Pulsar magnetospheres with arbitrary geometry in the force-free approximation

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Summary. Some general properties are derived of force-free configurations describing pulsar magnetospheres in the wave-zone, well beyond the light cylinder. No restrictions are imposed on geometry, or on the ‘parallel current function’ ψ. The work is confined to solutions free of magnetic monopoles, and possessing finite flux of energy to infinity. It is found that application of the obvious inertial constraint v < c removes a considerable part of the arbitrariness inherent in the force-free approximation. All solutions then have the following properties. (a) They possess magnetic neutral sheets. (b) The far field takes the form of an outwardly propagating vacuum wave (a radiation condition having removed the inwardly propagating solutions). (c) All charged particle speeds approach c at large distances, in such a way that their Lorentz factors increase roughly linearly with distance. (d) There always exist zones where counterstreaming of charged particles occurs (while their speeds are appreciably less than c). Some of these properties bring into question the validity of the force-free approximation.

1 Introduction

Progress in understanding the physics of pulsar magnetospheres has, to date, been limited almost entirely to the force-free approximation, in which charged particle inertia is neglected entirely. Recently though, the validity of the approach has been brought into question (Ardavan 1976; Mestel, Wright & Westfold 1976; Petvac, Petvac & Roberts 1975; Buckley 1976).

In this paper, some general properties are derived of solutions of the force-free equations at distances from the neutron star large compared with the light radius. The work parallels to some extent that in an earlier paper (Buckley 1976) which was restricted to axi-symmetric geometry. It must be emphasized here, as there, that no global solutions are shown to exist. However, if they do exist, then they will have the properties described.

We shall require of an acceptable solution, that it possess finite flux of energy to infinity, and that there be no magnetic monopoles associated with the system. The latter requirement imposes, as we shall see, important limitations on the field topology. It is interesting to note that the only global solution of the force-free equations found so far (Michel 1974) does involve such a monopole. We shall assume that the fields and all their gradients are continu-
ous everywhere. Discontinuities of various kinds can exist but, as discussed by Buckley (1976) they themselves would render dubious the force-free approximation. Finally, we shall assume that some magnetic field lines extend from the neutron star to infinity. Thus the system is not totally isolated magnetically from its surroundings. Also, within the force-free approximation, this assumption ensures that some charged particles can escape from the system.

In Section 2 it is demonstrated that any solution will possess at least one magnetic neutral surface. The proof consists (a) in showing that for any magnetic system, force-free or not, satisfying the above requirements, the toroidal component of \( \mathbf{B} \) vanishes somewhere, and then (b) that the limitation \( v < c \) within the force-free approximation beyond the light cylinder, implies that \( \mathbf{B} \) itself vanishes there.

Section 3 considers the asymptotic behaviour of the equations at distances large compared with the light radius \( r_L = c/\Omega \) (\( \Omega \) being the angular speed of the neutron star). Possible field configurations are severely restricted by the requirement \( v < c \). In spherical coordinates, the \( r, \theta, \) and \( \phi \) components of \( \mathbf{B} \) decrease respectively as \( r^{-2}, r^{-3}, \) and \( r^{-1} \) while \( \psi \) (a function defined in equation (14) below) must be of order unity. The field lines are, asymptotically, Archimedean spirals wound on the cones \( \theta = \text{constant} \). The far field represents an electromagnetic wave propagating outward everywhere (the corresponding inward wave solution having been eliminated with a radiation condition). The fundamental frequency of this wave is \( \Omega \), but any combination of harmonics may be present. Previous work on the non-symmetric force free equations has also highlighted these waves (Henriksen & Norton 1975; Mestel et al. 1976), but their work was restricted to situations where \( \psi \) took different constant values in various regions of space.

In fact, Mestel et al. (1976) were able to deduce some global properties of their restricted system. They proved that the requirements (a) outward going solution at large distances, and (b) solution regular at the light cylinder, are actually incompatible, and that in consequence, the force-free approximation must break down at the light cylinder. It is of course possible that the same will prove to be true for the general force-free equation, in which case the work described in this paper would have no relevance. In the absence however of any evidence for or against this conjecture, it seems worthwhile to analyse the general asymptotic behaviour at large distances with the tacit assumption that some solutions can be continued in a regular manner inward to the light cylinder.

Particle velocities are asymptotically radial and approach \( c \) in such a way that their Lorentz factors \( \gamma = (1 - v^2/c^2)^{-1/2} \) increase linearly with \( r \). (This of course implies that, at sufficiently large distances the neglect of particle inertia will not be justified.)

In Section 4, it is demonstrated that the 'outward' character of the solution (in a sense to be defined) is preserved to all orders in \( r_L/r \), and a formal method is described of constructing a solution starting from two arbitrary field coefficients. It is demonstrated also that all solutions possess zones in which counter-streaming of charged particles occurs along the magnetic field.

Several of the general properties derived render questionable the applicability of the force-free approximation, and this is discussed in the final section. Mathematical details are relegated to appendices.

2 Magnetic neutral sheets

Consider first any magnetic field configuration (force-free or not), in which all the field lines originate in a central body (the neutron star), and in which some at least of the field lines ('open' field lines) extend to indefinitely large distances from the star. Since magnetic
monopoles have been excluded from consideration, it is clear that there must exist at least one surface on which \( B_r \), the radial component of \( B \), vanishes. Further, \( B_r \) must be positive on some open lines, and negative on others. It is then obvious from a sketch that \( B_\phi \), the azimuthal or toroidal component, cannot be of the same sign everywhere. A formal proof of this statement is contained in Appendix 1. (It is also clear that it is not true if all field lines are allowed to close.) We conclude that there must exist a surface on which \( B_\phi \) vanishes.

