Non-LTE transfer — III. Asymptotic expansion for small $\epsilon$

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Summary. Radiative transfer with complete frequency redistribution in the full- or half-space with interior sources is considered in the limit of small $\epsilon$ (probability of collisional destruction). It is shown that a suitably scaled source function satisfies for $\epsilon \to 0$ a singular equation. In the full-space case with a Doppler profile this reads

$$S(\tau) - B(\tau) = -(-\Delta)^{1/2} S(\tau)$$

where $S$ and $B$ are the (scaled) source function and thermal source function, $(-\Delta)^{1/2}$ is the square-root of the negative of the Laplacian; this equation replaces the classical diffusion equation for coherent scattering. In the half-space case, the asymptotic expansion gives only the interior source function (vanishing at the surface) and there is an additional boundary layer. The numerical aspects are considered in a companion paper.

1 Introduction

Coherent radiative transfer in arbitrary geometry simplifies considerably in the limit of small $\epsilon$: there is first an interior solution in which, after rescaling the optical depth by a factor $\epsilon^{-1/2}$, the intensity becomes isotropic and satisfies a diffusion equation, and second a boundary layer of thickness $O(1)$ which can be locally approximated by a half-space problem (matched to the interior solution); all the emergent radiation emanates from the boundary layer (Papanicolaou 1975; Bensoussan, Lions & Papanicolaou 1976).

These properties do not survive when complete frequency redistribution (CRD) is allowed; basically because the photons may escape in the far line-wings. It is the aim of this paper to show that an asymptotic expansion can nevertheless be performed in the CRD case. Our technique bears some resemblance to the one used by Ivanov (Ivanov 1973, pp. 390–392) in the conservative case ($\epsilon = 0$) for a finite slab of thickness $\tau_0 (\tau_0 \to \infty)$. In both cases one obtains a singular integral equation for the leading term of the asymptotic expansion. Numerical solutions of this integral equation will be found in a companion paper (Frisch & Froeschlé 1977) henceforth referred to as Paper IV. Only the one-dimensional case is treated here; extension of the method to multi-dimensional situations is in principle easy. We also hope to extend the treatment to include partial redistribution.
We state now the CRD problem more precisely (see also Avrett & Hummer 1965); the coherent scattering (CS) problem will also be given for comparison.

The equation of transfer reads

\[ \mu \frac{\partial I(\tau, \mu, x)}{\partial \tau} = (1 - \epsilon) \phi(x) \left\{ I - \int_{-\infty}^{+\infty} dx' \int_{-1}^{+1} d\mu' \frac{1}{2} \phi(x') I(\tau, \mu', x') \right\} + \epsilon \phi(x) \left\{ I - B(\tau) \right\}, \]  

(1.1)

\( I, B, \tau \) and \( \mu \) are standard notations; \( x \) is a non-dimensional frequency, \( \phi(x) \) is a normalized frequency profile (henceforth we shall assume a Doppler profile \( \phi(x) = \pi^{-1/2} \exp(-x^2) \)), \((1 - \epsilon)\) is the scattering probability and \( \epsilon \) the destruction probability (typically \( 10^{-4} \) in astrophysical applications). In the CS-case the frequency variable and the \( \phi \)-factor are dropped. Only the full-space and half-space problems are considered; there is no incoming radiation.

Introducing the source function

\[ S(\tau) \equiv eB(\tau) + (1 - \epsilon) \int_{-\infty}^{+\infty} dx' \int_{-1}^{+1} d\mu' \frac{1}{2} \phi' I'(\tau), \]  

(1.2)

with

\[ \phi' = \phi(x'); \quad I'(\tau) = I(\tau, \mu', x'). \]  

(1.3)

We obtain from (1.1) the following integral equation

\[ S(\tau) = eB(\tau) + (1 - \epsilon) \int_{\mathcal{A}} K_{1}(\tau - \tau') S(\tau') \, d\tau', \]  

