THE LIGHT VARIATION OF 1 MONOCEROTIS

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SUMMARY

1338 'v' observations were obtained of the δ Scuti variable 1 Mon in 1972/73. Frequency analysis revealed a spectrum with 11 frequencies of the form $nf_1 \pm mf_L$, where $f_1 = 7.346132$ c/d and $f_L = 0.12911$ c/d. The frequency spectrum was discussed both in terms of the non-linear superposition of waves of frequencies $f_1 - f_L, f_1$ and $f_1 + f_L$ and in terms of the tidal modulation of a single wave of frequency $f_1$ with a frequency $f_L$. It was concluded that both theories might produce the observed frequencies, but that the most probable cause was the superposition of three non-sinusoidal, interacting waves.

I. INTRODUCTION

1 Monocerotis (HR 2107) is a known δ Scuti variable with a period of variation of about 0.136 and a strongly variable amplitude light curve implying the existence of at least two periodicities. Millis (1) has recently published a list of accurate $UBV$ photoelectric observations of this star, taken in 1971 and 1972; references to earlier work on 1 Mon will be found in his paper. Millis' principal conclusion was that the amplitude variation of 1 Mon could be explained in terms of the simultaneous excitation of two modes of period $0.13613$ and $0.13377$.

Independently we had undertaken an investigation of 1 Mon in an attempt to understand the multiple periodicities present in δ Scuti variables. Amongst stars assigned to the δ Scuti class 1 Mon has the largest amplitude with a total range of 0.36 mag in $V$ at maximum amplitude. Potentially therefore it presented the most favourable case for studying multiple periodicities because the signal to noise ratio was large.

One of us (RRS) has recently worked on the detection and frequency analysis of the small amplitude light and velocity variations of several Beta Canis Majoris pulsating variables; the application of the same frequency analysis computer program to the large and variable amplitude light curves of 1 Mon promised to yield much detail regarding the frequency spectrum of its light variations.

2. OBSERVATIONS

Photometric observations were made on 25 nights in 1972 December and in 1973 January and February using the 24-in. reflector at Siding Spring Observatory, Coonabarabran, NSW. Two photometers were used, RSS used a 10-channel photometer (uncooled) with dichroic filters to obtain simultaneous $uvb$ measurements of which only the $v$ data is reported here. RRS used a single channel photometer consisting of a Corning 3384 (yellow) filter and an uncooled EMI 6945S photomultiplier. The observing procedure was to alternate between program star and comparison star with an integration time of 60 or 80 s.
Reduction techniques were those already detailed by Shobbrook et al. (2) and (3). The comparison stars employed were HR 2039, 2150 and occasionally 2071. For the yellow (\( \beta \)) observations the mean rms deviations from the final fit of these stars for each night were \( \omega \cdot 0033 \), \( \omega \cdot 0029 \) and \( \omega \cdot 0032 \) mag, respectively. We expect that the light variation of 1 Mon has been measured to a similar precision of between \( \omega \cdot 002 \) and \( \omega \cdot 005 \) mag, depending on the quality of the night.

The \( \beta \) data were converted to the natural system of the EMI 6094S photomultiplier and Corning 3384 filter, so that all measurements are on the same system. The original magnitudes were expressed as \( \beta (1 \text{ Mon} - \text{HR 2039}) \) but the final data used for analysis is in the form of intensities with respect to \( \omega \cdot 000 \) as the mean of all observations.

In total we obtained 1338 \( \beta \) observations spanning 58 days. The sum of the lengths of the 25 nights was 5.35 days, which corresponds to nearly 40 cycles of the main period with an average of 34 observations per cycle. On reflection it was probably not necessary to obtain such a high density of observations per cycle. With 20 observations per cycle almost as many periodicities could have been found. However, one parameter that is crucial to the analysis of multiperiodic variables is the time density of cycles. This must be as high as possible. We achieved this by observing on each night for as long as possible and on as many consecutive nights as possible. In this way the usual problem of one cycle per day period ambiguities (aliases) was reduced to a minimum.

Copies of the data may be obtained on request from either author or from the Royal Astronomical Society, Variable Star File Number IAU27 RAS-31.

3. THE FREQUENCY SPECTRUM OF THE LIGHT VARIATIONS

The data were analysed using a frequency analysis computer program, originally written by D. Herbison-Evans and since modified by N. R. Lomb. This computer program has been described in detail in previous papers (Shobbrook, Lomb & Herbison-Evans (3), Section 5 and Shobbrook & Lomb (4)).

The highest peak in the frequency spectrum of the 1 Mon intensities occurred at \( f_1 = 7.3462 \) cycles per day (c/d), with an amplitude of \( \omega \cdot 08 \) (or 8 pc)*. Removal of a sinusoid of this frequency from the data (preshifting) reduced the rms deviation of the observations about the mean intensity from \( \omega \cdot 078 \) in the original data to \( \omega \cdot 053 \). The next highest peak occurred at \( f_2 = 7.4753 \) c/d, with an amplitude of \( \omega \cdot 06 \); the rms deviation about the mean was then reduced to \( \omega \cdot 027 \). These were the frequencies discovered by Millis (1). The corresponding periods are \( \omega \cdot 13612 \) and \( \omega \cdot 13377 \).

At this point in the reductions, the computer storage of 26K was insufficient to perform a least squares solution for the two frequencies, amplitudes and phases. The data were divided into two sets with the first, third, fifth etc. observations on one tape and the alternate observations on the other. Preshifting each data set with a least squares solution for the \( f_1 \) and \( f_2 \) frequencies, amplitudes and phases yielded a third peak at \( f_3 = 7.2171 \) c/d, with an amplitude of \( \omega \cdot 023 \). The rms deviation was then reduced to \( \omega \cdot 021 \). \( f_3 \) and \( f_4 \) are, within the errors, equally spaced from \( f_1 \) by \( f_{4} = 0.1291 \) c/d, corresponding to a long period of \( 7.46 \).