Next, consider the particle dynamics. Larmor gyration is associated with random motions, and is omitted in the force-free approximation. The particle velocity \( v \) is then given in general by the formula

\[
v = u_E + v_\| b,
\]

where

\[
\begin{align*}
u_E &= E \times B / B^2, \\
u_\| &= \text{the usual 'E x B' drift velocity, } b = \text{a unit vector along } B, \text{ and } v_\| = \text{the component of } v \text{ along } B, \text{ is unrestricted in this approximation. In our case, the electric field } E \text{ is determined by}
\end{align*}
\]

\[
E = -v_R \times B,
\]

where \( v_R = \Omega \times \mathbf{r} \), is the velocity of rigid body corotation. (Faraday's law for a system stationary in the rotating frame implies that the gradient of an arbitrary scalar \( \Phi \) can be added to (2). (Mestel 1971; Endean 1972). Since (2) is valid as it stands in the highly conducting neutron star, and since it is required that \( E \cdot B = 0 \) be zero in the force-free approximation, it is established that \( \Phi \) is identically zero, and that (2) is valid everywhere. It is precisely the vanishing if \( \Phi \) that enables a single equation for \( B \) (equation (4) below) to be established.) It follows that

\[
u_E = v_R - (v_R \cdot b) b.
\]

It is interesting to note that the Lorentz factor \( \gamma_E = (1 - u_E^2 / c^2)^{-1/2} \), can be expressed in the form

\[
\gamma_E^2 = B^2 / (B \cdot \tilde{B}),
\]

where \( \tilde{B} \) is the 'derived' field given in equation (5) below. Thus it is a constraint on any forcefree model that \( B \) and \( \tilde{B} \) form an acute angle with one another everywhere.

One must impose \( v < c \) as an additional constraint within force-free theory, and it is obviously necessary that both \( u_E \) and \( v_\| \) separately not exceed \( c \). While \( v_\| \) is arbitrary, \( u_E \) is fixed once the fields are known. We have seen that there exists a surface on which \( B_\phi \) and therefore \( v_\| \) vanishes. Beyond the light cylinder therefore, (3) predicts that \( u_E > c \), which is unacceptable. (The view of field lines rotating rigidly with the star, and of particles being confined to move on them as beads on a wire, renders these observations clear (Endean 1972).) To avoid this situation it is necessary that where \( B_\phi \) vanishes, either there are no particles, or \( B \) as a whole vanishes. In the former case, we would have a local vacuum. However, then the physical basis of equation (1) vanishes, and the force free approximation would be, at any rate locally, devoid of meaning. The existence of such regions of local vacuum has in fact been postulated in the corotating part of the magnetosphere (Holloway 1973). In the latter case, \( u_E \) is undefined. We conclude that magnetic neutral surfaces must exist. Although, as discussed later, the force-free approximation becomes questionable near such a surface, its breakdown is not as serious as would be the case if \( u_E \) were allowed to exceed \( c \). In that case, particle inertia would play an essential role.
In the axi-symmetric case (Buckley 1976), it was shown that magnetic neutral surfaces exist without invoking the constraint \( v < c \), and it is possible that the same is true in general. No proof of this has been found though, and its pursuit would seem to be superfluous.

In the next section it will appear that the existence of such a surface imposes severe restrictions on the possible large-distance behaviour of the fields and particles.

3 General properties of the solution at large distances

We first list the relevant force-free equations. These were derived in their full generality by Endean (1974) and Mestel (1973). They have been discussed by these authors, and by Buckley (1976), and references therein. We consider, as is usual, the system to be stationary in a frame of reference rotating with the neutron star. The magnetic field is determined by the fundamental equation

\[
\nabla \times \vec{B} = \psi \vec{B},
\]

with

\[
\vec{B}_p = \left(1 - \frac{r^2 \sin^2 \theta}{r_L^2}\right) \vec{B}_p,
\]

\[
\vec{B}_t = B_t.
\]

We shall use spherical polar coordinates \((r, \theta, \phi)\). The 'parallel current' function, \( \psi \) is restricted only in that it be constant on magnetic field lines. \( p \) and \( t \) refer to the poloidal and toroidal components of any vector.

Expressions will be needed for the charge density \( \rho_e \), current density \( j \), and the radial component \( S_r \) of the Poynting vector. Formulae suitable for present purposes are

\[
\rho_e = \frac{1}{\Omega \mu_0 r \sin \theta} \left( \nabla \times \vec{B} - \psi \vec{B}\right)_\phi,
\]

\[
j = \rho_e \Omega + \frac{\psi}{\mu_0} \vec{B},
\]

or

\[
j_p = \psi \frac{\vec{B}_p}{\mu_0},
\]

\[
j_\phi = \frac{\left( \nabla \times \vec{B}\right)_\phi}{\mu_0},
\]

\[
S_r = -\frac{\Omega r \sin \theta}{\mu_0} B_r \vec{B}_\phi.
\]

The electric field \( E \) and associated drift velocity are given in equations (2) and (3).

We shall from now on, scale the distance \( r \) in terms of the light radius \( r_L \). The components of (1) are

\[
\left(r \sin \theta - \frac{1}{r \sin \theta}\right) \frac{\partial B_\theta}{\partial \phi} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta B_\phi) = \psi B_r,
\]
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\[
\left( r \sin \theta - \frac{1}{r \sin \theta} \right) \frac{\partial B_r}{\partial \phi} + \frac{1}{r} \frac{\partial}{\partial r} (r B_\phi) = -\psi B_\theta ,
\]

(7)

whilst \( \nabla \cdot \mathbf{B} = 0 \) reads

\[
\frac{\sin \theta}{r} \frac{\partial}{\partial r} (r^2 B_r) + \frac{\partial}{\partial \theta} (\sin \theta B_\theta) + \frac{\partial B_\phi}{\partial \phi} = 0.
\]

(8)

For a solution to be physically acceptable we impose two conditions.

(a) \( B_r \) will decrease at least as fast as \( r^{-2} \).

(b) The flux \( \mathcal{F}_E \) of energy from the system is to be finite (and in general non-zero). The latter which is obtained from \( S_r \) (there being no other mechanism for energy loss in the force-free approximation) is from (6e)

\[
\mathcal{F}_E = -\frac{\Omega r^3}{\mu_0} \int d\theta \, d\phi \sin^2 \theta (B_\phi B_r r^{-3}),
\]

integrated over a sphere surrounding the system. (That \( \mathcal{F}_E \) is independent of the radius of this sphere follows, as it must do, from the field equations.) The product \( B_\phi B_r \) is required therefore, to decrease as \( r^{-3} \).

