COSMICRAY STREAMING—II
EFFECT OF PARTICLES ON ALFVÉN WAVES

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SUMMARY

Alfvén waves, generated by streaming cosmic rays, may decay via a cascade of forward and backward waves, caused by an interaction with sound waves. The analysis of this process by Chin & Wentzel is extended, and limits are placed on the validity of various approximate solutions. In a later paper, these will be used to give self-consistent solutions to the cosmic ray diffusion problem.

INTRODUCTION

In Skilling 1975a (Paper I), the problem of the motion of cosmic rays in a given field of Alfvén waves was re-examined, and it was found possible that Fermi acceleration of particles between forward and backward waves might play a significant role in the movement of cosmic rays down density gradients. To treat this effect correctly one needs a self-consistent analysis involving the mutual interactions between the particles and the two types of wave. Such an analysis, based on Paper I and the present paper, is to be presented in Skilling 1975b (Paper III).

Streaming cosmic rays generate 'forward' Alfvén waves propagating along the magnetic field in the direction of the streaming (Lerche 1967; Kulsrud & Pearce 1969; Wentzel 1969a). While these waves are useful in that they scatter the cosmic rays and confine them, the waves must somehow be removed if an equilibrium state is to be reached. Linear damping mechanisms such as viscosity and resistivity seem to be unimportant, and ion-neutral frictional damping, favoured by Kulsrud & Pearce (1969) and used in the self-consistent solution of Skilling (1971), becomes inefficient in highly ionized regions. One is led to consider wave–wave interactions as a means of removing forward Alfvén waves. The simplest (3-wave) reaction $A + A \rightarrow X$ does not work, because the selection rules force the product 'X' to be itself a forward Alfvén wave, and the probability of that reaction turns out to be zero. Higher order (4-wave) reactions $A + A + A \rightarrow A$ or equivalently $A + A \rightarrow A + A$, are inherently less probable than the simpler reactions, and this approach does not seem to be fruitful. It may be possible to run a series of 3-wave interactions between Alfvén and fast magnetosonic waves (Galeev & Oraevskii 1963, Karplyuk, Oraevskii & Pavlenko 1973), both of which can be generated by streaming cosmic rays. Although this mechanism ought to be investigated further, it has not found favour in the literature in the present context, and it will not be treated in this paper.
The accepted mechanism for removing Alfvén waves in a highly-ionized region is one in which the Alfvén mode is the upper quantum state in a reaction of the type $A \rightarrow X + Y$, driven by stimulated growth of $X$ and/or $Y$. Here the $X$ and $Y$ photons must be less energetic (corresponding to smaller frequency $\omega$) than the original $A$ photon, and the only plausible candidates are sound photons. We assume that the sound speed is less than the Alfvén speed, otherwise the Alfvén mode would be the lowest quantum state and this type of decay would be forbidden. In principle $A \rightarrow S + S$ can occur, but the $S$ waves must have relatively large and nearly opposite k-vectors, and the reaction is improbable. One is forced to consider the $A \rightarrow A + S$ reaction. The analysis in this paper follows that of Chin & Wentzel (1972, hereafter referred to as C & W), with a more accurate treatment of the case when sound waves are only lightly damped. Approximate solutions are obtained in certain well-defined parameter domains.

THE WAVE CASCADE

In a ‘$\beta > 1$’ plasma, in which the Alfvén speed $v_A = |\omega_A/k|$ exceeds the sound speed $v_S = |\omega_S/k|$, Alfvén waves travelling in opposite directions can interact with a sound wave when the appropriate selection rules are satisfied. For the reaction

$$A_+ \rightleftharpoons A_- + S$$

where $(+(-))$ refers to Alfvén waves travelling forwards (backwards) along the direction of the unperturbed magnetic field $B_0$, the selection rules

$$k_+ = k_- + k_S, \quad \omega_+ = \omega_- + \omega_S$$

ensure that $(k_+)/(k_-) = -(v_A + v_S)/(v_A - v_S) \simeq -(1 + 2v_A/v_S)$. An exactly similar reaction $A_- \rightleftharpoons A_+ + S^{\ast}, \quad S^{\ast}$ being a backward sound wave, can occur to remove the backward Alfvén wave, and a sequence or cascade of interacting Alfvén waves can be set up (Fig. 1), each wave differing from its neighbour by a factor $(v_A + v_S)/(v_A - v_S)$ in $|k_1|$.