* The computer program makes, necessarily, a solution for sinusoidal variations and the term 'amplitude' therefore refers to the amplitude of the sine wave (\( \omega \) in \( A \sin 2\pi f ) \); this is sometimes called the semi-amplitude in variable star analysis.
Since the mean nightly intensities were found to vary over a range of 0.02, thus contributing significantly to the rms deviation, their values, calculated by the procedure described at the beginning of Section 4, were subtracted from the data. A search was then made in the region of twice the above frequencies in order to search for Fourier components of the three waves. The highest peak in this region occurred at $f_1 + f_2 = 14.8215$ c/d with an amplitude of 0.005. Note that this frequency may also be described as $2f_1 + f_L$. Subsequently, peaks at $2f_1$, $2f_2$, and $2f_3$ were also found. Subdivision of the data into four separate sets (of about 335 observations each) enabled a least squares solution to be performed for the seven frequencies, amplitudes and phases thus far discovered. These are the first seven listed in Table I. The values and errors listed are the averages of the values and standard deviations from the four sets of solutions. The errors are strictly only ‘estimated’ values, because from this and previous experience with 0.002 and 0.003 mag accuracy photometry, we suspect a correlation over three or four variable star measurement intervals (10–20 min) due, probably, to short term sky transparency variations. Consequently, the average of the four standard deviations (sd) will be close to a true sd for the whole data.

### Table I

Parameters of the sinusoidal components of the light variations in 1 Monocerotis, from observations of Shobbrook and Stobie

<table>
<thead>
<tr>
<th>Frequency description</th>
<th>Frequency (c/d)</th>
<th>Amplitude* (pc)</th>
<th>Hel. JD of max. light</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $f_1$</td>
<td>7.43620 ± 0.0008</td>
<td>8.73 ± 0.08</td>
<td>2441681 +</td>
</tr>
<tr>
<td>(2) $f_2$ or $f_1 + f_L$</td>
<td>7.47533 ± 0.00012</td>
<td>6.24 ± 0.08</td>
<td>0.7548 ± 0.0003</td>
</tr>
<tr>
<td>(3) $f_3$ or $f_1 - f_L$</td>
<td>7.2172 ± 0.0003</td>
<td>2.24 ± 0.08</td>
<td>0.8063 ± 0.0008</td>
</tr>
<tr>
<td>(4) $f_1 + f_2$ or $2f_1 + f_L$</td>
<td>14.8223 ± 0.0004</td>
<td>2.01 ± 0.08</td>
<td>0.7324 ± 0.0004</td>
</tr>
<tr>
<td>(5) $2f_1$</td>
<td>14.6934 ± 0.0004</td>
<td>1.50 ± 0.08</td>
<td>0.7140 ± 0.0005</td>
</tr>
<tr>
<td>(6) $2f_2$ or $2f_1 + 2f_L$</td>
<td>14.9494 ± 0.0008</td>
<td>0.92 ± 0.08</td>
<td>0.7493 ± 0.0009</td>
</tr>
<tr>
<td>(7) $2f_3$ or $2f_1 - 2f_L$</td>
<td>14.565 ± 0.0002</td>
<td>0.50 ± 0.08</td>
<td>0.757 ± 0.0002</td>
</tr>
<tr>
<td>(8) $3f_1$</td>
<td>22.035 ± 0.0002</td>
<td>0.42 ± 0.08</td>
<td>0.755 ± 0.0002</td>
</tr>
<tr>
<td>(9) $2f_1 + f_2$ or $3f_1 + f_L$</td>
<td>22.164 ± 0.0002</td>
<td>0.38 ± 0.08</td>
<td>0.768 ± 0.0003</td>
</tr>
<tr>
<td>(10) $f_1 + 2f_2$ or $3f_1 + 2f_L$</td>
<td>22.296 ± 0.0003</td>
<td>0.25 ± 0.08</td>
<td>0.735 ± 0.0005</td>
</tr>
<tr>
<td>(11) $f_2$</td>
<td>1.000±</td>
<td>0.35</td>
<td>0.61;†</td>
</tr>
<tr>
<td>(12) $f_3$</td>
<td>0.1291±</td>
<td>0.50</td>
<td>2.12</td>
</tr>
<tr>
<td>(13) $f_4$</td>
<td>12.105±</td>
<td>0.30</td>
<td>0.767</td>
</tr>
</tbody>
</table>

* To convert these amplitudes to intensities on the Johnson V system, they should be divided by 1.04.

† This period of 1 day probably derives from the reductions rather than the star.

‡ The reality of this period is questionable (Section 3).

The remaining frequencies were found from a search through the spectrum, particularly at frequency values corresponding to $nf_1 ± mf_L$, where $n$ and $m$ are integers and $f_L$ is the weighted mean of $f_2 - f_1$ and $f_1 - f_3$. Three peaks at such values were found between 3.3 and 33 c/d (periods 0.3 and 0.03); these are the 8th, 9th and 10th frequencies listed in Table I. Although there are other peaks at, for instance, $2f_1 - 2f_L$ and $4f_1 + f_L$, their amplitudes are less than 0.0025, which is about three times the amplitude errors. This is the level at which we chose to cut off the frequency search as all peaks above this level have at least a 99 per cent chance of being real. Apart from the groups of peaks discussed below, these are the highest peaks between periods of 20 days and 0.065.
The 1°-00 period listed in Table I has its minimum near midnight (local time) and is presumably due to small reduction inaccuracies, although this period has never appeared in the analysis of other short period variables reduced by the same methods by one of us (RRS). The amplitude appears significant but since all observations lie within about 0°-2 of the minimum, a one-day cycle is obviously ill-defined.