(1.4)

where \( \mathcal{A} \) is the full space and \( \mathcal{A}^+ \) in the half-space case. The kernel \( K_{1}(\cdot) \) is given by

\[ K_{1}(\tau) = \int_{-\infty}^{+\infty} dx \int_{0}^{1} d\mu \frac{\phi(x)}{2} \exp \left( -|\tau| \frac{\phi(x)}{\mu} \right) \]  

CRD

(1.5)

or

\[ K_{1}(\tau) = \int_{0}^{1} \frac{d\mu}{2\mu} \exp \left( -|\tau| \frac{1}{\mu} \right). \]  

CS

(1.6)

The asymptotic expansion of \( K_{1}(\cdot) \) for large \( \tau \) is (Avrett & Hummer 1965)

\[ K_{1}(\tau) \sim 1/(4\pi^2 \ln (\tau/\sqrt{\pi})^{1/2}), \quad \tau \to +\infty \]  

CRD

(1.7)

\[ K_{1}(\tau) \sim \exp(-\tau/\tau), \quad \tau \to +\infty. \]  

CS

(1.8)

Notice that \( K_{1}(\cdot) \) is always even positive and normalized

\[ \int_{-\infty}^{+\infty} K_{1}(\tau) \, d\tau = 1. \]  

(1.9)

2 Asymptotic expansion in the full space

The starting point is the integral equation (1.4) for the source function. Using (1.9) we rewrite (1.4) as

\[ e[S(\tau) - B(\tau)] = (1 - \epsilon) \int_{\mathcal{A}} d\tau' K_{1}(\tau - \tau')[S(\tau') - S(\tau)]. \]  

(2.1)

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Let us introduce the scaled quantities
\[ \tilde{\tau} = \tau / h(\epsilon); \quad \tilde{S}(\tilde{\tau}) = S(\tau); \quad \tilde{B}(\tilde{\tau}) = B(\tau), \]
(2.2)

where the scaling factor is still to be determined. From (2.1) and (2.2) we obtain an equation which can be written in two different ways
\[ \tilde{S}(\tilde{\tau}) - \tilde{B}(\tilde{\tau}) = (1 - \epsilon) \int_{\tilde{\tau}}^{h(\epsilon)} K_1[(\tilde{\tau} - \tilde{\tau}') h(\epsilon)] \tilde{S}(\tilde{\tau}') - \tilde{S}(\tilde{\tau}) \]  
\[ d\tilde{\tau}' \]
(2.3)
\[ = (1 - \epsilon) \int_{\tilde{\tau}}^{h(\epsilon)} K_1(\sigma) \left[ \frac{\sigma}{h(\epsilon)} \right] \tilde{S}(\tilde{\tau} + \frac{\sigma}{h(\epsilon)}) - \tilde{S}(\tilde{\tau}) \]  
\[ d\sigma. \]
(2.3)'

In the CS-case we use (2.3)', Taylor expand \( \tilde{S}(\tilde{\tau} + \sigma/h) - \tilde{S}(\tilde{\tau}) \) to second order to obtain
\[ \tilde{S} - \tilde{B} = \frac{1 - \epsilon}{2} \frac{2}{e^2(h(\epsilon))^2} \int_{-\infty}^{+\infty} \sigma^2 K_1(\sigma) d\sigma + O(h^{-4}). \]
(2.4)

We now take \( h(\epsilon) = e^{1/2} \) and assume that \( \tilde{B}(\tilde{\tau}) \) has a finite limit for fixed \( \tilde{\tau} \) as \( \epsilon \to 0 \). (This means that \( B(\cdot) \) is actually a function of \( \epsilon \) and \( \tau \) with typical scale of variation proportional to \( h(\epsilon) \)); letting \( \epsilon \to 0 \) we obtain the well-known diffusion equation
\[ \tilde{S} - \tilde{B} = C\Delta \tilde{S}; \quad C = \frac{1}{2} \int_{-\infty}^{+\infty} \sigma^2 K_1(\sigma) d\sigma, \]
(2.5)

where the Laplacian \( \Delta \) is defined as \( \partial^2 / \partial \tilde{\tau}^2 \).