The only physically reasonable way in which (a) and (b) can be satisfied simultaneously is for \( B_r \) and \( B_\phi \) to decrease respectively as \( r^{-2} \) and \( r^{-1} \). This behaviour of \( B_\phi \) is reminiscent of a radiation field. We might expect \( B_\theta \) also to vary ultimately as \( r^{-1} \), but this can be ruled out by considering the drift velocity \( \mathbf{u}_E \). From (3), the \( \theta \) component of this is

\[
u_{E\theta} = -\Omega r \sin \theta b_\phi b_\theta.
\]

(10)

Where \( \mathbf{b} = \mathbf{B}/B \). It is clear that unless \( B_\theta \) is of order \( r^{-2} \) at most, the product \( b_\phi b_\theta \) will be of order unity at large \( r \). Thus \( \nu_{E\theta} \) and hence \( \nu \) would exceed \( c \) beyond the light cylinder. It is readily seen that this could not be mitigated by the term \( \nu \cdot \mathbf{b} \) in (1) without \( \nu \cdot \mathbf{b} \) itself exceeding \( c \). We conclude that \( B_\theta \) must decrease at least as fast as \( r^{-2} \). It now follows readily from (7), that \( \psi \) can increase with \( r \) no more rapidly than linearly. (The fact that \( \psi \) is constant on field lines, and that some of the latter extend to infinity does \textit{not} prove that \( \psi \) must be of order 1 at large \( r \). The (artificial) example \( \psi = r \tan (\phi + r) \) is of the order \( r \) at large \( r \), but describes spiral field lines all of which extend to infinity.)

At large \( r \) therefore we express \( \mathbf{B} \) and \( \psi \) in the form

\[
B_r = \frac{D_0}{r^2} + O(r^{-3}), \quad B_\theta = \frac{E_0}{r^2} + O(r^{-3}), \quad B_\phi = \frac{F_0}{r} + O(r^{-2}), \quad \psi = \psi_{-1} r + O(1).
\]

(11)

It must be emphasized that \( D_0 \) etc. are \textit{not} supposed independent of \( r \), and the same will apply to all similar field coefficients introduced later. Rather, they and all their derivatives are assumed to be of order unity at large \( r \). This assumption clearly involves no loss of generality as regards the coefficients themselves, but the assumption over their derivatives may rule out some solutions. These are likely to be of a pathological nature and are not thought to be significant.

Terms of leading order (called zero order henceforth) in (7), (8) then involve only the
leading coefficients in (11), and are as follows (we use comma and suffix to indicate partial derivatives).

\[
\begin{align*}
\sin \theta \ E_{0,\phi} &= \psi_{\perp} D_0, \\
\sin \theta \ D_{0,\phi} + F_{0,r} &= -\psi_{\perp} E_0, \\
-\sin^2 \theta \ E_{0,r} &= \psi_{\perp} F_0, \\
\sin \theta \ D_{0,r} + F_{0,\phi} &= 0.
\end{align*}
\]

To reduce these equations we introduce a 'stream function', (or single component vector potential) \( P \) such that (12d),

\[
F_0 = P_{,r}, \quad D_0 \sin \theta = -P_{,\phi}.
\]

Though the field itself is of course a single valued function of position, \( P \) need not be. Neither need it be of order unity in \( r \) at large \( r \). Rather, it can in general involve \( \phi \), and \( r \) linearly. (9a) and (c) show that

\[
E_{0,\phi}/P_{,\phi} = E_{0,r}/P_{,r},
\]

and hence the functional dependence of \( E_0 \) is reduced to

\[
E_0 = E_0(P, \theta).
\]

The same two equations show also that

\[
\psi_{\perp} = -\sin^2 \theta \ E'_0,
\]

where a prime associated with a function of \( P, \theta \) indicates \( \partial/\partial P \). With (13), (14), (15) the remaining equation (9b) involves \( P \) alone,

\[
P_{,\phi\phi} - P_{,rr} = -\sin^2 \theta \ E_0 E'_0.
\]

So we are at this stage, in principle allowed to specify \( E_0 \) arbitrarily as a function of \( P \). Then (16) restricts the \( r, \phi \) (though not \( \theta \)) dependence of \( P \). Though very little can be said in general about the solution of the non-linear wave equation (16), we shall see that the requirement \( u_\mathbb{E} < c \) everywhere drastically restricts the possible dependence of \( P \) on \( r, \phi \), and renders the general solution of (16) superfluous.

When we calculate \( u_\mathbb{E} \) and \( \gamma_\mathbb{E} \) using (3) or (3a) we find that the latter is of order unity at large \( r \), and that

\[
\gamma_\mathbb{E}^2 = (P^2_{,r} - P^2_{,\phi} - E_0^2 \sin^2 \theta)/P^2_0 + O(r^{-1}).
\]

The factor in the numerator of \( \gamma_\mathbb{E}^2 \) must be everywhere non-negative. We have seen (Section 2) that a magnetic neutral sheet always exists, and here, at large \( r \), we shall have \( P_{,r}, P_{,\phi}, \) and \( E_0 \) vanishing simultaneously. Using (16), (17) and the latter fact, it is shown in Appendix 2 that it is impossible to render \( \gamma_\mathbb{E}^2 \) everywhere non-negative unless the leading term in (17) is identically zero.

It follows that \( \gamma_\mathbb{E}^2 \) is of order \( r \) (at least) and so all charged particles become relativistic at sufficiently large distances. The dependence of \( P \) on \( r, \phi \) is limited to the form

\[
P = P(\eta, \theta) \quad \text{where} \quad \eta = r + \phi \cos \alpha.
\]

\( \alpha \) could, as far as equation (16) goes, be a function of \( \theta \). However if it was so dependent, the derivatives of the zero order field coefficients with respect to \( \theta \) would be of a higher order
in $r$ than the coefficients themselves, which is contrary to the original expansion hypothesis.