The final rms deviation about the mean intensity of 1 Mon when the first 11 frequencies in Table I and the nightly mean intensities are subtracted is 0°-0087 (or 0°-87 pc); this is still nearly three times the photometric errors. Two groups of peaks remain in the frequency spectrum, around 6-7 and 12° c/d. These are not aliases of each other but within each group the 1 c/d aliases are of similar heights. The mean amplitudes are 0°-003 to 0°-004, but they vary from zero to 0°-008 from night to night. There is a periodic variation of the amplitudes at the frequencies near 6-7 c/d with the 7°-746 long period (which may be due to residuals from 1f1, 1f2 and 1f3); this is not true of the frequencies near 12° c/d. It is possible that one of these latter frequencies is an unstable pulsation of the star. The corresponding period (0°-0826) of the highest peak is 0°-607 of 1/11 (0°-13612); this is similar to the ratios 0°-635, 0°-600 and 0°-600 derived or quoted by Fitch (5) for period pairs in the δ Scuti stars DQ Cep, δ Sct and CC And respectively. In CC And, Fitch notes that this overtone is not amplitude modulated, unlike the main period; the same is true in 1 Mon. We identify this frequency near 12° c/d as 1f4.

Removal of further frequencies lowers the rms by only 0°-0001 or 0°-0002 per frequency, so it seems certain that there is some irregularity in the amplitudes (or amplitude variation) at frequencies already mentioned.

Fig. 1. The frequency spectrum of 1 Mon, plotted from the parameters listed in Table I. The abscissae scales of the top, centre and bottom row are in the ratio 1 : 2 : 3.

Fig. 1 shows schematically the frequency spectrum of the observations and Fig. 2 presents nine of the 25 light curves, with the smooth curves derived from the first 11 frequencies, amplitudes and phases (plus mean nightly intensities) listed in Table I. Deviations are clearly seen, but appear to be smaller (in comparison with the amplitude) than those remaining after Fitch's 19 frequency solution for the light variation of CC And (his Fig. 3).
Fig. 2. Light curves for 1 Mon obtained over an 11-day observing period. The continuous curves are computer fitted from the parameters listed in Table I. The ordinate and abscissa zero points are arbitrary, but the zero of heliocentric $JD$ is marked by the vertical lines and the $JD$ is 24441600 plus the number against each graph. The 7-7-day variation in the amplitudes is clearly seen.
4. AMPLITUDE AND PHASE VARIATIONS OF THE PRIMARY WAVE

By primary wave we mean the wave consisting of the Fourier components \( f_1 \), \( 2f_1 \), \( 3f_1 \), etc. It should be stressed at this point that we interpret the harmonics of \( f_1 \) as simply the Fourier representation of a single non-sinusoidal wave of frequency \( f_1 \).

The amplitudes and phases of the \( f_1 \), \( 2f_1 \), \( 3f_1 \) and \( 4f_1 \) sinusoids and the mean nightly intensities were computed by least squares fitting to the observations of each night separately. These nine parameters were then plotted as a function of the \( 7.746 \) period in Figs 3 and 4. Fig. 3 shows the mean intensity variation which ranges over only about 1 pc; the standard deviations of the plotted points vary from 0.5 to 2 pc. The parameters of this variation are the twelfth set listed in Table I.

Fig. 4 illustrates the amplitude and phase variations of the Fourier components of the \( f_1 \) wave with the \( 7.746 \) period. The smooth curves were drawn through the Fig. 4(a) graphs by eye. For the amplitude variations of the harmonics, the \( f_1 \) curve was scaled down to pass through the maximum and minimum. For the phase variations of the harmonics the \( f_1 \) curve was scaled up by factors of 2, 3 and 4 for the \( 2f_1 \), \( 3f_1 \) and \( 4f_1 \) waves, respectively. In Fig. 4, the filled symbols and the open symbols represent alternate cycles of the \( 7.746 \) period, so that the curves also indicate the variation with a double cycle of 15.492 days. This is relevant to the discussion of a possible tidal modulation explanation of the frequency spectrum (Section 9).

Significant deviations may be seen from the six scaled curves in Fig. 4(b), (c) and (d), but in general it is evident that the amplitudes of the harmonics vary in phase with that of the \( f_1 \) sinusoid and the phase variations are proportional to the frequency of the harmonic. As Fitch (5) mentions in the case of CC And, this is the result of a variation in the time of occurrence of a non-sinusoidal wave of frequency \( f_1 \).
The most obvious difference between the plotted points and the scaled curves in Fig. 4(b), (c) and (d) lies in the amplitude variation of the $2f_1$ harmonic. That the scaled curve does not fit the observed variation means that the primary wave changes shape with amplitude. To study this effect further, define a 'non-sinusoidal parameter' equal to

\[
\frac{\text{amplitude of sinusoidal variation at } 2f_1}{\text{amplitude of sinusoidal variation at } f_1}
\]

The 'non-sinusoidal parameter' is closely related to Walraven's (6) $S$-distortion. If this parameter is plotted against the long period phase (Fig. 5(a)), there is a significant variation, with maxima (corresponding to maximum deviation from a sinusoid) at both the maximum and the minimum of the $7^d-746$ amplitude variation (see Fig. 4(a)). Minima occur near the upper and lower ends of the abrupt phase decrease (compare Figs 4(b) and 5(a)). Fig. 5(a) includes data from Millis' observations (Section 5).

Fig. 5(b) and (c) show clearly the change in the shape of the $f_1$ wave with amplitude for both our and Millis' data. The non-sinusoidal parameter is a minimum when the amplitude of the $f_1$ sinusoid is about 6.5 pc (light range 13 pc). This result was quite unexpected as we considered that the lower the amplitude of the pulsation the more sinusoidal the motion would be. Although this is true in the case of singly periodic variables it apparently is not necessarily true in the case of multiperiodic variables. This phenomenon is further discussed in Sections 8 and 9.

