THE MASS SPECTRUM OF INTERSTELLAR CLOUDS
AND THE ASSUMPTION OF TOTAL COALESCEENCE

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SUMMARY

We have investigated some modifications of the statistical mechanical model for the formation and disruption of interstellar clouds originally proposed by Field and Saslaw. We conclude that the asymptotic nature of the mass spectrum (i.e. $N(m) \approx m^{-3/2}$), under the assumptions of total coalescence and uniform $\sigma v$, is independent of the redistribution of the larger clouds into the smaller ones and the time dependence of the problem. We also describe some preliminary results of more detailed model calculations.

I. INTRODUCTION

It is the purpose of this paper to expand on the statistical mechanical model proposed by Field & Saslaw (1965) for the formation and disruption of interstellar clouds. Their preliminary model was further developed by Field & Hutchins (1968) who considered more realistic cross-sections, by Penston et al. (1969) who added the velocity parameter and by Nakano (1966) who considered several different cross-sections. Herein we show that the asymptotic nature of the mass spectrum ($N(m) \approx m^{-3/2}$) depends only on the assumption of total coalescence and uniform $\sigma v$ and not on the time dependence or redistribution aspects of the problem.

We shall exhibit below unique, stable, time independent solutions to this problem. These equilibrium solutions provide a background against which the non-unique time dependent models can be more easily interpreted. Our primary goal in establishing the more general formalism was to describe processes related to star formation within an object such as the Orion nebula. The evolution of a gas cloud of this nature may be separable into a period during which mass segregation occurs, followed by the development of protostars. Mass conservation would be a good approximation in this situation. Mass conservation would be applicable for interstellar clouds if the time to approach equilibrium ($T_e$) were very much less than a rotation period of the galaxy. An estimation of $T_e$ will be given in the last section.

We define $N(m, t)$ to be the number of clouds of mass $m$ per unit volume at the time $t$. For simplicity we assume the mass parameter is discretized by integers according to

$$m_j = jm \quad j = 1, 2, 3, \ldots,$$

and refer to clouds of mass $i$ (i.e. $m_i = im$) and $j$. We also abbreviate $N(m = m_k, t)$ as $N_k(t)$. One can easily show that

$$\frac{dN_k(t)}{dt} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} H_{ij}^k N_i(t)N_j(t)\langle \sigma v \rangle_{ij},$$

where $H_{ij}^k$ are the cross-sections.

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where $\langle \sigma v \rangle_{ij}$ is the value of $\sigma v$ averaged over the velocity distribution. The upper limit of $n$ means that there is some largest mass $m_n$ which we will consider. To simplify the notation the symbol

$$\sum_i$$

will be used as an abbreviation for

$$\sum_{i=1}^n.$$

$H_{ij}^k$ has the form

$$H_{ij}^k = H_{jk}^k = P_{ij}^k - \delta_{ik} - \delta_{jk} + R_{ij}^k,$$  \hspace{1cm} (3)

where $P_{ij}^k$ is the conditional probability that when $i$ collides with $j$ there is (are) a fragment(s) of mass $k$. Redistribution is represented by the $R_{ij}^k$ term discussed in more detail below. For total coalescence between clouds $P_{ij}^k$ has the form

$$P_{ij}^k = \delta_{i+j,k}.$$  \hspace{1cm} (4)

In a closed system the total mass must be conserved and there is, therefore, a constraint on the $H_{ij}^k$, \textit{viz.}

$$\sum_{k=1}^{2n} kH_{ij}^k = 0 \hspace{1cm} \forall \hspace{0.5cm} i, j \in \{1, n\},$$  \hspace{1cm} (5)

since

$$\sum_k kN_k(t) = M/(mV) = \text{constant}$$  \hspace{1cm} (6)

where $M$ is the total mass of clouds in the volume $V$. Because objects of mass larger than $n$ may be produced by collisions of objects with masses less than or equal to $n$ these large bodies must be assumed to be disrupted creating a feedback into lower mass objects. Field and Saslaw, following a suggestion due to Oort (1954), assumed these large mass objects generated the appropriate number of clouds of unit mass. Thus $R_{ij}^k$ can contribute only to $H_{ij}^k$ when $k = 1$. It is this assumption we want to pay particular attention to.

