Nearly collisionless spherical accretion

Mitchell C. Begelman

Institute of Astronomy, Madingley Road,
Cambridge CB3 0HA

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Summary. A fluid-like gas accretes much more efficiently than a collisionless gas. The ability of an accreting gas to behave like a fluid depends on the relationship of the mean free path of a gas particle at $r \rightarrow \infty$ ($\lambda_{\infty}$), to the typical length scales associated with the star–gas system. We examine this relationship in detail by generalizing the model of Danby & Camm to cases where $\lambda_{\infty}$ is finite. For constant collision cross-section we find evidence for a rapid changeover from collisionless to fluid-like accretion flows when $\lambda_{\infty}$ drops below a certain value, but for hard Coulomb collisions, the transition is more gradual, and is sensitive to the adiabatic index of the gas at $r \rightarrow \infty$.

To these results must be added the effects of the substantial cusp of bound particles, which always develops in a system with arbitrarily small but non-zero cross-section. The density run in such a cusp depends on the collision properties of the particles, but it is always steeper than the ‘zero flow’ solution described by Bahcall & Wolf. ‘Loss-cone’ accretion from the cusp may in some cases exceed the accretion rate predicted from our generalization of Danby & Camm’s theory.

Our result may apply to the accretion of grains and the dynamics of stellar cusps around massive black holes, as well as providing a cautionary tale for those who accept the fluidity of all accreting plasmas as gospel truth.

1 Introduction

When one estimates the rate at which interstellar material accretes onto a star, the result usually depends strongly on whether the gas is treated as a fluid, or as a mainly collisionless aggregate of particles. Using the isentropic fluid equations, Hoyle & Lyttleton (1940), and Bondi (1952) (hereafter BHL) showed that a star may swallow fluid-like gas at the impressive rate

$$\dot{M}_{\text{BHL}} = 4\pi r_{\text{ac}}^2 \sqrt{\frac{\gamma k T_{\infty}}{m}} \rho_{\infty}$$

where $r_{\text{ac}} = GMm/\gamma k T_{\infty}$ is the accretion (Bondi) radius, $\gamma$ is the adiabatic index of the gas and $\alpha$ varies between 1.12 and 0.25 as $\gamma$ varies between 1 and 5/3. For a completely
collisionless gas, Eddington (1926) found that particles accrete only as rapidly as their hyperbolic orbits, fixed at \( r \to \infty \), carry them across the stellar surface. Eddington’s ideas were formalized by Danby & Camm (1957) (hereafter DC), who wrote down a distribution function for particles in the gas, and used it to compute an accretion rate \( \dot{M}_{\text{DC}} \) which is lower than \( \dot{M}_{\text{BHL}} \) by a factor \( (r_*/r_{\infty})^2 \), where \( r_* \) is the radius of the star. For a white dwarf immersed in typical interstellar medium \( (r_*/r_{\infty}) \leq 10^{-6} \). Clearly, changing the collision frequency can alter \( \dot{M} \) drastically.

Of course, DC and BHL treat only the extremes of collision frequency. To find out how \( \dot{M} \) responds to changes in collision frequency, we must study the Boltzmann equation, and the kinetic theoretic approach seems preferable to a hydrodynamical treatment. Unfortunately, it is immediately clear that DC’s initial assumptions do not allow for the addition of even the smallest finite collision cross-section \( \sigma \). A gas is basically collisionless on a given scale if a typical particle can traverse that scale without suffering a collision. In an accretion problem, the natural scale is the distance from the star, \( r \), and the condition for collisionlessness is that \( t_{\text{dyn}}/t_{\text{coll}} \sim (\rho \sigma)(v_{\text{rel}}/\bar{v}) \ll 1 \), where \( v_{\text{rel}} \) is the mean relative speed of two particles, and \( \bar{v} \) is the rms speed. When this relation holds, the distribution function must depend only on constants of the orbital motion (Jeans’s theorem) to \( 0(t_{\text{dyn}}/t_{\text{coll}}) \). DC assume that the distribution is Maxwellian infinitely far from the star, and obtain, in the limit of spherical symmetry

\[
f_{\text{DC}} = \rho_\infty \left( \frac{m}{2\pi kT_\infty} \right)^{3/2} \exp \left( -\frac{E}{kT_\infty} \right) \quad E > 0
\]

(1)

where \( E = v^2/2 - GM/r \) is the energy per unit mass.