5.1 Comparison with Millis' $V$ observations

During the analysis of our observations, Millis (x) published 341 observations of 1 Mon, taken on two nights in 1971 and 15 in 1972—one and two years earlier than our data. As already mentioned, he found only the frequencies we have called $f_1$ and $f_2$ ($=f_1+f_1$). Frequency analysis of his data (Johnson $V$ magnitudes converted to intensities) yielded essentially the same frequency spectrum as that obtained from our own data. We list in Table II the frequencies, amplitude and times of light maximum from his 1972 observations only. Fig. 6(a) and (b) derived from Millis' data are equivalent to Fig. 4(a) and (b) derived from our data. The intensity scale in Fig. 6(a) and (b) has been multiplied by a factor 1.04 to convert

<table>
<thead>
<tr>
<th>Frequency description</th>
<th>Frequency (c/d)</th>
<th>Amplitude* (pc)</th>
<th>Hel. JD of max. light</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>$7.34628 \pm 0.00005$</td>
<td>$8.15 \pm 0.10$</td>
<td>$0.4186 \pm 0.0002$</td>
</tr>
<tr>
<td>$f_2$</td>
<td>$7.47494 \pm 0.00015$</td>
<td>$6.02 \pm 0.10$</td>
<td>$0.4133 \pm 0.0003$</td>
</tr>
<tr>
<td>$f_3$</td>
<td>$7.2176 \pm 0.0004$</td>
<td>$2.18 \pm 0.10$</td>
<td>$0.4108 \pm 0.0009$</td>
</tr>
<tr>
<td>$f_1 + f_2$</td>
<td>$14.8212 \pm 0.0005$</td>
<td>$1.85 \pm 0.10$</td>
<td>$0.4079 \pm 0.0005$</td>
</tr>
<tr>
<td>$2f_1$</td>
<td>$14.6930 \pm 0.0006$</td>
<td>$1.55 \pm 0.10$</td>
<td>$0.4094 \pm 0.0006$</td>
</tr>
<tr>
<td>$2f_2$</td>
<td>$14.9518 \pm 0.0017$</td>
<td>$0.62 \pm 0.10$</td>
<td>$0.4066 \pm 0.0015$</td>
</tr>
<tr>
<td>$2f_3$</td>
<td>$14.5632 \pm 0.0018$</td>
<td>$0.50 \pm 0.10$</td>
<td>$0.4064 \pm 0.0020$</td>
</tr>
</tbody>
</table>

* These amplitudes are on an intensity system derived from Millis' $V$ observations.
Millis' intensities to our \( \nu \) intensity scale. The smooth curves were directly transferred from Fig. 4(a) and (b). It is apparent from a comparison of Tables I and II and Figs 4 and 6 that we have obtained almost identical results in the analysis of \( \iota \) Mon with the two separate data sets.

**Fig. 4(a)–(d).** Amplitude and phase variations of the \( f_1, 2f_1, 3f_1 \) and \( 4f_1 \) waves during the 7.746 period. Alternate cycles of this period are represented by filled and open circles. The ordinate phases are \( \phi \) in the relation \( 2\pi f(t - T) + \phi \) where \( T = \text{Hel. JD} 2441681.7620 \). Consequently, the higher the value of \( \phi \) on any night the earlier the time of arrival of the wave with respect to its computed time of arrival assuming a constant \( \phi \). Zero on the 7.746 phase is Hel. JD 2441681.928. Unless indicated by standard deviation error bars, the errors on the plotted points are smaller than the size of the symbols.
Fig. 4 (c)–(d)
5.2 Analysis of Millis' \((B-V)\) observations

We give here the results of our analysis of Millis' \((B-V)\) data, taken in early 1972 at the same time as his \(V\) observations. Table III gives the intensity amplitude in \((B-V)\) for the same seven frequencies as those listed for \(V\) in Table II. The frequencies were assumed and a least squares solution made for the amplitudes and phases. In Table IV, Millis' \(V\) and \((B-V)\) amplitudes and phases are compared at

### Table III

Parameters of the sinusoidal components of the \((B-V)\) colours of \(\chi\) Monocerotis (on an intensity scale) from Millis' observations

<table>
<thead>
<tr>
<th>Frequency (c/d)</th>
<th>Amplitude (pc)</th>
<th>Hel. JD of bluest colour</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.346</td>
<td>3.20 ± 0.10</td>
<td>0.4167 ± 0.0003</td>
</tr>
<tr>
<td>7.475</td>
<td>2.24 ± 0.10</td>
<td>0.4130 ± 0.0005</td>
</tr>
<tr>
<td>7.018</td>
<td>0.73 ± 0.10</td>
<td>0.4117 ± 0.0014</td>
</tr>
<tr>
<td>14.821</td>
<td>0.60 ± 0.10</td>
<td>0.4070 ± 0.0010</td>
</tr>
<tr>
<td>14.603</td>
<td>0.46 ± 0.10</td>
<td>0.4070 ± 0.0012</td>
</tr>
<tr>
<td>14.952</td>
<td>0.30 ± 0.10</td>
<td>0.4030 ± 0.0018</td>
</tr>
<tr>
<td>14.563</td>
<td>0.16 ± 0.10</td>
<td>0.4055 ± 0.0033</td>
</tr>
</tbody>
</table>
each frequency. The $\Delta(B-V)/\Delta V$ amplitude ratios are essentially the same for all frequencies and the phase differences between the colour and $V$ intensity waves are zero within the errors, with the exception of that of the $f_1$ sinusoid. This latter

difference may be due to a possible real difference in the shapes of the $f_1$ wave in $V$ and $(B-V)$, which is permitted by the values and errors of $\Delta(B-V)/\Delta V$ for the $f_1$ and $2f_1$ waves.