We turn now to the CRD-case where the above procedure does not work because the constant \( C \) in (2.5) would be infinite. Let us instead start from equation (2.3). Using (1.7) we find that the explicitly \( \epsilon \)-dependent part of the integral in (2.3) has a finite non-zero limit if and only if \( h(\epsilon) \) has a leading term proportional to \( (\epsilon \sqrt{\ln \epsilon})^{-1} \). Specifically, we have for fixed \( \tilde{\tau} \) and \( \tilde{\tau}' (\tilde{\tau} \neq \tilde{\tau}') \)
\[ \lim_{\epsilon \to 0} (1 - \epsilon) \frac{h(\epsilon)}{e^{1/2}} K_1[(\tilde{\tau} - \tilde{\tau}') h(\epsilon)] = \frac{1}{4(\tilde{\tau} - \tilde{\tau}')^2}, \]
(2.6)

where we take
\[ h(\epsilon) = (\epsilon \sqrt{\ln \epsilon})^{-1}. \]
(2.7)

We now interchange (without justification at this point), the integral in (2.3) and the limit \( \epsilon \to 0 \), use (2.6) and assume that \( \tilde{S}(\tilde{\tau}) \) and \( \tilde{B}(\tilde{\tau}) \), which are \( \epsilon \)-dependent, have finite limits; we then obtain (\( P \) means Cauchy principal value)
\[ \tilde{S}(\tilde{\tau}) - \tilde{B}(\tilde{\tau}) = P \int_{\tilde{\tau}}^{1/2} \frac{1}{4(\tilde{\tau} - \tilde{\tau}')^2} [\tilde{S}(\tilde{\tau}') - \tilde{S}(\tilde{\tau})] d\tilde{\tau}'. \]
(2.8)

Notice that the contribution from \( \tilde{\tau}' \approx \tilde{\tau} \) will be finite if, say, \( \tilde{S} \) is twice continuously differentiable (\( C^{1+\alpha} \) Hölder continuity for some \( \alpha > 0 \) would be sufficient). By Fourier transformation it may be shown that the singular integral operator
\[ f(\tau) \to P \int_{\tau} \frac{1}{(\tau - \tau')^2} [f(\tau) - f(\tau')] d\tau', \]
(2.9)
is, within a constant of proportionality, the square root of the negative of the Laplacian. Such a constant may be incorporated into the definition of $\theta(\tau)$, so that the equation for $\tilde{S}(\tau)$ reads

$$\tilde{S}(\tau) - \tilde{B}(\tau) = -(-\Delta)^{1/2} \tilde{S}(\tau).$$

(2.10)

The solution of (2.10) gives us by the change of variable (2.2) the leading term of the asymptotic expansion of the source function $S(\tau)$ for small $\epsilon$. Although equation (2.10) looks very similar to the diffusion equation (2.5), it must be stressed that the operator $(-\Delta)^{1/2}$ is non-local and cannot be approximated numerically by a simple finite difference operator. However, (2.10) is readily solved by Fourier Transformation (F.T.) to give

$$\tilde{S}(k) = \tilde{B}(k)/(1 + |k|),$$

(2.11)

where $\tilde{S}(k)$ and $\tilde{B}(k)$ are the F.T. of $\tilde{S}(\tau)$ and $\tilde{B}(\tau)$.

To justify the interchange of $\epsilon \to 0$ limit and integration, leading to (2.8), we must estimate the contributions to (2.3) and (2.8) coming from $|\tau - \tau'|$ $\theta(\epsilon) < 1$ where the asymptotic form of $K_1(\cdot)$ cannot be used. It is easily shown that these contributions go to zero with $\epsilon$ if we assume the uniform boundedness of $\partial^2 \tilde{S}/\partial \tau^2$ or, at least of the Hörder norm $C^{1+\alpha}$.

Remarks

(1) The scaling factor $\theta(\epsilon)$ is known as the ‘thermalization length’ in the literature. The present formalism provides an unambiguous definition of this quantity.

(2) The logarithmic correction in $K_1(\cdot)$ does not appear in the asymptotic equation (2.10) but only in the thermalization length (2.7).