![Fig. 1. The Alfvén cascade.](https://example.com/fig1.jpg)

If Alfvén waves are generated, perhaps by cosmic rays, in some particular range of wavenumbers, then waves of significantly different wavenumber can be produced by repeated reactions of this type. In particular, Alfvén waves can cascade down in energy to indefinitely long wavelengths, with each individual reaction being driven by stimulated emission from the upper state. Alfvén photons are conserved in this process.

BASIC EQUATIONS

C & W have investigated these reactions using the wave–wave interaction techniques of Sagdeev & Galeev (1969). They derive their rate equations (27), (33)
and (34), which, for the $A_+ \leftrightarrow A_- + S$ reaction, we re-write as

$$
\frac{\partial N_+}{\partial t} = -\frac{1}{2} \frac{4\pi}{v_A + v_S} \frac{\hbar v_A^3 k^3}{4U_M v_S} (N_+ N_- + N_+ N_S - N_- N_S) \tag{1}
$$

$$
\frac{\partial N_-}{\partial t} = +\frac{1}{2} \frac{4\pi}{v_A - v_S} \frac{\hbar v_A^3 k^3}{4U_M v_S} (N_+ N_- + N_+ N_S - N_- N_S) \tag{2}
$$

$$
\frac{\partial N_S}{\partial t} = +\frac{1}{2} \frac{2\pi}{v_A} \frac{\hbar v_A^3 k^3}{4U_M v_S} (N_+ N_- + N_+ N_S - N_- N_S) \tag{3}
$$

where

$$
N_\pm(k)dk = \text{number density of Alfvén photons in wavenumber range } dk,
$$

$$
N_S(2k)dk = \text{number density of sound photons in wavenumber range } 2dk \text{ about the value } 2k, \text{ and}
$$

$$
U_M = B_0^2/2\mu_0 = \text{ambient magnetic energy density}.
$$

To satisfy the selection rules, $N_+$ is evaluated at $k(1 + v_S/v_A)$, $N_-$ at $k(1 - v_S/v_A)$ and $N_S$ at $2k$. Here and in the following analysis, $k$ is to be interpreted as $|k|$; this accounts for the extra factor of $\frac{1}{2}$ in the reaction rates above, as photons with positive and negative $k_1$ have been grouped together under the same (positive) label $k$.

Each sound wave interacts only with one pair of Alfvén waves, so that (3) gives the evolution of sound waves directly. An Alfvén wave, however, can undergo two reactions, either gaining energy or losing energy. Adding these two, and also inserting damping rates $\sigma_\pm(k)$ due to some extraneous process (perhaps cosmic ray interactions), gives

$$
\frac{\partial N_+}{\partial t} + \sigma_+ N_+ = \frac{\pi \hbar v_A^3 k^3}{2U_M v_S} \left( \frac{4v_S}{v_A} N_+ \frac{\partial}{\partial k} (k^2 N_-) + (N_S + N_S^*)(N_- - N_+) \right.
$$

$$
+ \frac{v_S}{v_A} N_+ - N_- \frac{\partial}{\partial k} \left[ k^4 (N_S - N_S^*) \right] + \frac{2v_S}{v_A} (N_S^* - N_S) k \frac{\partial N_-}{\partial k} \right) \tag{4}
$$

$$
\frac{\partial N_-}{\partial t} + \sigma_- N_- = \frac{\pi \hbar v_A^3 k^3}{2U_M v_S} \left( \frac{4v_S}{v_A} N_- \frac{\partial}{\partial k} (k^2 N_+) + (N_S + N_S^*)(N_+ - N_-) \right.
$$

$$
+ \frac{v_S}{v_A} N_+ - N_- \frac{\partial}{\partial k} \left[ k^4 (N_S - N_S^*) \right] + \frac{2v_S}{v_A} (N_S^* - N_S) k \frac{\partial N_+}{\partial k} \right) \tag{5}
$$

where $N_+$ and $N_-$ are now evaluated at the same wavenumber $k$, $N_S$ and $N_S^*$ being evaluated at $2k$, and where terms of second order in $v_S/v_A$ have been dropped from the right-hand sides. The corresponding equations for forward and backward sound waves, with extraneous damping rate $\sigma_S$, are