Fig. 6(a) and (b). Amplitude and phase variations of the $f_1$ and $2f_1$ waves during the 7.746 period for Millis' data. Alternate cycles are represented by filled and open triangles. The zero points of the phase variations are at the same dates as for those in Fig. 4.
6. FREQUENCIES DERIVED FROM COMBINED DATA

The frequencies derived in Tables I and II can be improved by combining both sets of data. From the Julian Dates of maximum light given in Tables I and II we found that we could accurately count cycles between the two dates for the first six frequencies. For the remaining frequencies the cycle count was uncertain. These refined frequencies are listed in the upper half of Table V.

**Table V**

*Frequencies in 1 Monocerotis light variation derived from combined data*

<table>
<thead>
<tr>
<th>Frequency description</th>
<th>Frequency (c/d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>f₁</td>
<td>7.346132 ± 0.000009</td>
</tr>
<tr>
<td>f₂</td>
<td>7.475281 ± 0.000013</td>
</tr>
<tr>
<td>f₃</td>
<td>7.21718 ± 0.00004</td>
</tr>
<tr>
<td>f₁ + f₂</td>
<td>14.82136 ± 0.000004</td>
</tr>
<tr>
<td>2f₁</td>
<td>14.60227 ± 0.000004</td>
</tr>
<tr>
<td>2f₂</td>
<td>14.95050 ± 0.000011</td>
</tr>
<tr>
<td>f₅</td>
<td>0.12911 ± 0.00003</td>
</tr>
<tr>
<td>2f₅</td>
<td>14.56315 ± 0.000008</td>
</tr>
<tr>
<td>3f₁</td>
<td>22.03840 ± 0.00003</td>
</tr>
<tr>
<td>2f₁ + f₂</td>
<td>22.16751 ± 0.000006</td>
</tr>
<tr>
<td>f₁ + 2f₂</td>
<td>22.29662 ± 0.00009</td>
</tr>
</tbody>
</table>

One question which can now be answered is whether the first three frequencies in Table V form an equal frequency split. This assumption is implicit in the \((f₁, f₅)\) frequency description. The value of \(f₁\), was derived from the first three frequencies in Table V. From Table V, \(f₂ − f₁ = 0.12915 ± 0.00002\) and \(f₁ − f₃ = 0.12895 ± 0.00005\). Although the two frequency differences are separated by 3 sd the values are so close compared to the whole spectrum of possible values they could have taken, we consider that we were justified in assuming that the frequency split was exact. A weighted mean value of \(f₂ − f₁\), and \(f₁ − f₃\) leads to \(f₅ = 0.12911 ± 0.00003\).

Although improved frequencies cannot be derived directly from Tables I and II beyond the first six frequencies we can derive improved frequencies if the frequency identification in Table I is correct. With \(f₁\) given as above and \(f₁\) from Table V improved values have been calculated for the remaining frequencies and these are listed in the lower half of Table V.
All frequencies in Tables I and II agree to within two standard deviations with the improved frequencies in Table V (with the exception of $2f_1$ in Table I which is 2.5 sd away from the improved value in Table V). Moreover the frequencies in Table V satisfy their frequency description to within 2 sd. This confirms a posteriori that our original frequency identification was correct.

Because the frequency split is exact, the beat period of the $f_1$ and $f_2$ waves will equal the beat period of the $f_1$ and $f_3$ waves. It also means that the relative phase relationship of these two beat waves is invariant. Define $T_{12}$ as the Julian Date of maximum amplitude in the beat of the $f_1$ and $f_2$ waves and similarly for $T_{13}$. Then from the elements in Table I we obtain $T_{12} = 2441683.55 \pm 0.03$, $T_{13} = 2441684.82 \pm 0.06$ and hence $(T_{13} - T_{12})_{1973} = 1.27 \pm 0.09$. Similarly, from the elements in Table II we obtain $T_{12} = 2441335.11 \pm 0.03$, $T_{13} = 2441335.86 \pm 0.06$ and hence $(T_{13} - T_{12})_{1972} = 0.75 \pm 0.09$. Since the difference $T_{13} - T_{12}$ is invariant we can take the mean of the above values giving $T_{13} - T_{12} = 1.01$ days or $0.13$ in phase of the beat period.

It is interesting to note that the amplitude at the frequencies $f_2$ and $f_3$ total to that at $f_1$ to within 0.3 pc. Therefore the difference in the times of maximum amplitude of the $(f_1,f_2)$ and $(f_1,f_3)$ waves is the cause of (a) the non-zero minimum in the amplitude variation of $f_1$ (Figs 4(a) and 6(a), top graphs), and (b) the value of $106^\circ$ for the phase change near this amplitude minimum, rather than 180$^\circ$ which would occur if the amplitude reached zero (as discussed by Millis).

7. SUPERPOSITION OR MODULATION?

The generally accepted cause of the variable amplitude light curves of some $\delta$ Scuti stars is that of the superposition of the light variations from two or more modes of pulsation, radial or non-radial, in the star itself. However, Fitch (5) has presented evidence, particularly for the star CC And, suggesting that the variable amplitudes are mainly caused by tidal modulation of a single pulsation. The detailed frequency spectrum of CC And shows many similarities to that of $\iota$ Mon.

Taking account of the fact that Fitch's long period is a double cycle, then his $2f_1$ is equivalent to our $f_L$. From Table V of Fitch's paper the eight largest amplitude frequencies identified in CC And were (in our frequency notation) $f_1$, $f_1 - f_L$, $f_1 + f_L$, $2f_1$, $f_3$, $2f_1 - f_L$, $2f_1 + f_L$ and $f_1$. These frequencies are listed in order of decreasing amplitude. The remaining frequencies in his Table V are of amplitude less than 0.0025 at which point we terminated our frequency search. All of these frequencies (with the exception of $2f_1 - f_L$) are common to the frequency analysis of CC And and $\iota$ Mon although the amplitude order may differ. This suggests that the cause of the variable amplitude light curves is the same in both stars. Because $\iota$ Mon has a larger signal to noise ratio than CC And it presents a more favourable case for studying whether the light curve may be interpreted as superposition of a number of waves or tidal modulation of one wave. For this reason we have discussed in some detail in the next two Sections how the observations fit the superposition and modulation pictures.

