ('noise')
was smoothed out by hand. The crinkles were strongest in regions of low density-
slope, and the effects of plotter-pen vibration (as judged from smooth-curve
plots) could make only a negligible contribution. The redrawn diagrams, which
respectively correspond to the image contact prints reproduced as Plate I(a)–(d),
are given in Fig. 21(a)–(d). The curves labelled 1, 2, 3, ... in these figures corre-
spend to optical densities $\sigma \cdot 1$, $\sigma \cdot 2$, $\sigma \cdot 3$, ... respectively, above the reference level
given by the border elements of the scanned areas (substantially the sky back-
ground), and so represent image-intensities on an arbitrary fixed scale.

6. COMPARISON OF OBSERVED AND MODEL IMAGES;
CONCLUDING REMARKS

The most interesting feature which appeared in the computed model images,
namely the 'sparkle' of size comparable with that of the Airy disc, can be
pictured as resulting from the presence in the entry pupil of a fairly large number of
small areas, distributed more or less at random over the aperture, in each of
which the slope of the perturbed wave-front is everywhere nearly perpendicular
to any preselected direction nearly parallel to the direction of the object star. When the
preselected direction is varied, the displacement contribution from the corre-
sponding small areas varies; when this contribution approaches a local maximum,
diffraction effects may produce a 'sparkle' in the star image. Comparison with the
effect of a randomly perforated screen placed in the unperturbed light beam
suggests that the brightest region of the 'sparkle' should be comparable in size
with the Airy disc of the unscreened telescope (compare G. Curtès 1962). How-
ever, these considerations are not precise enough by themselves to give us more
than vague general indications. For example, they take no account of the variation
of intensity from point to point of the perturbed wave front in the entry pupil
('scintillation'), and they do not provide a means of forming a quantitative idea
of the relations between the statistical properties of the complex perturbations
in the entry pupil on the one hand and the size, brightness and distribution of the
image 'sparkles' on the other.

Some progress towards forming such an idea is made possible by the computed
figures of Sections 2 and 3, together with isophote diagrams based on the already
described observation of images of $\alpha$-Aur (Capella). Comparing Figs 18 and 19
with Fig. 21(a)–(d), and remembering that the first two are computed for quasi-
monochromatic light, the last four obtained from observations in polychromatic
light (in which the wavelength spread will cause considerable filling-in of deep
intensity valleys and a considerable contrast-reduction in fine detail generally),
we see that the observed images agree pretty well in their general structural
characteristics with the model image for $\tau = 6$, $M = 2$, $N = 40$, $\sigma = 15$. (As
explained in Section 2, the notation $N = (20)$ indicates a particular treatment of
equation (9) in the seven cases with $N = 40$.) The computed image with $\sigma = 10$
(Fig. 18) does not agree with the observed images so well as that with $\sigma = 15$,
because its central peak is still a dominating feature, easily distinguishable from the
rapidly forming and evanescing 'sparkles'. Possibly a value of $\sigma$ somewhat larger than 15 would produce a still closer resemblance to the observed images.

The tentative conclusion is that the model perturbations here analysed, though clearly too simple in statistical characterization to represent with any fidelity those encountered in using large telescopes, are able to reproduce and in a certain sense to explain some of the more important effects of imperfect 'seeing' on telescopic star images. Further, the analysis indicates a possible influence of the focal ratio of a large telescope on its photographic performance and suggests that a focal ratio near to $F/3$ may perhaps offer some photographic advantages in large telescope used with fast emulsions in an unsteady atmosphere.

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