$$
\frac{\partial N_S}{\partial t} + \sigma_S N_S = \frac{\pi \hbar v_A^3 k^3}{4U_M v_S} \left( N_+ N_- + \frac{v_S}{v_A} k \left( N_+ \frac{\partial N_+}{\partial k} - N_+ \frac{\partial N_-}{\partial k} \right) \right.
$$

$$
+ N_S \left[ N_+ - N_- + \frac{v_S}{v_A} k \frac{\partial}{\partial k} (N_+ + N_-) \right] \tag{6}
$$

$$
\frac{\partial N_S^*}{\partial t} + \sigma_S N_S^* = \frac{\pi \hbar v_A^3 k^3}{4U_M v_S} \left( N_+ N_- + \frac{v_S}{v_A} k \left( N_- \frac{\partial N_-}{\partial k} - N_- \frac{\partial N_+}{\partial k} \right) \right.
$$

$$
+ N_S^* \left[ N_- - N_+ + \frac{v_S}{v_A} k \frac{\partial}{\partial k} (N_+ + N_-) \right]. \tag{7}
$$
These are the equivalents of equations (38) to (41) of C & W, but with no assumption of power-law spectra and with some extra terms of order $v_A/v_S$ retained: specifically these are the third and fourth terms on the right of (4) and (5), and the $O(v_A/v_S)$ terms in (6) and (7). When the sound waves are heavily damped ($v_S$ large) the sound wave intensities become small, and the extra terms are unimportant. However, the sound wave intensities can rise when $v_S$ is small. A large population of sound waves forces the Alfvén waves towards a symmetrical distribution ($N_+ \simeq N_-$), and the last terms in each of (4)–(7) can become comparable to the earlier terms. The damping of sound waves is expected to become weak at long wavelengths (since $v_S \propto k^2$ for thermal conductivity or other diffusive process) and the extra terms become relevant in this limit.

It is convenient to change from the quantum description of the waves to a dimensionless classical description by defining

$$J_\pm(k) = N_\pm(k) \hbar^2 v_A / U_M$$

intensity (kinetic + magnetic) of Alfvén waves per unit logarithmic wavenumber range (with $+ve$ and $-ve$ wavenumbers combined into the positive label $k$), relative to the ambient magnetic energy, and

$$J_S(2k) = N_S(2k) \hbar^2 v_S / U_M$$
similar intensity of sound waves.

Equations (4)–(7) become

$$\frac{\partial J_+}{\partial t} + \sigma_+ J_+ = \frac{\pi v_A k}{2} \left( 4 k \frac{\partial J_1}{\partial k} - \frac{v_A^2}{4v_S^2} (J_{S+} + J_{S-}) (J_1 - J_{S-}) \right)$$

$$- \frac{v_A}{4v_S} (J_S - J_{S*}) k^3 \frac{\partial J_1}{\partial k} + \frac{v_A}{4v_S} k \frac{\partial}{\partial k} \left[ (J_1 - J_{S-}) (J_S - J_{S*}) \right]$$

(8)

$$\frac{\partial J_-}{\partial t} + \sigma_- J_- = \frac{\pi v_A k}{2} \left( 4 k \frac{\partial J_1}{\partial k} + \frac{v_A^2}{4v_S^2} (J_{S+} + J_{S*}) (J_1 - J_{S-}) \right)$$

$$+ \frac{v_A}{4v_S} (J_S - J_{S*}) k^3 \frac{\partial J_1}{\partial k} + \frac{v_A}{4v_S} k \frac{\partial}{\partial k} \left[ (J_1 - J_{S-}) (J_S - J_{S*}) \right]$$

(9)

$$\frac{\partial J_S}{\partial t} + \sigma_S J_S = \frac{\pi v_A k}{4} \left( 4 k \frac{\partial J_1}{\partial k} + \frac{4v_S}{v_A} \frac{\partial}{\partial k} \left[ \frac{v_A}{v_S} (J_{S+} - J_{S-}) \right] \right)$$

$$+ J_S \left[ \frac{v_A}{v_S} (J_{S+} - J_{S-}) + k^3 \frac{\partial}{\partial k} \left[ \frac{v_A}{v_S} (J_{S+} + J_{S-}) \right] \right]$$