8. SUPERPOSITION

In this section we will show how the frequency spectrum of $\iota$ Mon may be explained on the hypothesis of the non-linear superposition of three non-sinusoidal waves. The three basic waves are of frequency $f_1$, $f_2$ and $f_3$. If the three basic waves

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are non-sinusoidal then this explains the existence of the harmonics $2f_1$, $2f_2$, $2f_3$, $3f_1$, etc., in the frequency analysis. To explain the remaining frequencies we require to postulate that the superposition is non-linear (i.e. the waves interact). The existence of this interaction in r Mon is apparent from Fig. 5 of Millis (1) which is a plot of maximum and minimum light as a function of the long-period phase. If the superposition were linear then the upper and lower curves would be of equal amplitude. This interaction is also clearly shown in other multiperiodic variables and has been investigated previously by Walraven using his $M$-distortion curve (e.g. AI Vel, Walraven (6), Figs 3 and 5).

To give an example of the non-linear superposition envisaged consider two sinusoidal waves of frequency $\omega_1$ and $\omega_2$ (radians per day). Linear superposition of these two waves

$$ S(t) = a_1 \cos \omega_1 t + a_2 \cos \omega_2 t $$

may be written as

$$ S(t) = A \cos (\omega_1 t + \Phi) $$

where

$$ A = (a_1^2 + a_2^2 + 2a_1a_2 \cos \phi)^{1/2} $$

$$ \Phi = \tan^{-1} [a_2 \sin \phi/(a_1 + a_2 \cos \phi)] $$

$$ \phi = (\omega_2 - \omega_1) t. $$

The term $A \cos (\omega_1 t + \Phi)$ expresses the combined pulsation which the star experiences. Any non-linearity in the superposition will be primarily dependent on the combined pulsation. For example, when the combined amplitude $A$ is a minimum we expect the non-linear effects to be a minimum and when $A$ is a maximum we expect the non-linear effects to be a maximum. Also the phase of the non-linear effects will be tied in to the phase of the combined pulsation. This situation is analogous to that found in models of singly periodic stars. When the pulsation is initiated at low amplitude the motion approximates a sinusoidal variation. However, as the amplitude increases the motion becomes more and more asymmetric with a steep rising branch. Mathematically this non-linear effect can be expressed by modifying equation (2) to

$$ S(t) = A \cos (\omega_1 t + \Phi) + b(A) \cos 2(\omega_1 t + \Phi) $$

where $b(A)$ is a monotonic increasing function of $A$. In expression (4) we have neglected for the moment terms in $3(\omega_1 t + \Phi)$, $4(\omega_1 t + \Phi)$, etc. which may also arise.

It can readily be shown from the Fourier expansion of the term $b(A) \cos 2(\omega_1 t + \Phi)$ in expression (4) that this term will cause the frequencies $2\omega_1 \pm n\omega_1$ (where $\omega_1 = \omega_2 - \omega_1$) to appear in the frequency analysis. The amplitude of each term will depend on $a_1$, $a_2$ and the function $b(A)$. From the detailed form of expressions (3) and (4) it can be shown that the amplitude of the terms $2\omega_1 + n\omega_1$ and $2\omega_1 - n\omega_1$ will not in general be of equal amplitude. Similarly if we had considered the term in $3(\omega_1 t + \Phi)$ this will cause frequencies $3\omega_1 \pm n\omega_1$ to appear in the frequency analysis.

The above argument is not changed if we have a non-linear superposition of three waves of frequency $\omega_1$, $\omega_2$ and $\omega_3$ provided that $\omega_2 - \omega_1 = \omega_1 - \omega_3 = \omega_L$. This is because an expression identical to equation (2) can be derived for the linear superposition of three waves of equally spaced frequency. The coefficients
A and $\Phi$ are slightly more complicated, being given by

$$
\begin{align*}
A &= (a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 \cos \phi + 2a_1a_3 \cos \theta + 2a_2a_3 \cos (\phi + \theta))^{1/2} \\
\Phi &= \tan^{-1} \left[ \frac{(a_2 \sin \phi - a_3 \sin \theta)}{(a_1 + a_2 \cos \phi + a_3 \cos \theta)} \right] \\
\phi &= (\omega_2 - \omega_1) t \\
\theta &= \phi - \alpha
\end{align*}
$$

where $\alpha$ is the lag of the maximum amplitude in the beat of the $(\omega_1, \omega_3)$ waves relative to maximum amplitude in the beat of the $(\omega_1, \omega_2)$ waves.

Thus we can account for all the frequencies identified in Table I with the exception of the long-period term $f_L$. This term implies that the mean intensity over one cycle is varying as a function of the long period phase. The $f_L$ term could not have been predicted by the present kind of non-linear interaction since expression (4) has zero mean intensity independent of the long-period phase. However, the existence of this term is not too surprising as it is conceivable that the mean intensity of a multiperiodic variable could depend for example on the total amplitude of the motion. Note that maximum light for the $f_L$ term occurs on JD 2441683.12 which is close to maximum amplitude of the principal beat period occurring on JD 2441683.55 (see Section 6). In other words the mean intensity is a function of the amplitude and has a maximum when the amplitude is at a maximum.