2. REDISTRIBUTION ONLY INTO THE SMALLEST MASS

Let us assume, as Field and Saslaw did, that $\langle \sigma v \rangle_{ij}$ is independent of $i$ and $j$. Then, if $x_k = sN_k$ and $T = \langle \sigma v \rangle_{ij} t/s$ we can rewrite the equation governing the time development of the $x_k$ as

$$\frac{dx_k}{dT} = \frac{1}{2} \sum_i \sum_j x_i x_j H_{ij}^k, \hspace{1cm} k = 1, 2, \ldots, n,$$  \hspace{1cm} (7)

where $s$ can be chosen such that

$$\sum_k x_k = 1.$$  \hspace{1cm} (8)

We shall show that there exists one solution of equations (7) and (8) with $dx_k/dT = 0$ for all $n$ values of $k$ and these are precisely those already obtained for the time-independent part of $N(m, t)$ by Field and Saslaw. Theirs, however, were not equilibrium solutions but decayed as $1/t$ for large $t$. 
One can see that for redistribution into the smallest mass only

\[ x_2 = \frac{1}{2}x_1^2 - x_2, \]  
\[ x_3 = x_1x_2 - x_3, \]  
\[ x_4 = \frac{1}{2}x_2^2 + x_1x_3 - x_4, \text{ etc.,} \]

so that in equilibrium

\[ x_2 = \frac{1}{2}x_1^2 = \left(\left(2 \cdot 2 - 3\right)/2\right)x_1x_1, \]  
\[ x_3 = \frac{1}{2}x_1^3 = \left(\left(2 \cdot 3 - 3\right)/3\right)x_1x_2, \]  
\[ x_4 = \frac{1}{2}x_1^4 = \left(\left(2 \cdot 4 - 3\right)/4\right)x_1x_3, \]  
\[ x_5 = \frac{1}{2}x_1^5 = \left(\left(2 \cdot 5 - 3\right)/5\right)x_1x_4, \]

or

\[ x_p = \left(\left(2p - 3\right)/p\right)x_1x_{p-1} = \left(\left(2p - 3\right)!!/p!\right)x_1^p. \]  

In the Appendix the proof of this induction is contained as a corollary to a more general theorem we will use below. We have not written down the \( dx_1/dT \) equation because it does not facilitate the analysis but will instead obtain \( x_1 \) from the mass conservation constraint,

\[ F_n(x_1) = S_n(x_1) + x_1 - 1 = \sum_{p=2}^{n} \frac{(2p-3)!!}{p!} x_1^p + x_1 - 1 = 0. \]  

It is clear that because there is only one sign change there exists one real positive root. Thus the solution is unique. Furthermore, for \( n = 1 \), \( x_1 = 1 \) and for \( n = \infty \) from the expansion of \((1 - 2z)^{1/2}\) we find \( x_1(\infty) = \frac{1}{2} \). A simple \textit{reductio ad absurdum} argument shows that \( x_1(n) > x_1(n + 1) \) and since \( F_n(1) > 0 \), \( F_n\left(\frac{1}{2}\right) < 0 \)

\[ \lim_{n \to \infty} x_1(n) = \frac{1}{2}^+. \]  

The expression \( x_p = (2p - 3)!! x^p/p! \) has a minimum (for \( x > \frac{1}{2} \)) at

\[ (1 + x)/(2x - 1) + 1 \geq p \geq (1 + x)/(2x - 1), \]  

but it is not clear that there is a minimum in the mass spectrum, i.e. that

\[ n \geq 3x_1(n)/(2x_1(n) - 1). \]

We shall now show that this is not the case for any value of \( n \geq 1 \). We first observe

\[ S_{n+1}(x) = S_n(x) + \frac{(2n-1)!!}{(n+1)!} x^{n+1} = \left(\frac{1}{2}\right)x^2 + 2xS_n(x) - 3 \int_0^x S_n(u) \, du. \]  