The coefficients are given in terms of $P$ by

$$D_0 \sin \theta = -\cos \alpha P, \eta,$$
$$E_0 \sin \theta = -\sin \alpha P, \eta,$$
$$F_0 = P, \eta,$$
$$\psi_{-1} = \sin \alpha \sin \theta P, \eta \eta/P, \eta.$$  \hspace{1cm} (19)

It is clear that $\alpha$ is the angle in the meridian plane made by $B_p$ and $u_E$ with the radial direction. Indeed we see from (3) that

$$u_E/c = \cos \alpha + O(r^{-1}),$$
$$u_{E\theta}/c = \sin \alpha + O(r^{-1}),$$
$$u_{E\phi}/c = O(r^{-1}).$$  \hspace{1cm} (20)

Next we show that the mass continuity equations for the charged particles cannot be satisfied unless $\sin \alpha = 0$ (i.e. radial streaming). First, we note that (6a) demands that the charge density be of the form

$$\rho_c = \frac{e_0 \sin \theta E_0 r}{r} + O(r^{-2}).$$  \hspace{1cm} (21)

It follows that, if $E_0 r \neq 0$, at least one of the number densities $N_\pm$ of charged particles must decrease no faster than $r^{-1}$. Let us suppose that $N_+$ goes as $N_+ r^{-1}$ at large $r$. (If $N_+$ were of order unity at large $r$, the same argument would apply.) The outward positively charged particle flux is then, at sufficiently large $r$

$$\int N_+ u_{Er} r^2 d\Omega$$

over a sphere

$$= r \cos \alpha \int N_+ \sin \theta d\theta d\phi,$$  \hspace{1cm} (22)

$\alpha$ being constant. Since $N_+$ must be positive, this quantity will increase without limit as $r$ increases, which is clearly impossible. (With a density going as $r^{-1}$, it is obviously necessary that $u_{Er}$ would need to take on both signs over the sphere. With constant $\alpha$ this is not possible.)

So the densities must fall as $r^{-2}$ at least, and so $E_0 r$ must vanish. Thus $E_0$ is independent of $r$ and hence of $\eta$. (19) then implies either that $P, \eta$ is independent of $\eta$, in which case the far field is axisymmetric, or $\sin \alpha$ must be zero. The first possibility having been dealt with elsewhere (Buckley 1976), we conclude that $\sin \alpha$ must vanish. Of the possible values $0$ and $\pi$ of $\alpha$, the latter corresponds to inward particle flow everywhere, and this we reject. We conclude that $\alpha$ is zero, and we have radially outward flow in all directions at sufficiently large $r$. Equation (19) then shows that $E_0$ and $\psi_{-1}$ must vanish.

We conclude that, at large $r$, $B_\theta$ is $0(r^{-2})$, $\psi$ is $0(1)$, and

$$F_0 = -D_0 \sin \theta = H_0(\eta, \theta)$$

say, where

$$\eta = \phi + r.$$  \hspace{1cm} (23)
(H₀ is now used in place of P, η.) This shows that the far field lines are Archimedean spirals wound on the cones θ = constant. (23) describes an outwardly propagating electromagnetic wave with frequency Ω (and speed c). This can be seen if the explicit time dependence is recovered through the substitution φ → φ − Ωt; Then the 'phase' variable η in dimensional terms, becomes

$$\eta = \phi + \frac{\Omega}{c} (r - ct).$$  (24)

The wave magnetic field is of course B_φ while the corresponding electric field is from (2),

$$E_θ \sim -\frac{cD₀ \sin θ}{r} = \frac{cF₀}{r},$$  (25)

while

$$B_φ \sim \frac{F₀}{r}.$$  

Such low-frequency waves can be thought of as the extreme relativistic limit of Alfvén waves when the mass density is vanishingly small.

The function H₀ is restricted by the requirement of zero magnetic flux from the system. This is

$$\int dθ \ dφ \ H₀(φ, θ) = 0,$$  (26)

integrated over all directions. Otherwise H₀ is arbitrary within the expansion restrictions mentioned above. The flux of energy (9) becomes

$$Φ_E = \frac{Ωr₁^2}{μ₀} \int dθ \ dφ \ \sin θ \ H₀^2(φ, θ),$$  (27)

which is of course automatically positive.

To proceed further we must go to next order in r⁻¹. We write

$$B_r = \sum_{n=0} D_n r^{n+2}, \quad B_θ = \sum_{n=1} E_n r^{n+2}, \quad B_φ = \sum_{n=0} F_0 r^{n+1}, \quad Ψ = \sum_{n=0} \Psi_n r^n.$$  (28)

where D₀ and F₀ are restricted by (23). To first order, equations (7) and (8) yield

$$\sin θ \ E₁, φ + \frac{1}{\sin θ} (\sin θ \ F₀) , θ = \Psi_0 D₀,$$

$$\sin θ \ D₁, φ + F₁, r = 0,$$  (29)

$$(\sin^2 θ \ D₀), θ - \sin^2 θ \ E₁, r = \Psi_0 F₀,$$

$$\sin θ \ D₁, r + F₁, φ = 0.$$  

Introduce in place of r, φ the phase variables

$$ξ = φ - r,$$

$$η = φ + r.$$  (30)
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Then (29b, d) imply that
\[ F_1 = G_1(\xi, \theta) + H_1(\eta, \theta), \]
\[ \sin \theta D_1 = G_1(\xi, \theta) - H_1(\eta, \theta). \]  \hspace{1cm} (31)

\( G_1 \) and \( H_1 \) are, so far, arbitrary functions of their arguments. Also, with (23), (26a,c) imply that \( E_1 \) and \( \psi_0 \) are independent of \( \xi \), and that
\[ \sin^2 \theta E_{1,\eta} = -\psi_0 H_0 - (\sin \theta H_0)_{,\theta}. \]  \hspace{1cm} (32)

We next show that \( G_1 \), the 'inward' component of the first-order field, must be zero. This argument again entails examination of the drift speed \( u_E \). This is known to be unity at large \( r \), so we must examine its behaviour to next order \( r^{-1} \). (3) or (3a) yields for the corresponding Lorentz factor
\[ \gamma^{-2}_E = \frac{4G_1}{H_0} \frac{1}{r} + O(r^{-2}). \]  \hspace{1cm} (33)

Hence \( G_1 \) and \( H_0 \) must never be of opposite sign. It is shown in Appendix 3 that this is impossible unless \( G_1 \) vanishes.
\[ G_1 = 0. \]  \hspace{1cm} (34)

So (31) the Archimedean spiral character of the far field is carried to higher order, and (33) shows that \( \gamma_E \) is \( O(r) \) for large \( r \). Thus, all charged particles will gain energy from the field proportionately to their distances from the star. The inevitable breakdown of the force-free approximation at sufficiently large \( r \) is examined elsewhere (Buckley 1977).