On the superposition hypothesis Figs 4(a) and 6(a) represent plots of the amplitude $A$ and the phase $\Phi$ as a function of the long-period phase $\phi$. Fig. 7 shows how closely superposition can represent the observed amplitude and phase variations of the primary sinusoidal wave in 1 Mon (our observations). $A$ and $\Phi$ have been plotted as a function of the phase $\phi$ for the two-wave model (dashed line) and the three-wave model (solid line) taking coefficients from the frequency analysis in Table I; $a_1 = 0.087$, $a_2 = 0.062$, $a_3 = 0.022$ and $\alpha = 59^{\circ}$ from Section 6. Clearly the three wave superposition model accurately represents the observed variations apart from a slight discrepancy in the predicted phases for the three lowest amplitude points. The two wave model is a fair fit to the observations only because the first two waves are of considerably larger amplitude than the third wave.

On the basis of superposition the non-sinusoidal parameter defined in Section 4 and plotted in Fig. 5 is simply the value of $b(A)/A$ in expression (4). Thus although we cannot predict theoretically what form the function $b(A)$ should take it can be determined from the observations in Fig. 5(b) and (c). As mentioned in Section 4 one surprising result of the light variation of 1 Mon is that the non-sinusoidal parameter $(b(A)/A)$ does not tend to zero as the total amplitude $(A)$ tends to zero. One tentative explanation of this phenomenon is that the non-linear effects, instead of instantaneously adjusting to the combined amplitude of the pulsation, have a finite response time. For example consider the superposition of two waves of equal amplitude. When the two waves are exactly out of phase the combined amplitude $A$ is zero. However, if the non-linear effects have a finite decay time then $b(A)$ may not be zero. In this situation $b(A)/A$ tends to infinity as $A$ tends to zero.

Thus far the whole superposition picture has been built on the existence of three non-sinusoidal waves whose superposition is non-linear. One of the intriguing questions which arises is to which modes of pulsation do these three waves correspond? Two of the most relevant points here are (1) the existence of close frequencies, and (2) the three waves forming an equal frequency split.
frequencies are close this excludes any interpretation of the waves all being low order radial modes. On the other hand the existence of the equal frequency split immediately suggests non-radial modes of a star perturbed by rotation (Ledoux (7)). Ledoux showed that for a slowly rotating star the frequencies of the non-radial modes are given by the relation

$$\sigma_{k,l,m} = \sigma_{k,l,0} \pm m(1 - C_{k,l}) \Omega$$

where $\sigma$ is the frequency of the mode, $C$ is a constant depending on the structure of the star and $\Omega$ is the angular velocity of the star. The subscripts $k$, $l$, $m$ describe the non-radial mode. Using this relation we can check if the frequency split in

![Graph](image)

**Fig. 7.** The dashed and continuous curves represent the two-wave and three-wave linear superposition models, respectively. The plotted points are the observed amplitude and phase variations of the $f_2$ wave with the 7.746 period (i.e. Fig. 4(a)). The horizontal and vertical arrows denote the zero points of the phases $\Phi$ and $\phi$, respectively.

1 Mon is consistent with the observed value of $v \sin i$. Taking the simplest non-radial modes of $k = 0$, $l = 2$ and $m = 0, 2$ then we predict the three frequencies $\sigma$, $\sigma + 2(1 - C) \Omega$ and $\sigma - 2(1 - C) \Omega$. The value of $C$ is not critically dependent on the model or on the mode and we adopt $C = 0.15$. Hence equating $2\pi f_L$ and $2(1 - C) \Omega$ the rotational period of 1 Mon is found to be 13 days. With an assumed radius of $R/R_\odot = 3.7$ this leads to $v = 15$ km s$^{-1}$. This may be compared with the observed value of $v \sin i = 10-15$ km s$^{-1}$ (Danziger & Dickens (9)).
9. TIDAL MODULATION

In this situation the assumption is that the amplitude and phase variations of the \( f_1 \) wave from night to night are due to the effect of tides raised by a secondary star on the primary (the latter assumed to be the variable). Since there are two tides on the primary, the modulation would occur twice per orbital period. If \( \gamma \) Mon has a secondary, the orbit must be nearly circular, since there is very little difference between alternate cycles of the \( 7^{d} 746 \) period (represented by the filled and open symbols in Figs 4, 5, 6). The only evidence we can find suggesting that there are two different cycles in a \( 15^{d} 492 \) period is in the phase variation of the \( 2f_1 \) harmonic (lower graph, Figs 4(b) and 6(b)). Between phases \( 0.6 \) and \( 0.8 \) of the \( 7^{d} 746 \) period, the phase change with the open symbols appears considerably more abrupt than that with the filled symbols. One of Millis' nights of observation provided confirmation of the low phase angle value for \( 2f_1 \) at the long-period phase near \( 0.66 - 0.67 \). For the tidal explanation, therefore, we assume a two-cycle wave in a \( 15^{d} 492 \) period. In order to show the two-cycle variation, corresponding to either Fig. 4, 5 or 6 the filled symbols should be plotted with one-half of their phase values shown in these figures and the open symbols with half their \( 7^{d} 746 \) phase plus \( 0.5 \).

The frequency spectrum of a sine wave of frequency \( f_1 \), which is amplitude and phase modulated by a sine wave of frequency \( f_L \), may be written in the general form:

\[
A \{1 + k' \sin (2\pi f_1 t + \phi_L') \} \sin \{2\pi f_1 t + \phi_1 [1 + k \sin (2\pi f_1 t + \phi_L)]\}
\]

If we add also the harmonic terms \( 2f_1, 3f_1, 2f_L \), etc., we can predict the existence of all the frequencies listed in Table I, plus of course many more which would be expected to have lower amplitudes.

The main problem here is the absence of the frequencies \( f_1 \pm 2f_L \). On the modulation theory, since the \( 2f_1 + 2f_L \) term is present, it is surprising not to find at least \( f_1 + 2f_L \).