If we repeatedly differentiate and set \( x = \frac{1}{2} \) we find

\[ S_n(\frac{1}{2}) = \frac{1}{2} - (2n-1)!!/(2n)!! \]  

\[ \frac{d^n S_n(x)}{dx^n} \bigg|_{x=1/2} = -\delta p + \frac{1}{2p} - \frac{1}{2} (2n-1)!!/(2n-2p)!! \]

\[ (2j + 1)!! = (2j + 1)!!/(2j)!!, \quad (2j)!! = 2^j j!. \]
Hence, if we write \( x_1(n) = \frac{1}{2} + y_n, y_n > 0 \), equation (12) can be written

\[
I = \sum_{p=1}^{n} \binom{n}{p} (2y_n)^p / (2p - 1) \tag{16a}
\]

where \( \binom{n}{p} \) is the combinatorial symbol. From equation (13b) we see that there exists a physical minimum if, and only if \( y_n > 3/2(2n - 3) \). Using the fact that \( 1 / (2p - 3) > 1 / (2p), p > 1 \) and rewriting the factorials we find that the assumption there is a physical minimum is valid only if

\[
\frac{1}{2} > \sum_{p=2}^{n} \frac{(3/2)^p}{p} \frac{p - 1}{p^p} \prod_{k=1}^{n} (1 - k/n). \tag{16b}
\]

The right-hand side of equation (16b) is easily seen to be an increasing function of \( n \), and at \( n = 4 \) it is larger than 65/128. Hence, there is never a minimum in the mass spectrum (for \( n = 1, 2, 3 \) see Table I) when \( n \) is finite.

### Table I

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<th>( n )</th>
<th>( x_1(n) )</th>
</tr>
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</tr>
<tr>
<td>3</td>
<td>0.650629</td>
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<td>0.548375</td>
</tr>
<tr>
<td>10</td>
<td>0.543455</td>
</tr>
<tr>
<td>( \infty )</td>
<td>( \frac{1}{2} )</td>
</tr>
</tbody>
</table>

If \( n \) is infinite and we use Stirling’s formula to expand the factorials in \( x_p \) we find

\[
x_p \simeq (1 + \frac{3}{2}p) p^{-3/2}(2\sqrt{\pi}), \tag{17}
\]

so that the mass spectrum is always monotonic decreasing.

### 3. More General Redistributions

We now turn to the question of the effect of the particular form of redistribution. Suppose all large mass clouds are redistributed into only mass \( q \), then clearly,

\[
x_{pq} = ((2p - 3)p) x_q x_{(p-1)q}, \quad p = 2, 3, \ldots, \quad [n/q] \tag{18}
\]

all other values of \( x_p \) being zero. Here and subsequently we are using the notation that \([u]\) is the greatest integer less than or equal to \( u \).

A more reasonable assumption would be to redistribute a fraction \( r \) into the mass 2 and a fraction \( 1 - r \) into the mass 1. We would then have, in equilibrium,

\[
x_3 = x_1 x_2, \tag{19a}
\]

\[
x_4 = x_2 \left( \frac{1}{2} x_2 + x_1 \right)^2, \tag{19b}
\]

\[
x_5 = x_1 x_2 \left( \frac{3}{2} x_2 + x_1 \right)^2, \tag{19c}
\]

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\[ x_8 = x_2 \left( \frac{1}{2} x_2^2 + 3 x_2 x_1^2 + x_1^4 \right), \]  
\[ x_7 = x_1 x_2 \left( \frac{5}{2} x_2^2 + 5 x_2 x_1^2 + x_1^4 \right), \text{ etc.} \]  
\[ (19d) \]
\[ (19e) \]

The previous solution for the \( r = 0 \) redistribution case was of the form

\[ x_p = A_p x_1 x_{p-1} \quad \text{with} \quad A_p = (2p-3)/p. \]  
\[ (20) \]

The most general form linear in \( x_{p-1} \) and \( x_{p-2} \) for arbitrary \( r \) is

\[ x_p = A_p x_1 x_{p-1} + B_p (2x_2 - x_1^2) x_{p-2}, \]  
\[ (21) \]

where \( A_p \) in equation (21) remains \( (2p-3)/p \) and the coefficient \( 2B_p \) measures the contribution to \( x_p \) from \( x_{p-2} \) of clouds of mass 2 not created by the coalescence of clouds of mass 1. In the Appendix we find that \( B_p = (p-3)/p \) so that we must solve a linear, homogeneous, second order difference equation with linear coefficients. The theory of this type of difference equation is well developed (Batchelder 1927) and the exact solutions can be expressed in terms of hypergeometric functions. The asymptotic solution is shown to be of the form

\[ x_p = c^p p^d \left( s + s'/p + s''/p^2 + \ldots \right), \]  
\[ (22) \]