4 Higher order fields and counter-streaming surfaces

We shall next show that the independence of \( \xi \) (equation (30)) exhibited by the lower order field coefficients, occurs at all orders, and we then describe a method which enables in principle the field to be constructed as a series in powers of \( 1/r \), in terms of two arbitrary functions. By analogy with the two lowest orders, we shall rewrite the expansion (28) in the form
\[ \sin \theta B_r = \sum_{0} (G_n - H_n) r^{-n-2}, \]
\[ B_0 = \sum_{1} E_n r^{-n-2}, \]  \hspace{1cm} (35)
\[ B_\phi = \sum_{0} (G_n + H_n) r^{-n-1}, \]
\[ \psi = \sum_{0} \psi_n r^{-n}, \]
where we recall that \( G_0 \) and \( G_1 \) are zero, and that \( \psi_0, H_0, E_1, \) and \( H_1 \) are independent of \( \xi \).
Using $\xi, \eta$ as variables to replace $\phi, r$, equations (7), (8) at second order yield
\begin{equation}
\sin^2 \theta E_{2, \eta} = -\psi_1 H_0 - \psi_0 H_1 - (\sin \theta H_1)_{,\theta},
\end{equation}
\begin{equation}
E_{2, \xi} = 0,
\end{equation}
\begin{equation}
4G_{2, \eta} = -\psi_0 E_1 - (\sin \theta E_1)_{,\theta} - \frac{H_0 \eta}{\sin^2 \theta},
\end{equation}
\begin{equation}
4H_{2, \xi} = \psi_0 E_1 - (\sin \theta E_1)_{,\theta} + \frac{H_0 \eta}{\sin^2 \theta} - 2H_1.
\end{equation}

(36b, a) show that $E_2$ and $\psi_1$ are independent of $\xi$ (all terms other than $\psi_1$ in (36a) being so independent). The right-hand side of (36d) being independent of $\xi$, integration would yield, if $H_{2, \xi}$ were not zero, an unacceptable linear dependence of $H_2$ on $\xi$. With $H_{2, \xi} = 0$, (36d) then determines $H_1$ in terms of ‘known’ quantities. Finally (33b) represents $G_2$ as the sum of a ‘known’ function of $\eta$ and $\theta$, and an arbitrary function of integration $G_2(\xi, \theta)$ say. Similarly, at the third order we find that the latter function has to vanish if $G_3$ is not to involve a term linear in $\xi$. Then in succession we find that $G_3, E_3$ and $H_3$ are independent of $\xi$, and further, that $E_2, \psi_1, H_2$ are determined explicitly in terms of quantities of lower order. The procedure carries on at all subsequent orders, and we conclude that all field coefficients are independent of $\xi$, the ‘inward-going’ phase variable. It is not true however, that the $G_n$ are zero if $n > 2$, and the field deviates from its Archimedian spiral character at the higher orders.

The complete hierarchy of equations at general order is given in Appendix 4, together with a procedure for its formal integration in terms of $H_0, \psi_0$ alone, which can be taken (with one proviso discussed in the Appendix) to be arbitrary. Note that only one integration with respect to $n$ (for $G_n$) is necessary at each order. Note also that the equation for $\psi_n$ shows that the latter will, in general, diverge when $H_0 = 0$ (as $H_0$ must do on some surface by (26)). This implies that $\psi$ itself will diverge on some surface, and that therefore $B$ must (and will automatically) vanish there (equation 4). Hence this locates (in principle) the neutral surface.

Examples of fields are readily constructed but the algebra involved rapidly gets beyond bounds. The only general feature worthy of note would seem to be the rapid build-up of toroidal harmonics. The question of the global validity of such solutions remains of course, open.

We mention the formula for the Lorentz factor $\gamma_E$ in terms of the field coefficients. From (3a) this reduces to
\begin{equation}
\gamma_E^2 = \left[ 4 \frac{G_2}{H_0} \sum \frac{E_1^2}{H_0^2} \frac{1}{\sin^2 \theta} \right] - 2 + 0(r^{-3}).
\end{equation}

The coefficient here (which can actually be shown to vanish when $H_0 = 0$) must itself never become negative, but this does not appear to restrict possible solutions very much.

Finally, we show that there exist additional surfaces on which there is azimuthal counter-streaming of charged particles. This was shown to be the case for axisymmetric geometry (Buckley 1976), and remains true in general. It is first shown that surfaces exist where $\psi$ vanishes. The proof is due to Endean (1975, private communication).

Let $\mathcal{C}$ be a closed curve lying entirely in a magnetic neutral surface, and everywhere sufficiently far from the star that the field is dominated by its asymptotic limit. Further, let $S$ be a surface bounded by $\mathcal{C}$, similarly far from the star, and everywhere oriented in such a way that its outward normal makes an acute angle with the radius vector from the stellar.

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centre. On $S$, $H_0$ everywhere has the same sign, so that the radial component $B_r$ of the field will also be of the same sign everywhere. Since $\mathcal{G}$ lies in the neutral surface, it follows that

$$0 = \oint_{\mathcal{G}} B \cdot dl = \oint_{\mathcal{G}} \tilde{B} \cdot dl,$$

$$= \int_S \nabla \times \tilde{B} \cdot dS \quad \text{by Stokes' theorem},$$

$$= \int_S \psi B \cdot dS \quad \text{by the fundamental equation}. \quad (38)$$

But $B \cdot dS$ has the same sign at every point on $S$. It follows that $\psi$ must change sign along some curve in $S$. By varying the position of $S$ we can trace out a surface on which $\psi$ vanishes.