3 Asymptotic expansion in the half-space (only CRD)

3.1 Interior expansion

We rewrite (1.4) as

$$\epsilon[S(\tau) - B(\tau)] = (1 - \epsilon) \int_0^\infty K_1(\tau - \tau') [S(\tau') - S(\tau)] d\tau' - (1 - \epsilon) S(\tau) \int_{-\infty}^0 K_1(\tau - \tau') d\tau'.$$

(3.1)

Using essentially the same procedure as in the previous section we obtain for $\epsilon \to 0$, with the same scaling factor $\theta(\epsilon) = (\epsilon \sqrt{-\ln \epsilon})^{-1}$ as above,

$$\tilde{S}(\tau) - \tilde{B}(\tau) = P \int_0^\infty \frac{1}{4(\tilde{\tau} - \tilde{\tau}')^2} [\tilde{S}(\tau') - \tilde{S}(\tau)] d\tau' - \frac{\tilde{S}(\tau)}{4\tau}.$$  

(3.2)

Upon integration by parts, (3.2) may be rewritten

$$\tilde{S}(\tau) - \tilde{B}(\tau) = P \int_0^\infty \frac{1}{4(\tau' - \tau)} \frac{\partial \tilde{S}(\tau')}{\partial \tau'} d\tau'.$$

(3.2)'

The singular integral operators appearing in (3.2) and (3.2)' are not reducible merely to a fractional power of the Laplacian as in the full-space problem. Similar operators appear in Ivanov’s (1973, p. 392) treatment of the conservative slab of thickness $\tau_0(\tau_0 \to \infty)$.* Like

*The analog of (3.2) has been solved numerically for a Doppler profile by Ivanov and co-workers (Ivanov 1976, private communication).
any Wiener–Hopf equation, (3.2)' can be solved exactly by analytic function techniques (Case & Zweifel 1967; Mushkelishvili 1953). More surprising is the fact that (3.2)' can still be solved by such techniques in the non-translationally invariant case when, for example, a finite slab is used (finite upper limit in the integral of (3.2)') and/or a non-uniform $\epsilon$ is used (this gives an additional $\bar{e}(\tau)$ factor in the lhs of (3.2)'). The reason for this 'miracle' is that (3.2)' is essentially a Cauchy integral* which may be obtained from the boundary values on $\mathbb{R}^+$ of a suitable analytic function. Since the main object of this paper is not exact solutions, we shall not elaborate on this point.

3.2 Surface boundary layer

It may be shown by Wiener–Hopf techniques, that for $\tau \to 0$, the solution of the interior equation (3.2) behaves like $\bar{S}(\tau)$ so that $\partial^2 \bar{S}/\partial \tau^2$ is not uniformly bounded up to $\tau = 0$. From this we infer that (3.2) is valid only for $\tau > \eta > 0$ where $\eta$ is arbitrary. The fact that

$$\lim_{\tau \to 0} \bar{S}(\tau) = 0$$

is not surprising, since it is known that the surface value of the source function is $O(\epsilon^{1/2})$ (whatever the frequency profile, cf. for example Frisch & Frisch 1975). It is clear that when $\tau$ and $\tau'$ are $O(1)$ the asymptotic form of $K_1$ cannot be used so that (3.2) is not valid. Actually, there is a boundary layer of thickness $O[h(\epsilon)]$ where another asymptotic expansion holds. Its leading term is obtained by writing $S(\tau) = \epsilon^{1/2} \bar{S}(\tau)$; this gives, for $\epsilon \to 0$ (Mihalas 1970, p. 367; Ivanov 1973, p. 231)

$$\bar{S}(\tau) = \int_0^\infty K_1(\tau - \tau') \bar{S}(\tau') d\tau'.$$

This equation determines $\bar{S}(\tau)$ up to a multiplicative constant which may in principle be obtained by matching boundary and interior asymptotic expansions for $1 \ll \tau \ll h(\epsilon)$ (Cole 1968).