Substituting this expression into equation (21) and equating the coefficients of different powers of \( p \) successively equal to zero yields

\[ c^2 - 2x_1 c + x_1^2 - 2x_2 = 0, \]  
\[ (23a) \]
\[ d = -3/2, \]  
\[ (23b) \]
\[ s'/s = \frac{3}{8} (2(x_1^2 - 2x_2) - cx_1)/(cx_1 + 2x_2 - x_1^2). \]  
\[ (23c) \]

One would normally expect that corresponding to the two roots of the quadratic equation for \( c \) one would obtain two different values of \( d \) as is the case for \( s'/s \). However, all the \( c \) dependent terms originally in equation (23b) cancel each other. Preceding in this way we find

\[ x_p = c_1 p^{-3/2} s_1 (1 - 3(x_1 - 2(x_2)^{1/2})/(8p(2x_2)^{1/2}) + \ldots \]  
\[ + c_2 p^{-3/2} s_2 (1 + 3(x_1 + 2(x_2)^{1/2})/(8p(2x_2)^{1/2}) + \ldots \],  
\[ (24) \]

where \( c_1 = x_1 + (2x_2)^{1/2}, c_2 = x_1 - (2x_2)^{1/2} \). We can evaluate \( s_1 \) and \( s_2 \) by observing that when \( r \) is zero so is \( c_2 \) and the first expression must be the asymptotic expansion of equation (11). By comparison we find \( s_1 \) to be \( 1/2 \sqrt{\pi} \). In contrast \( r = 1 \) implies \( x_1 = 0, c_1 = -c_2 \) and equation (24) reduces, in lowest order, to

\[ x_p = c_1 p^{-3/2} (s_1 + (-1)p s_2), \]  
\[ (25) \]

which will vanish appropriately if \( s_2 = s_1 \). For \( n \) finite \( x_1 \) and \( x_2 \) are functions of \( r \) and \( n \) which can be obtained by combining the mass conservation equation with either \( x_1 = 0, \) or \( x_2 = 0 \). The \(-3/2\) behaviour of the spectrum and the absence of a minimum persists.

The next generalization would be to redistribute into the masses 1, 2 and 3 and one finds that

\[ x_p = A_p x_1 x_{p-1} + B_p (2x_2 - x_1^2) x_{p-2} + C_p (x_3 - x_1 x_2) x_{p-3} \]  
\[ (26) \]

with \( A_p, B_p \) as before and \( C_p = (2p-3)/p \). Proceeding in a fashion analogous to
the above one obtains a cubic equation for $c$ while $d = -3/2$ persists independent of $c$. We have also explicitly done the calculations for redistribution into mass 1 and 3 alone, and 2 and 3 alone. The $-3/2$ nature of the spectrum remains. We did not explicitly verify the absence of a minimum in these latter two cases. We suspect further generalizations of this type will yield similar results. We feel, therefore, that the $-3/2$ behaviour and absence of a minimum follow from the two assumptions (i) total coalescence and (ii) $\langle \sigma v \rangle_{ij}$ is independent of $i$ and $j$.

4. SOME FURTHER MODELS AND THE EQUILIBRIUM TIME

The model we have investigated here (total coalescence and uniform $\langle \sigma v \rangle_{ij}$) has the advantage over a more realistic model of being one which can be discussed completely analytically with or without mass conservation and/or with or without time dependence. It also gives us a lower bound on the index $(-3/2)$ of any possible mass spectrum which is a power law. The formalism has also been applied to a more general discussion of the problem of interstellar clouds and we have constructed many detailed models for $P_{ij}^k$ by incorporating as much of the physics of cloud–cloud collisions as possible. We may briefly mention that the next less restrictive probabilistic possibility favouring coalescence is that when $j, i (j > i)$ collide the result is that the part of $i$ swept out by $j$ sticks to $j$, the remainder (the 'ear') going free. When $i = j$ we assume three fragments, the central region of coalescence and the two ears (see Figs 1 and 2). The simplest geometry for the