The proof that on such surfaces there must exist azimuthal counter-streaming is similar to that applicable to the axisymmetric case. From (6) we have, when $\psi = 0$,

$$\rho_e = \frac{1}{\Omega \mu_0 r \sin \theta} (\nabla \times B)_\phi \quad \text{(39)}$$

$$j_\phi = \rho_e v_R.$$

From our expansion, we have, reinserting the dimension $r_L$ of $r$

$$\begin{align*}
(\nabla \times B)_\phi &= \frac{r_L}{r^3} \left( E_{1,\eta} + \left( \frac{H_0}{\sin \theta} \right)_{,\theta} \right) + O(r^{-4}), \\
&= -\frac{2 \cos \theta}{\sin^2 \theta} \frac{r_L^2}{r^3} + O(r^{-4}). \quad \text{(40)}
\end{align*}$$

Here, (29) has been used for $\partial E_1/\partial \eta$. The quantity $\psi_0$, while not zero when $\psi = 0$, is clearly of order $1/r$ there, and so does not contribute to the leading term. Thus

$$\rho_e = -\frac{2 \Omega e_0 \cos \theta}{\sin^3 \theta} \frac{H_0}{r^4} \left( \frac{r_L}{r} \right)^4 + O(r^{-5}). \quad \text{(41)}$$

Since the surface where $\psi = 0$ is distinct from the neutral sheet (where $\psi = \infty$), $H_0$ cannot vanish on it, and so $\rho_e \neq 0$. Hence the net charge density does not vanish.

In terms of densities and velocities, (39) is therefore

$$N_e v_{\phi^+} - N_e v_{\phi^-} = (N_e - N_\pi) v_R \neq 0. \quad \text{(42)}$$

Since $v_R > c$ beyond the light cylinder, (42) can be satisfied with $|v_{\phi^\pm}| < c$ only if neither $N_e$ nor $N_\pi$ vanishes, and $v_{\phi^+}, v_{\phi^-}$ are significantly different. (There is also poloidal counter-streaming on such surfaces, but as $B$ is predominantly toroidal at large distances, the expected two-stream instability will be more important along $B$ (that is azimuthally) than across it.)

The example used for the axisymmetric case is adequate here. Thus $N_e$ and $N_\pi$ may be of the form

$$N_e = N_0 \left( \frac{r_L}{r} \right)^3 + N_\pi \left( \frac{r_L}{r} \right)^4 + O(r^{-5}), \quad \text{(43)}$$
with $N_1 \neq N_1$. (42) then implies that $N_0 \neq 0$, and that

$$v_{\phi+} - v_{\phi-} \propto c \sin \theta \frac{N_1 - N_1}{N_0} = \frac{2c \Omega e_0}{e} \cos \theta H_0 \frac{\cos \theta H_0}{\sin^2 \theta N_0}.$$  \hspace{2cm} (44)

The fact that, at large $r$, $v_{\phi+}$ and $v_{\phi-}$ differ by a quantity of order unity in $r$ indicates, since $v_r$ itself approaches $c$ at large $r$, that we will, ultimately attain speeds in excess of $c$. Thus there is a powerful reason for questioning the validity of the force-free approximation in such regions apart from considerations of the two-stream instability.

Conclusions

Some general properties have been considered of the far fields of pulsars within the force-free approximation. Although no proof is available that any globally applicable solution exists, it will, if it does, have these properties. No geometrical restrictions have been imposed. Physical requirements are (a) absence of magnetic monopoles, (b) finite flux of energy to infinity, and (c) no particle speed must exceed that of light. Constraint (c) is not automatically present in a force-free approach. The properties are summarized as follows:

(a) Magnetic neutral surfaces always exist.
(b) The far field always takes the form of an outwardly propagating electromagnetic wave with the neutron star's rotation speed as fundamental frequency.
(c) The $\phi$, $r$, and $\theta$ components of $B$ decrease with radial distance as $r^{-1}$, $r^{-2}$ and $r^{-3}$ respectively.
(d) Charged particles eventually stream radially away from the star in such a way that their Lorentz factors, and hence energies, increase linearly with distance.
(e) The outwardly propagating nature of the field is maintained to all orders in $r^{-1}$.
(f) There always exist surfaces on which occurs azimuthal counter-streaming of charged particles.

All the conclusions reached for the axisymmetric case (Buckley 1976), continue to apply in general geometries. Neutral, and counterstreaming surfaces will almost certainly involve plasma instabilities which are not contained within the force-free formalism. The approximation is therefore, brought into question, and should probably be supplemented by the effects of particle pressure, and anomalous dissipation.

In addition, since the relativistic mass of charged particles is predicted to increase without limit as $r$ increases, a distance will eventually be attained at which the inertial terms in the equations of motion are no longer negligible. This effect is analysed elsewhere (Buckley 1977).

Finally it must again be emphasized that no globally reasonable solutions are considered in this work. If, as has been proved for a very limited class of radiating free-force models (Mestel et al. 1976), it were to occur that particle inertia is essential close to the light cylinder, then the results of the present paper would cease to have relevance.

References

Appendix 1

Let the magnetic field be represented in the form (valid for any divergence-free field)

\[ \mathbf{B} = \nabla \alpha \times \nabla \beta \]  

(A1)

where the field lines are determined by the intersections of the surfaces \( \alpha = \) constant, and \( \beta = \) constant. \( \alpha \) or \( \beta \) may be multivalued functions of position in that they may involve the azimuthal coordinate \( \phi \) linearly rather than periodically. (For axisymmetric systems with non zero toroidal component it is easy to see that one of \( \alpha, \beta \) must be of this form.) It is clearly possible, by choosing suitable surfaces for \( \alpha, \beta \) to make one of them, \( \beta \) say, single-valued in space.