Comparison of the exact solution of (1.4) for finite (small) $\epsilon$ with the leading terms of the interior and boundary asymptotic expansions are made in the companion Paper IV on the numerical aspects.

Remark on the Lorentz profile

For the Lorentz profile $\phi(x) \propto (1 + x^2)^{-1}$, the kernel $K_1(\tau)$ goes like $\tau^{-3/2}$ at large $\tau$, the $(\bar{\tau} - \tau')^{-2}$ factor in (2.8) and (3.2) becomes a $(\bar{\tau} - \tau')^{-3/2}$, the $(-\Delta)^{1/2}$ in (2.10) becomes a $(-\Delta)^{1/4}$, the thermalization length $h(\epsilon) \propto \epsilon^{-2}$ and the boundedness of the second derivative of $\bar{S}(\tau)$ required to derive the asymptotic equation is changed into the boundedness of the first derivative (more precisely Hölder continuity $C^{0, \alpha}$ with $\alpha > 1/2$). Other profiles may be handled as well.

4 The specific intensity (half-space)

Upon using (1.1) and (1.2) we obtain

$$\frac{\partial I}{\partial \tau} = \phi(I - S),$$

(4.1)

*This property holds only for profiles such that the $K_1$ function satisfies

$$\lim_{\lambda \to \infty} \frac{K_1(\lambda \tau)}{K_1(\lambda)} = \tau^{-2}.$$
which is solved, taking into account boundary conditions, to give

\[ I(\tau, \phi, \mu) = \int_{-\infty}^{\infty} \exp \left( -\frac{\tau - \tau'}{\mu} \phi \right) \frac{\phi}{\mu} S(\tau') \, d\tau', \quad (4.2) \]

where the upper limits \( \infty \) and 0 correspond to \( \mu > 0 \) and \( \mu < 0 \). To simplify we use \( \phi \) rather than \( x \) as frequency variable.

### 4.1 Interior Expansion

Defining \( \tilde{I}(\tau, \phi, \mu) = I(\tau, \phi, \mu) \) we obtain

\[ \tilde{I}(\tau, \phi, \mu) = \int_{-\infty}^{\infty} \exp \left( -\frac{\tau - \tau'}{\mu} \phi h(\epsilon) \right) \frac{\phi h(\epsilon)}{\mu} \tilde{S}(\tau') \, d\tau', \quad (4.3) \]

where \( \tilde{S}(\tau) \) is the solution of the (3.2) interior problem. For fixed \( \tau, \phi \) and \( \mu \), we obtain, letting \( \epsilon \to 0 \)

\[ \lim_{\epsilon \to 0} \tilde{I}(\tau, \phi, \mu) = \tilde{S}(\tau). \quad (4.4) \]

If instead we put a scaling factor on \( \phi \), define

\[ \tilde{\phi} = \phi h(\epsilon); \quad \tilde{I}(\tau, \tilde{\phi}, \mu) = I(\tau, \phi, \mu) \quad (4.5) \]

and then let \( \epsilon \to 0 \) for fixed \( \tau, \tilde{\phi} \) and \( \mu \), we obtain the non-trivial result

\[ \lim_{\epsilon \to 0} \tilde{I}(\tau, \tilde{\phi}, \mu) = \int_{-\infty}^{\infty} \exp \left( -\frac{\tau - \tau'}{\mu} \tilde{\phi} \right) \frac{\tilde{\phi}}{\mu} \tilde{S}(\tau') \, d\tau'. \quad (4.6) \]

Equation (4.4) means that the interior photons have an isotropic, flat (in frequency) distribution up to frequencies \( x \) such that \( \phi(x) = O[h^{-1}(\epsilon)] \). For larger frequencies, according to (4.6), there is a non-trivial, generally non-isotropic distribution. This is in contrast with the coherent scattering case which yields an isotropic distribution everywhere in the interior.