The various figures in this paper may be explained qualitatively on the tidal modulation theory. In Figs 4 and 6, if the modulation is considered as that of the whole non-sinusoidal wave \( f_1 \), in general all harmonics will be amplitude modulated in phase with each other and the size of the phase modulation of each will be proportional to the frequency of the harmonic. As we have already observed, discrepancies due to the change in shape of the \( f_1 \) wave with amplitude are to be expected.

As noted by Fitch, Christy (8) has shown that the amplitude, phase and shape of the light and velocity variations of the Cepheids are determined in a highly non-linear fashion by the physical parameters in and above the H and He ionization zones. It is reasonable to suppose that the tidal bulges will disturb these ionization zones, which drive the pulsations. Consequently, minima in the light amplitude (at phase \( 0.67 \) of the \( 7^{d} 746 \) period or \( 0.33 \) and \( 0.83 \) of the \( 15^{d} 492 \) period) are interpreted as times when the tidal bulges pass the line of sight. This is also the situation envisaged by Fitch (5) for \( \sigma \) Sco, a \( \beta \) Canis Majoris pulsating variable, in his Fig. 4.

The variation of the non-sinusoidal parameter (Fig. 5) presents a problem. It is, as might be expected, at a maximum when the \( f_1 \) wave has its maximum amplitude, but there is another sharp maximum when each tidal bulge faces us. At this time, the light range is at its smallest value of only 5 pc. Although, presumably, the
disturbance of the ionization zone is a maximum at this time (and the phase of the $f_1$ wave is changing most rapidly), it is not obvious why the low amplitude light curve should be markedly less sinusoidal than, for instance, when the bulge is asymmetrically placed on the visible hemisphere. The $f_1$ wave is, in fact, most nearly sinusoidal at phases $0.55$ and $0.90$, when the tidal bulge is about $30^\circ$ off the central meridian and the light range is about $13$ pc. It should be noted that these minima in the non-sinusoidal parameter are also determined by the increase in the parameter towards the high amplitude light curve region centred on phase $0.25$.

In the modulation case, the change in mean brightness of $\epsilon$ Mon with a frequency $f_L$ is simply interpreted as a variation due to aspect changes of the tidally distorted primary (Fig. 3), i.e. $\epsilon$ Mon is an ellipsoidal variable. Comparison of Figs 3 and 4 show the minima occur when the tidal bulges are facing us, as expected in such a variable.

Finally, note that the phase variation in the $f_1$ wave to be expected from the effect of light travel time across a circular orbit, inclination $90^\circ$, of a binary system with masses $1.5 M_\odot$ (F2II primary) and $0.5 M_\odot$ (assumed secondary) and $15^h 5$ orbital period, is about $1^h 5$. This is small compared to the $106^h$ observed.

10. DISCUSSION

In Section 8 we showed how the sinusoidal component of the light variation of $\epsilon$ Mon may be explained quantitatively by the superposition of three sinusoidal waves of frequencies $f_1, f_2$ and $f_3$. The rest of the spectrum follows (qualitatively) from the assumption of non-linear superposition of the three non-sinusoidal waves $f_1, f_2$ and $f_3$. The difficulties lay in explaining the behaviour of the non-sinusoidal parameter at minimum amplitude and in the existence of the long period intensity variation with $f_L$.

In Section 9, tidal modulation of a single non-sinusoidal wave of frequency $f_1$ was found to account for the amplitude and phase variations of the Fourier components of $f_1$ and to produce the observed frequencies. As there is no quantitative theory of tidal modulation, the observed effects (seen also with a somewhat lower signal to noise ratio in CC And by Fitch) have to be accepted *prima facie* as evidence for tidal modulation. The difficulties here lay in explaining the peak in the value of the non-sinusoidal parameter at minimum amplitude (as in the superposition case) and in the non-existence of the frequencies $f_1 \pm 2f_L$. This latter problem constitutes a serious objection to the tidal modulation explanation, both in $\epsilon$ Mon and in CC And (Fitch (5)). Since $f_1 \pm 2f_L, f_1 \pm 3f_L$, etc. are not present, there is implied a fortuitous modulation of the shape and amplitude of the $f_1$ wave (lower graphs, Figs 4(a) and 6(a)). Had the phase modulation been of a different form, the extra sidebands would have been present, and the simple combination of $f_1$, $f_1+f_L$ and $f_1-f_L$ would not have accounted for the variation as well as they do (Fig. 7).

We have discussed both theories in order to present fully the evidence for both alternatives for $\epsilon$ Mon. There are no other $\delta$ Scuti stars for which there is such a large signal-to-noise ratio for so many periods. In practice, the tidal modulation theory may be proved or disproved by verifying the existence or non-existence of a $15^h 492$ orbital velocity variation. With this period, and appropriate masses for the stars, the orbital velocity variation of the primary should be of the order of
100 km s$^{-1}$. From the radial velocities of 1 Mon published by Jones (10) the two nights JD 2439103 and JD 2439139 contain sufficient data to enable the radial velocity amplitude, 2K, and the mean velocity to be determined for each night. The mean velocity on these two nights and the particular phases of the supposed 15$^{th}$-492 orbital period restrict the orbital velocity variation to less than 4 km s$^{-1}$. The inclination of the orbit cannot explain this low limit to the orbital velocity variation since any tidal modulation effect tends to zero as the orbital inclination tends to zero. Thus the radial velocity observations do not support any orbital velocity variation which is required by the tidal modulation theory. Instead the amplitude of the radial velocity observations is consistent with the amplitude of the light variation using Leung & Wehlau’s (11) value of 2K/$\Delta b = 62$ km s$^{-1}$ mag$^{-1}$ for $\delta$ Scuti stars. Overall, we consider that the light variation of 1 Mon is most likely caused by the superposition of three non-sinusoidal waves, combining non-linearly.

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