(10)

$$\frac{\partial J_{S*}}{\partial t} + \sigma_{S*} J_{S*} = \frac{\pi v_A k}{4} \left( 4 k \frac{\partial J_1}{\partial k} + \frac{4v_S}{v_A} \frac{\partial}{\partial k} \left[ \frac{v_A}{v_S} (J_{S+} - J_{S-}) \right] \right)$$

$$+ J_{S*} \left[ - \frac{v_A}{v_S} (J_{S+} - J_{S-}) + k^3 \frac{\partial}{\partial k} \left[ \frac{v_A}{v_S} (J_{S+} + J_{S-}) \right] \right].$$

(11)

**STEADY-STATE SOLUTIONS**

In a steady state, these simultaneous differential equations for the spectra can be further simplified by again using $v_S/v_A$ as a small parameter, extracting the...
dominant terms on the right-hand sides in various régimes, and equating them to the damping terms on the left. Given a set of damping rates $\sigma$, we define two dimensionless variables

$$A = \frac{I_+ - I_-}{I_+ + I_-}, \quad D = \frac{4\sigma S}{\pi\nu_A (I_+ + I_-)}$$  \hspace{1cm} (12)

representing respectively the asymmetry of the Alfvén waves ($-1 \leq A \leq 1$) and the relative importance of the damping of sound waves. The orders of magnitude of $A$ and $D$, as shown in the (logarithmic) plot Fig. 2, determine how the equations (8)–(11) simplify. Since $\sigma S \propto k^2$, $D$ is expected to increase with $k$ in most solutions. Correspondingly, the cascade down to lower $k$ can be thought of as running from right to left in Fig. 2.

![Diagram](https://academic.oup.com/mnras/article-abstract/173/2/245/971335)

**Fig. 2. Domains in which the steady state equations simplify.** Shaded region is excluded; I: Solution obtained from (22) and (23); II: Solution is (25) and (26); III: Solution obtained from (28), (29) and (30); IV: Solution from (32) and (34), or (33) and (35).

The sound intensities can be extracted from (10) and (11) as

$$J_S = \frac{4\sigma S}{\pi\nu_A k} \frac{k^3}{k^2} \left( \frac{\partial}{\partial k^2} I_+ - \frac{\partial}{\partial k^2} I_- \right)$$  \hspace{1cm} (13)

$$J_S^* = \frac{4\sigma S}{\pi\nu_A k} \frac{k^3}{k^2} \left( \frac{-\partial}{\partial k^2} I_+ - \frac{-\partial}{\partial k^2} I_- \right)$$  \hspace{1cm} (14)

These are necessarily positive, so, since the numerators are both approximately $I_+ I_-$ and hence positive, we must have

$$\frac{\nu_A}{\nu_S} |I_+ - I_-| < \frac{4\sigma S}{\pi\nu_A k} \frac{k^3}{k^2} \frac{\partial}{\partial k^2} I_+ I_-$$  \hspace{1cm} (15)
in any steady state. In terms of $\mathcal{A}$ and $\mathcal{D}$ this reads

$$\frac{v_A}{v_S} |\mathcal{A}| < \mathcal{D} + 1 \quad (16)$$

on disregarding numerical factors and taking $k \partial / \partial k = O(1)$, since there is no reason to expect the relevant differential $k^3 \partial ((\mathcal{A}_+ + \mathcal{A}_-)/k^2) / \partial k$ to vanish (see discussion below under 'domain IV'). The condition (16) excludes the shaded region of Fig. 2. Physically, a substantially asymmetric distribution of Alfvén waves cannot be sustained if sound waves are only weakly damped ($\mathcal{D}$ small) and allowed to grow.

At the border of the excluded region, the sound wave intensity $\mathcal{S}$ increases without limit, which itself tends to produce a more symmetrical distribution of Alfvén waves. Although it is possible to set up growth rates $- \sigma_{\pm}$ so that the solution lies arbitrarily close to the excluded region, we do not expect this to occur in the cosmic ray context.