In spherical polars the toroidal component \( B_\phi \) is given by

\[ B_\phi = \frac{1}{r} (\alpha_r \beta_\theta - \alpha_\theta \beta_r). \]  

(A2)

Let us suppose that this vanishes nowhere (and so is without loss of generality, positive everywhere), and examine the consequences. Consider a meridian plane \( \phi = \) constant. Since \( B_\phi \) is everywhere positive, the Jacobian of the transformation \( (r, \theta) \rightarrow (\alpha, \beta) \) is everywhere unique. Let us replace \( \theta \) as independent variable by \( \alpha \), so that (A2) becomes

\[ B_\phi = -\frac{1}{r} \left( \frac{\partial \alpha}{\partial \theta} r, \phi \frac{\partial \beta}{\partial r} / \alpha, \phi \right), \]  

(A3)

it being understood that \( \theta \) is to be replaced by \( \alpha \) in the first term after the differentiation is performed. Again \( B_\phi \) being positive everywhere, the two factors in (A3) can themselves never vanish, so one is positive and the other negative everywhere. Suppose, again without loss of generality that \( (\partial \beta / \partial r)_{\alpha, \phi} \) is everywhere positive. \( \beta \) therefore increases with \( r \) along the intersection of any curve \( \alpha = \) constant with every meridian plane. Therefore, since \( \beta \) is a single valued function of position no line determined by the intersection of the surfaces \( \alpha = \) constant, \( \beta = \) constant (that is, no field line) can attain indefinitely large distances from the origin.

Conversely, if we suppose, as we have done, that some field lines do extend to indefinitely large \( r \), it follows that \( B_\phi \) must vanish somewhere.

Appendix 2

The function \( P \) is required to satisfy (16), and the constraint that follows from (17),

\[ P_{rr} - P_{,\phi \phi} = \sin^2 \theta E_0 E'_0, \]

\[ \Lambda = \frac{1}{4} (P^2_{rr} - P^2_{,\phi} - E_0^2 \sin^2 \theta) \gg 0 \text{ everywhere}. \]  

(A4)
The $\frac{3}{8}$ is inserted for later convenience in $\Lambda$, and $E'_0$ denotes $\partial E_0(P, \theta)/\partial P$. Introduce the phase variables $\xi, \eta$ (30) instead of $\phi, r$ and let $E'_0 \sin^2 \theta / 4 = \mathcal{E}(P, \theta)$. Then (A4) becomes

$$P, \eta \xi = -\frac{3}{8} \mathcal{E'},$$

$$\Lambda = -P, \eta P, \xi = \mathcal{E} > 0 \text{ everywhere.}$$

(A5)

Now calculate $\Lambda, \eta$ and $\Lambda, \xi$ from (A5a) to eliminate $\mathcal{E'}$

$$\Lambda, \eta = P, \xi G, \eta,$n

$$\Lambda, \xi = -P, \xi G, \xi,$$

(A6)

where

$$G = -P, \eta / P, \xi = (\partial \xi / \partial \eta)_{P, \theta}.$$

We know (Section 2), that there is at least one magnetic neutral sheet. Hence there must exist associated surfaces asymptotic to these sheets at large $r$, on which $P, r, P, \phi$ and $E_0$ vanish. Hence $P, \eta, P, \xi, \mathcal{E}$ and $\Lambda$ also vanish on such surfaces. Let $\mathcal{C}$ be a curve in the $\xi, \eta$ plane (Fig. 1) where such a surface intersects any cone $\theta = \text{constant}$. Since $\mathcal{E}$ vanishes on $\mathcal{C}$, it follows that $P$ is constant along $\mathcal{C}$, and hence (A6) that $G$ is the gradient of $\mathcal{C}$.

Choose an arc $QR$ on $\mathcal{C}$ such that $G$ is everywhere positive on it (the proof for a negative gradient is similar). $\Lambda$ is zero on $\mathcal{C}$ and must nowhere be negative. If therefore $QR$ is short (but still of finite length), $\Lambda$ must not decrease away from $\mathcal{C}$ anywhere within the rectangle $ABCD$ (Fig. 1) erected on the arc. Hence we have

on $A, \Lambda, \xi > 0$, on $B, \Lambda, \eta < 0$,

on $C, \Lambda, \eta > 0$, on $D, \Lambda, \xi < 0$.

(A7)

Hence

$$G_R - G_Q = \int_A G, \xi \, d\xi + \int_B G, \eta \, d\eta$$

$$= -\int_A \frac{\Lambda, \xi}{P, \xi} \, d\xi + \int_B \frac{\Lambda, \eta}{P, \xi} \, d\eta \text{ (by A6)}$$

$$< 0. \text{ (by A7)}$$

Figure 1. The curve $\mathcal{C}$ and its associated construction in the $\xi, \eta$ plane, that is used in Appendix 2.
Alternatively,

\[ G_R - G_Q = \int_C G,_{\eta} d\eta + \int_D G,_{\xi} d\xi \]

\[ = \int_C \Lambda,_{\eta} \frac{d\eta}{P^2,_{\xi}} - \int_D \Lambda,_{\xi} \frac{d\xi}{P^2,_{\xi}} \quad \text{(by A6)} \]

\[ > 0. \quad \text{(by A7)} \]

Hence

\[ G_R = G_Q. \quad \text{(A9)} \]

\( Q, R \) being arbitrary points on \( \mathcal{C} \), it follows that \( G \) must be constant on \( \mathcal{C} \), which must therefore be a straight line. It also follows from (A8) that \( \Lambda \) must be constant on the rectangle \( ABCD \), and therefore zero since that is its value on \( \mathcal{C} \). The rectangle itself being arbitrary, we conclude that \( \Lambda \) is identically zero. From (A6) then, \( P,_{\eta}/P,_{\xi} \) is independent of \( \xi \) and \( \eta \), and therefore \( P \) depends on \( \xi, \eta \) only through the combination \( \eta + \lambda \xi \) or \( r + \mu \phi \), where \( \lambda, \mu \) are functions of \( \theta \) alone. It then follows readily from the facts that \( \Lambda \) is zero, and that \( \mathcal{E} = E_0^2 \sin^2 \theta / 4 \) is non-negative, that the quantity \( \mu \) cannot exceed unity in magnitude. We write it therefore in the form \( \mu = \cos \alpha \).