### 4.2 Emergent Intensity

In contrast again to the coherent scattering case, there will be both boundary layer and interior contributions to the emergent intensity. For fixed \( \phi \) and \( \mu \), we obtain the line-core contribution

\[ \lim_{\epsilon \to 0} \frac{I(0, \phi, \mu)}{\sqrt{\epsilon}} = \int_{0}^{\infty} \exp \left( -\frac{\tau'}{\mu} \phi \right) S(\tau') \, d\tau', \quad (4.7) \]

where \( S(\tau) \) is the solution of the (3.3) boundary layer equation (matched to the interior solution). But if again we put a scaling factor on \( \phi \), we have the line-wing contribution

\[ \lim_{\epsilon \to 0} I \left( 0, \frac{\phi}{h(\epsilon)}, \mu \right) = \int_{0}^{\infty} \exp \left( -\frac{\tau'}{\mu} \frac{\phi}{h(\epsilon)} \right) \frac{\phi}{h(\epsilon)} \tilde{S}(\tau') \, d\tau', \quad (4.8) \]

where \( \tilde{S}(\tau) \) is the interior solution (clearly, at such frequencies the boundary layer is transparent).
The asymptotic expansion of the specific intensity is here derived from the asymptotic expansion of the source function. This is in contrast to the coherent case where the asymptotic expansion can be obtained directly from the equation of transfer (1.1) (Papanicolaou 1975). Essential for this expansion, is that for fixed \( \tau \), the null-space of the scattering operator on the rhs of (1.1) (with the \( x \)'s and \( \phi \)'s deleted), is the one-dimensional space of \( \mu \)-independent functions; such functions are normalizable with respect to the measure

\[
\int_{-1}^{+1} \frac{d\mu}{2}.
\]

In the CRD-case the null-space is made of functions independent of both \( x \) and \( \mu \) and such functions are not normalizable with respect to

\[
\int_{-\infty}^{+\infty} dx \int_{-1}^{+1} \frac{d\mu}{2}
\]

because the frequency integral diverges (unless frequencies are restricted to a finite interval). Equivalently, a photon which starts with some frequency distribution and is then subject to only scattering (no destruction, no further creation) will never reach a stationary frequency distribution; instead the frequency distribution will flatten out indefinitely (Field 1959).

That we cannot hope to derive an asymptotic expansion for the specific intensity by 'standard' methods, may also be seen by looking at the problem from a probabilistic view point. Coherent scattering has a well-known probabilistic interpretation (Sobolev 1963). The standard asymptotic expansion tells us that in the coherent case, the photon trajectory, which is a Markov Process continuous in space and discontinuous in direction, goes over for \( \varepsilon \to 0 \) into a diffusion Markov Process, still continuous in space. In the non-coherent case, the limiting Markov Process, if it exists, is likely to have jumps in all three variables \( x, \mu \) and \( \tau \); if we rescale all variables (including the time) a photon can disappear somewhere and immediately reappear elsewhere! It also implies that the equation satisfied by the asymptotic specific intensity, if such an equation exists, will not be local (i.e. differential) in \( \tau \) and therefore that we lose one of the classic advantages of working with intensities rather than with source functions.

5 Conclusion

Traditionally the smallness of \( \varepsilon \) (typically \( 10^{-4} \) in astrophysical applications) was thought to be a source of complication, particularly in the numerics. We hope that this paper has convinced the reader that the problem actually simplifies for small \( \varepsilon \) if we make an asymptotic expansion. Let us again summarize the main steps required:

(i) Find the thermalization length \( h(\varepsilon) \).

(ii) Find the interior asymptotic expansion, the leading term of which is the solution of (3.2) with \( \tilde{S}(0) = 0 \).

(iii) Find the boundary layer asymptotic expansion, the leading term of which is the solution of (3.3) which is determined up to a numerical factor.

(iv) Match interior and boundary layer to determine the numerical factor. Although in principle this requires knowing more than the leading term (Cole 1968), in practice the leading terms may often suffice (cf. Paper IV).

It is important to realize that interior and boundary layer calculations may be decoupled. For the interior calculation only the asymptotic form of \( K_1(\cdot) \) is needed (no need to
tabulate or to represent by sophisticated functions). As for the surface calculation it does not involve $B(\cdot)$ and can be done once for all for a given profile (cf. Paper IV).

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References

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