Instead, we proceed on the assumption that the inequality (16) is strongly satisfied. Then, from (13) and (14),

$$\mathcal{S} \simeq \mathcal{S}^* \simeq 4 \frac{\mathcal{A}_+ + \mathcal{A}_-}{4 \sigma_S - \frac{k^3}{\pi v_A k} \frac{\partial}{\partial k} (\mathcal{A}_+ - \mathcal{A}_-)} \quad (17)$$

$$\frac{\mathcal{S} - \mathcal{S}^*}{\mathcal{S} + \mathcal{S}^*} \simeq \frac{v_A}{v_S} \frac{(\mathcal{A}_+ - \mathcal{A}_-)}{4 \sigma_S - \frac{k^3}{\pi v_A k} \frac{\partial}{\partial k} (\mathcal{A}_+ - \mathcal{A}_-)} \ll 1. \quad (18)$$

Dropping the fourth terms on the right-hand sides of (8) and (9) because they are always negligible, substituting for the sound wave intensities, and cancelling common factors of $\mathcal{A}_+$ and $\mathcal{A}_-$, we arrive at

$$\frac{\sigma_+}{\pi v_A k} = 2k \frac{\partial \mathcal{A}_0}{\partial k} - v_A^2 \frac{4 \sigma_S}{v_S^2} \frac{(\mathcal{A}_+ - \mathcal{A}_-)^2}{4 \sigma_S - \frac{k^3}{\pi v_A k} \frac{\partial}{\partial k} (\mathcal{A}_+ - \mathcal{A}_-)^2} \quad (19)$$

$$\frac{\sigma_-}{\pi v_A k} = 2k \frac{\partial \mathcal{A}_+}{\partial k} + v_A^2 \frac{4 \sigma_S}{v_S^2} \frac{(\mathcal{A}_+ - \mathcal{A}_-)^2}{4 \sigma_S - \frac{k^3}{\pi v_A k} \frac{\partial}{\partial k} (\mathcal{A}_+ - \mathcal{A}_-)^2} \quad (20)$$

These two steady-state equations for the Alfvén waves alone could be used as they stand. Nevertheless, it is useful to simplify them further, which can be done in four major domains (Fig. 2).

FURTHER SIMPLIFICATIONS

'* Domain I'. Sound waves heavily damped.

$$\mathcal{D} > 1, \quad \mathcal{A} < \frac{v_S^2}{v_A^2} \mathcal{D} \quad (21)$$

or

$$\sigma_S \gg \frac{\pi}{4} k v_A (\mathcal{A}_+ + \mathcal{A}_-), \quad \sigma_S \gg \frac{v_A^2 \pi}{v_S^2} \frac{1}{4} k v_A (\mathcal{A}_+ - \mathcal{A}_-)$$
Only the first terms remain on the right of (19) and (20), yielding
\[ k \frac{\partial \mathcal{J}_+}{\partial k} = \frac{\sigma_-}{2\pi v_A k}, \quad k \frac{\partial \mathcal{J}_-}{\partial k} = \frac{\sigma_+}{2\pi v_A k}. \] (22)

These can be called the 'pure cascade' equations. They correspond to equations (46) of C & W, and have been used in the cosmic ray context by Wentzel (1969a, b, 1972). Interestingly enough, the cascade rate turns out to be independent of the sound speed \( v_S \).

The cascade equations imply that
\[ k \frac{\partial \mathcal{J}}{\partial k} = \frac{1}{\pi v_A k} \frac{\mathcal{J}_- - \mathcal{J}_+}{(\mathcal{J}_+ + \mathcal{J}_-)^2}. \]

We expect cosmic ray streaming to give a positive growth rate of forward Alfvén waves (\( \sigma_+ < 0 \)), whereas backward waves will be damped (\( \sigma_- > 0 \)). Thus the asymmetry of the waves will decrease as the cascade proceeds down to longer wavelengths. In such a situation, \( \mathcal{J} \) is expected to decrease from right to left in Fig. 2. One presumes that appropriate boundary conditions on the cascade will prevent the asymmetry \( \mathcal{J} \) or, worse, the backward intensity \( \mathcal{J}_- \), from attempting to become negative.

The sound waves are damped so heavily that they have no effect on the transitions, which are driven purely by stimulated emission among the Alfvén waves themselves. In fact the intensities of the sound waves (from (17)) are
\[ \mathcal{J}_S \simeq \mathcal{J}_{S \pm} \simeq \frac{\pi v_A k}{\sigma_S} \mathcal{J}_+ \mathcal{J}_-. \] (23)

Since \( \sigma_S \gg k v_A \mathcal{J}_+ \) in this domain, these sound intensities are smaller even than that of the backward Alfvén waves \( \mathcal{J}_- \). All the energy which an Alfvén photon loses as it cascades down in energy appears almost immediately as heat in the background medium.