This completes the required proof.

Appendix 3

We show that the quantity \( G_1(\xi, \theta) \) in (31) must vanish. We shall need two of the second-order field equations. These are similar to (36) but with the assumption \( G_1 = 0 \), of course relaxed. They are

\[ \sin^2 \theta E_{2,\xi} = \psi_0 G_1 - (\sin \theta G_1),_{\theta}, \]

\[ 4G_{2,\eta} = -\psi_0 E_1 - (\sin \theta E_1),_{\theta} - \frac{H_0,\eta}{\sin^2 \theta} + 2G_1. \quad \text{(A10)} \]

Of the terms on the right-hand side of (A10b), all but the last are known to be independent of \( \xi \). On integration (A10b) will yield an expression of the form

\[ G_2 = \text{(function of } \eta, \theta \text{)} + \frac{1}{2} G_1(\xi, \theta) \eta + \text{(function of } \xi, \theta \text{ of integration).} \quad \text{(A11)} \]

The unacceptable linear dependence of \( G_2 \) on \( \eta \) in the second term can be removed by the first only if \( G_1 \) is independent of \( \xi \). Hence the right-hand side of (A10a) is independent of \( \xi \). The unacceptable linear dependence of \( E_2 \) on \( \xi \) that would then result shows that this right-hand side must vanish,

\[ \psi_0 G_1(\theta) - (\sin \theta G_1(\theta)),_{\theta} = 0. \quad \text{(A12)} \]

Hence, if \( G_1 \) does not vanish, \( \psi_0 \) has to be independent of \( \eta \). Equation (32) (in the body of the paper) can then be cast in the form

\[ \sin^2 \theta G_{1,\eta} = - (\sin^2 \theta G_1 H_0),_{\theta}. \quad \text{(A13)} \]
Suppose we now integrate this equation over a cycle of \( \phi (\phi, \text{ or } \eta \text{ going from } 0 \text{ to } 2\pi) \). The left-hand side vanishes because \( E_1 \) must be a single-valued function of position. It follows that

\[
\left( \sin^2 \theta \, G_1 \tilde{H}_0 \right)_\theta = 0, \quad (A14)
\]

where

\[
\tilde{H}_0(\theta) = \frac{1}{2\pi} \int_0^{2\pi} H_0(\eta, \theta) \, d\eta. \quad (A15)
\]

(A14) shows that one or other of \( G_1 \) and \( \tilde{H}_0 \) must vanish because otherwise one at least of them would diverge when \( \sin \theta = 0 \) (the rotation axis). Suppose \( \tilde{H}_0 \) vanishes. It then follows that at each fixed value of \( \theta \), \( H_0(\eta, \theta) \) must take on both signs as \( \eta \) varies over 0 to 2\( \pi \). But \( G_1 \), being dependent only on \( \theta \), is constant as \( \eta \) varies in this way. Hence \( G_1/H_0 \) would take both signs as \( \eta \) so varied. But (33) requires, if \( u_E \) is not to exceed \( c \), that \( G_1/H_0 \) must never be negative. So \( \tilde{H}_0 \) cannot vanish, and thus \( G_1 \) must vanish. In this case, (A12) is an identity, and \( \psi_0 \) need not be independent of \( \eta \).

This completes the proof.

Appendix 4

The hierarchy of equations that results from (7), (8) with the expansions (35) reduces, when all dependence on \( \xi \) is dropped to

\[
4E_{n+1, \eta} = -\sum_{i=0}^{n-1} \psi_q E_{n-q} - (\sin \theta E_n)_\theta + \frac{(G_{n-1} - H_{n-1})_\eta}{\sin^2 \theta} + 2nG_n, \quad nH_n = -2G_{n+1, \eta} - (\sin \theta E_n)_\theta + nG_n, \quad (A16)
\]

\[
\frac{1}{2} n \sin^2 \theta E_{n+1} = \sum_{i=0}^{n-1} \psi_q G_{n+1-q} - (\sin \theta G_{n+1})_\theta + \frac{1}{2} \eta E_{n-1} + \frac{1}{2} \left( \frac{G_{n-1} - H_{n-1}}{\sin \theta} \right)_\theta, \quad nH_0 \psi_0 = -2 \sum_{i=0}^{n-1} \psi_{q, \eta} G_{n+1-q} - \sum_{i=0}^{n-1} \psi_q \sin \theta E_{n-q} + \sum_{i=0}^{n-1} q \psi_q (G_{n-q} - H_{n-q}).
\]

Equation (29) is used to start the solution. The following formal procedure for solving (A16) is then established:

(a) Choose \( H_0, \psi_0 \). The former must satisfy the zero flux requirement (26). \( \psi_0 \) must be restricted slightly as described under (b).

(b) (29) determines \( E_1 \) up to a function \( E_1(\theta) \) say of integration. There is however, an integrability constraint on \( \psi_0, H_0 \) that must hold of \( E_1 \) is not to involve terms linear in \( \eta \). The \( \eta \) average of the right-hand side of (29) must vanish for all values of \( \theta \).

(c) Similarly the right-hand side of (A16a) with \( n = 1 \) must have zero \( \eta \) average. This condition determines \( E_1(\theta) \), introduced at (b) up to a constant \( e_1 \) say. Then \( G_2 \) is determined up to a function of integration \( G_2(\theta) \).

(d) (A16b, c, d) then determines \( H_1, E_2, \psi_1 \) (with no integration necessary), in terms of \( G_2(\theta) \).

(e) (A16a) with \( n = 2 \) being required to have zero \( \eta \) average, a certain second-order linear differential equation results for \( G_2 \). The coefficients are in general, extremely complex. It seems likely that the constant \( e_1 \), introduced at (c) will be an eigenvalue for this equation, so unless it has one of a discrete set of values, convergence of \( G_2(\theta) \) at \( \theta = 0, \pi \) will not be secured. \( G_2(\theta) \) is then determinate up to a pair of constants of integration (possibly only one if one of the characteristic functions of the \( G_2 \) equation is ruled out on convergence grounds).

The entire cycle then repeats itself in an obvious way.