![Diagram](https://example.com/diagram.png)

**Fig. 1.** The two fragment model ($j > i$). The shaded part of $i$ sticks to $j$ and the remainder goes free. Arrows indicate directions of relative velocity.

clouds is that of spheres and the assumption that all clouds have the same mean density simplifies the problem into one of calculating the appropriate volumes. The volume calculation is now formally equivalent to the calculation of light intensities in the case of totally limb-darkened eclipsing binaries. We have solved the equilibrium problem under these assumptions with geometrical cross-sections for
Various maximum masses. After obtaining the numerical results we have least squares fit them to a power law curve. For \( n = 5, 10, 20, 40 \) the exponents of the power law curve were \(-2.83, -2.45, -2.11, \text{ and } -1.82\) respectively. We see that as \( n \) approaches infinity the mass spectrum appears to approach \( m^{-5/3} \) which is the result for total coalescence with geometric cross-section (Field and Saslaw for the time dependent case and Taff for the time independent case). As a measure of the goodness of fit of the numerical results to a curve of this type one can use a Students \( t \) test or, equivalently, the correlation coefficient. The correlation coefficients were \( 0.99, 0.98, 0.98, \text{ and } 0.98 \) respectively. If one allows three fragments in the case \( j > i \) (only for impact parameters larger than the one for internal tangency) rather than two it becomes increasingly difficult for large objects to successfully

![Diagram](https://academic.oup.com/mnras/article-abstract/160/1/89/2604659)

**Fig. 2.** The two fragment model (\( i = j \)). The shaded parts of \( i \) and \( j \) coalesce and both remaining outer pieces go free. Arrows indicate directions of relative velocity.

survive collisions with small ones. Calculations using the spherical model and geometrical cross-sections for \( n = 5, 10, 20 \) yield power law exponents of \(-5.51, -7.40, -9.51\) with correlation coefficients \( 0.97, 0.98 \text{ and } 0.98 \) respectively.

The question of the equilibrium relaxation time \( T_e \), mentioned above, can be approximately estimated in the following way. In equations (2) and (6) let us make the change of variables

\[
x_k = V\gamma mN_k/M, \quad (27a)
\]

\[
dT = \pi v(3m/4\pi\rho)^{2/3} Mdt/m\gamma V, \quad (27b)
\]

\[
\gamma = \sum_k kx_k > 1, \quad (27c)
\]

where \( M, m, V \) are as before, \( \rho \) is the mean interior cloud density and \( v \) is the mean cloud speed (assumed uncorrelated with mass). The \( x_k \) will then satisfy equations (7) and (8). Introducing the variable \( y_k \) by

\[
x_k(t) = x_k^0 + y_k(t) \quad (28)
\]

where \( x_k^0 \) is the equilibrium solution and linearizing in the usual fashion in the \( y_k \) one can show that the \( y_k \) time dependence is exponentially decreasing. If \( \lambda \) is the
absolute value of the smallest non-zero eigenvalue of the stability matrix then \( T \) will be given approximately by

\[
T = \left( \frac{V m \gamma / \Lambda \lambda}{\pi \nu (4 \pi \rho / 3 m)^{2/3}} \right).
\]

If we call \( \delta \rho \) the density contrast between the clouds and the intercloud medium and write the volume of a cloud as \( e^3 V \) then

\[
T < \left( \frac{e^3 \rho \gamma / \pi \nu}{(4 \pi V^{1/2}) / 3} \right)^{2/3}.
\]

As representative values we can take \( \delta \rho = 10, \epsilon = 1/10, \gamma / \lambda = 1, \nu = 10 \text{ km s}^{-1} \) and \( V = 5 \text{ kpc} \times (100 \text{ pc})^2 \), which will then imply \( T < 3 \times 10^7 \text{ yr} \) which is about \( 1/10 \) of a galactic rotation period. A fuller explanation of the various models and mathematics is in preparation (Taff).