'Domain II'. Sound waves moderately damped.
\[ \mathcal{D} > 1, \quad \frac{v_S^2}{v_A^2} \mathcal{D} < \mathcal{J} < \frac{v_S}{v_A} \mathcal{D} \] (24)
or
\[ \sigma_S \gg \frac{\pi}{4} k v_A (\mathcal{J}_+ + \mathcal{J}_-), \quad \frac{v_A}{v_S} \pi k v_A (\mathcal{J}_+ - \mathcal{J}_-) \ll \sigma_S \ll \frac{v_A^2}{v_S^2} \pi k v_A (\mathcal{J}_+ - \mathcal{J}_-). \]

The pure cascade terms in (19) and (20) disappear in favour of terms derived from the sound waves, yielding
\[ \frac{\sigma_+}{\pi v_A k} = \frac{v_A^2}{v_S^2} \frac{\pi v_A k}{4 \sigma_S} \mathcal{J}_- (\mathcal{J}_+ - \mathcal{J}_-) \]
\[ \frac{\sigma_-}{\pi v_A k} = \frac{v_A^2}{v_S^2} \frac{\pi v_A k}{4 \sigma_S} \mathcal{J}_+ (\mathcal{J}_+ - \mathcal{J}_-) \]
and thence
\[ \frac{\mathcal{J}_+}{\sigma_+} - \sigma_+ = \frac{v_S}{v_A} \frac{2}{\pi v_A k} \left( \frac{\sigma_S}{\sigma_- + \sigma_+} \right)^{1/2}. \] (25)
Here the wave intensities are determined by purely local (in $k$) conditions, and differential equations do not have to be solved. We may note that this domain cannot be entered unless $\sigma_+$ is indeed negative and $\sigma_-$ positive, as otherwise the required degree of asymmetry in the waves cannot be maintained. The sound intensities from (17) are

$$I_S \simeq I_S^* \simeq \frac{\pi v_A k}{\sigma_S} I_{+} I_{-}$$

which are still less than the backward Alfvén intensity $I_{-}$.

The net rate of input of energy from cosmic rays is $-(\sigma_+ I_{+} + \sigma_- I_{-})$ (relative to $U_M$, per unit log. bandwidth). To dominant order in $v_S/v_A$, this net input is zero, so that the energy $\sigma_S (I_S + I_S^*)$ which does appear as heat is small compared to the cosmic ray input and output $\sigma_+ I_{\pm}$. The cosmic rays take as much energy out from backward waves as they put into forward waves.

' Domain III'.

$$\mathcal{D} < 1, \quad \mathcal{A} \mathcal{D} > \frac{v_S^2}{v_A^2}$$

or

$$\frac{v_S^2}{v_A^2} \frac{\pi}{4} k v_A \left( \frac{I_{+} \cdot I_{-}}{I_{+} - I_{-}} \right)^2 \ll \sigma_S \ll \frac{\pi}{4} k v_A (I_{+} + I_{-}).$$

As in domain II, the pure cascade terms in (19) and (20) disappear, but the terms derived from the sound waves take a different approximation, giving

$$\frac{\sigma_+}{\pi v_A k} = -\frac{v_A^2}{v_S^2} \frac{4 \sigma_S I_{+} - I_{-}}{\pi v_A k} \left( -k^2 \frac{\partial}{\partial k} \frac{I_{+} + I_{-}}{k^2} \right)^{-2}$$

$$\frac{\sigma_-}{\pi v_A k} = +\frac{v_A^2}{v_S^2} \frac{4 \sigma_S I_{+} + I_{-}}{\pi v_A k} \left( -k^2 \frac{\partial}{\partial k} \frac{I_{+} + I_{-}}{k^2} \right)^{-2}$$

and thence

$$\frac{I_{+}}{I_{-}} = -\sigma_+$$

$$-k^2 \frac{\partial}{\partial k} \frac{I_{+} + I_{-}}{k^2} = \frac{2 v_A}{v_S^2} \frac{(\sigma_S (\sigma_+ + \sigma_-))^{1/2} I_{+} + I_{-}}{\sigma_+ - \sigma_-},$$

the sign of the derivative in (29) being fixed by remembering that $I_S$ in (17) must be positive. Again we see that $\sigma_+$ must be negative and $\sigma_-$ positive for this degree of asymmetry to be maintained, and $\sigma_+ I_{+} + \sigma_- I_{-} = 0$ ensures that there is little net input of cosmic ray energy to the spectrum of the waves.

The sound intensities (17) evaluate, using (29), to

$$I_S \simeq I_S^* \simeq \frac{2 v_S - \sigma_+ - \sigma_-}{v_A} \frac{I_{+} + I_{-}}{\sigma_+ - \sigma_- (\sigma_S (\sigma_+ + \sigma_-))^{1/2}}$$

' Domain IV'. Sound waves weakly damped.

$$\mathcal{D} < 1, \quad \mathcal{A} \mathcal{D} < \frac{v_S^2}{v_A^2}$$

or

$$\sigma_S \ll \frac{\pi}{4} k v_A (I_{+} + I_{-}), \quad \sigma_S \ll \frac{v_S^2}{v_A^2} \frac{\pi}{4} k v_A \left( \frac{I_{+} + I_{-}}{I_{+} - I_{-}} \right)^2.$$
Although the sound waves are only weakly damped ($\mathcal{D} < 1$), the asymmetry of the Alfvén waves is small enough to render the growth of sound waves unimportant, and the pure cascade

$$k \frac{\partial \mathcal{I}_\pm}{\partial k} = \frac{\sigma_-}{2\pi v_A k^2}, \quad k \frac{\partial \mathcal{I}_-}{\partial k} = \frac{\sigma_+}{2\pi v_A k^2}$$

is recovered. Since the wave asymmetry is bound to be small ($\mathcal{I}_+ \simeq \mathcal{I}_-$) when the cascade enters this domain, the only solution of (32) within the domain which has $\sigma_+ < \sigma_-$ is one in which the spectra are essentially flat.

$$\mathcal{I}_+ \simeq \mathcal{I}_- \simeq \text{constant.}$$

The sound wave intensities from (17) are

$$\mathcal{I}_S \simeq \mathcal{I}_S^* \simeq \frac{4\mathcal{I}_+ \mathcal{I}_-}{-k^3 \frac{\partial}{\partial k} \mathcal{I}_+ + \mathcal{I}_-}.$$  

We have argued that $\mathcal{I}_+ + \mathcal{I}_-$ will be nearly constant, so that the differential of $(\mathcal{I}_+ + \mathcal{I}_-)/k^2$ does not vanish. This confirms the position of the upper boundary $\mathcal{I} \sim v_s/v_A$ of this domain (see discussion following (16)). It also indicates that there will be approximate equipartition of energy between Alfvén and sound waves, for (34) simplifies to

$$\mathcal{I}_S \simeq \mathcal{I}_S^* \simeq \mathcal{I}_+ \simeq \mathcal{I}_-. $$

**CONCLUSION**

We have presented formulae for the steady-state behaviour of the spectra $\mathcal{I}_\pm(k)$ of Alfvén waves, when these are subject to given external inputs (−$\sigma_\pm$) and are removed by interactions involving sound waves. The pure cascade of Chin & Wentzel (1972) is recovered when sound waves are heavily damped, and the limits on its validity are made clear.

Each of the four domains in which the general formulae simplify have been defined in terms of the wave intensities. It is tempting to try to obtain the domains in terms of the given inputs instead, because then one could determine immediately in which domain the solution for the intensities would actually occur. Unfortunately, the equations for $\mathcal{I}_\pm$ (except in domain II) take the form of differential equations. Thus the specific solutions for $\mathcal{I}_\pm$ depend on suitable boundary conditions, and there is not necessarily a simple correspondence between the $\mathcal{I}_\pm$ and the $\sigma_\pm$.

Without more knowledge of the boundary conditions and more knowledge of the $k$-dependences of $\sigma_\pm$ than we wish to assume in this paper, it does not seem possible to obtain the domains in terms of simple conditions on $\sigma_\pm$.

In Paper III we shall investigate self-consistent solutions to the cosmic ray diffusion problem, using the formulae of Paper I to determine how the particles, and hence the interaction rates $\sigma_{ib}$, depend on the waves themselves. In particular, we shall discuss whether Fermi acceleration by the waves of the cascade is likely to play a role in the energetics of the escape of cosmic rays from the galaxy.

**REFERENCES**