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REFERENCES


APPENDIX

Here we shall demonstrate the proofs of equations (11) and (21) of the text using a method introduced by Field \& Saslaw (1965) in their discussion of a similar problem. It may be extended to the more general cases of redistribution discussed in the text.

Theorem

The solution of

\[
\frac{1}{3} \sum_{k=1}^{N-1} x_k x_{N-k} - x_N + x_1 \delta_{p1} + (x_2 - \frac{1}{3} x_1) \delta_{p2} = 0
\]  

(A1)

may be recursively obtained from

\[
x_p = \left( \frac{2p-3}{p} \right) x_1 x_{p-1} + \left( \frac{p-3}{p} \right) (2x_2 - x_1^2) x_{p-2}, \quad p \geq 3.
\]  

(A2)

One can easily see that equation (A1) is the same as equation (7) when all the time derivatives vanish and only \( R_{ij} \) and \( R_{ij}^2 \) are non-zero. For typographical convenience let us set

\[
a = 2x_1, \quad \text{(A3a)}
\]
\[
b = 2x_2 - x_1^2. \quad \text{(A3b)}
\]
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Proof: We define \( X(z) \) by

\[
X(z) = \sum_{j=1}^{\infty} z^p x_p
\]  

(A4)

for all values of \( z \) such that the series converges uniformly. Multiplying equation (A1) by \( z^p \) and summing over \( p \) we find

\[
X^2(z) - 2X(z) + az + bx^2 = 0.
\]  

(A5)

The correct root is easily seen to be the one with the minus sign. If we use the binomial theorem to expand the radical (with \((-1)!! = 1\)) we have the result

\[
X(z) = \sum_{j=1}^{\infty} F(j)z^j(a + bz)^j,
\]  

(A6)

where \( F(j) = (2j-3)!!/(2j)!! \). This is equivalent to

\[
X(z) = \sum_{j=1}^{\infty} F(j) \sum_{k=0}^{j} (k!) b^k a^{j-k} z^{j+k},
\]  

(A7)

where \((k!)\) is the combinatorial symbol. From the definition of \( X(z) \) we see that \( p! x_p \) is the \( p \)'th derivative of \( X(z) \) with respect to \( z \), evaluated at \( z = 1 \). It is obvious the only term that will survive the evaluation at zero, after differentiation \( p \) times, is that for which \( k = p - j \). However \( k \) is bounded above by \( j \) and below by \( 0 \) which implies

\[
2j \geq p \geq j,
\]  

(A8)

so that

\[
x_p = \sum_{j = [(p+1)/2]}^{p} F(j) (p-j!) b^{p-j} a^{j+2j-p}.
\]  

(A9)

We form the combination \((2p-3)ax_{p-1/2} + (p-3)bx_{p-2}\) and after separating off the term of highest order in \( a \) we find it equal to

\[
pF(p)a^p + \frac{(2p-3)}{2} \sum_{j = [(p+1)/2]}^{p-2} F(j) (p-1-j!) b^{p-j} a^{2j+2-p} \\
+ (p-3) \sum_{j = [(p-1)/2]}^{p-2} F(j) (p-2-j!) b^{p-j} a^{2j+2-p}.
\]  

(A10)

Making the change of variable \( j = k - 1 \) in both sums and taking advantage of the fact that if \( p \) is odd \([p/2] + 1 = [(p+1)/2]\), whereas if \( p \) is even we must add \( F(p/2)(p^2/2-1) b^{p/2} = 0 \) to the first sum to make the lower limit of summation \([[(p+1)/2]\) we find equation (A10) is the same as

\[
pF(p)a^p + \sum_{k = [(p+1)/2]}^{p-1} F(k) (p-k!) b^{p-k} a^{2k-p} ((2p-3)(2k-p) + 2(p-3)(p-k)) (2k-3)
\]  

(A11)

This is easily seen to be equal to \( px_p \).

Corollary: When, in addition, \( R_0 = 0 \), equation (11) is valid.

Proof: If there are no clouds of mass \( 2 \) being created by redistribution then \( x_2 = \frac{1}{4} x_1^2 \), or \( b = 0 \). The proof of the corollary is now an immediate consequence of the above theorem.