DC identify \( f_{\text{DC}} \) with the total distribution function, failing to recognize that while \( f_{\text{DC}} \) is the unique distribution function for unbound particles, their boundary conditions admit the addition of any distribution of bound particles \( (E < 0) \), as long as it satisfies Jeans’s theorem. If \( \sigma \) were identically zero, there would be no reason to assume that a bound cusp actually existed; but one would be hard put to justify DC’s choice of a Maxwellian at \( r \to \infty \), if there had not been the chance for collisional relaxation somewhere!

We show, in the following section, that the presence of an arbitrarily small but finite \( \sigma \) renders a bound cusp not only plausible, but inevitable. For \( \sigma \) sufficiently small, the cusp does not contribute appreciably to the accretion rate, which tends towards \( \dot{M}_{\text{DC}} \). In subsequent sections, we generalize the technique of DC to account for finite \( \sigma \), and use our generalization to examine the transition from partly collisionless to fully fluid-like accretion.

2 The bound cusp

2.1 THE INEVITABILITY OF A BOUND CUSP

Strictly speaking, \( f_{\text{DC}} \) applies only to inbound particles (radial velocity \( v_r < 0 \)). The distribution of outbound particles must vanish in the ‘loss-cone’ region of \((E,J^2)\)-space \((J = v_r r \) is the angular momentum per unit mass), where perihelia fall within a distance \( r_* \) of the stellar centre. However, the quantitative effects of the loss-cone can be made to vanish by taking \( r_* \to 0 \). Then, for arbitrarily small but finite \( \sigma \), a spherically symmetric distribution function must satisfy the Boltzmann equation

\[
\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + \left( \frac{v_r^2}{r} - \frac{GM}{r^2} \right) \frac{\partial f}{\partial v_r} - \frac{v_r v_\perp}{r} \frac{\partial f}{\partial v_\perp} = \langle \frac{\partial f}{\partial t} \rangle_{\text{coll}}
\]

(2)
with no local particle sinks (since \( r_s \to 0 \)), and hence with a collision integral having the form (Harris 1971)

\[
\left( \frac{\partial f(v_1)}{\partial t} \right)_{\text{coll}} = \frac{1}{m} \int d^3v_1' d^3v_2' W(v_1', v_2'|v_1, v_2) \cdot \{ f(v_1') f(v_2') - f(v_1) f(v_2) \}.
\]

(3)

\( W \) is the probability that a binary collision changes \( v_1, v_2 \) to \( v_1', v_2' \), and vice-versa. If we insert \( f \propto \exp (-av^2) \), \( 0 < v^2 < \infty \), then conservation of kinetic energy in elastic collisions requires that \( f(v_1') f(v_2') = f(v_1) f(v_2) \). This explains why the collision term vanishes locally for any true Maxwellian. If \( f = f_{\text{DC}} \), then \( f_1 f_2 = f'_1 f'_2 \) when both \( v_1^2 \) and \( v_2^2 \) exceed \( 2GM/r \) \((v_1, v_2, \text{as pre-collision states, automatically satisfy this})\). However, there are collisions in which one of a pair of unbound particles becomes bound. Then, either \( v_1' \) or \( v_2' \) fails to satisfy \( v^2 > 2GM/r \), and \( f_{\text{DC}}(v_1') f_{\text{DC}}(v_2') = 0 \). Since volumes of phase space are preserved in elastic collisions, there are more ways of depopulating each unbound orbit \( v_1(r) \) than of replenishing it, and \( [\partial f_{\text{DC}}(v_1)/\partial t]_{\text{coll}} \) is negative.

But the dependence of \( f_{\text{DC}} \) on \( E \) alone ensures that the left-hand side of equation (2) vanishes identically; therefore a correction to \( f_{\text{DC}} \), \( \delta(v_1) \), is necessary, even in the absence of a loss-cone. For \( v^2 > 2GM/r \), it is possible to require \( \delta \delta / \partial t = 0 \), and the Boltzmann equation yields \( \delta(v_1, v^2) \propto 2GM/r \sim 0 \left[ (t_{\text{dyn}}/t_{\text{coll}}) f_{\text{DC}} \right] \). However, when \( v^2 < 2GM/r \), \( f_{\text{DC}} = 0 \) and there is no steady-state with

\[
\int_{v^2 < 2GM/r} \delta(v_1) \, dv_1 \leq \rho_{\text{DC}}(r).
\]

This fact arises from the qualitative difference between bound and unbound orbits.

A time-independent spherically symmetric distribution may be expressed in terms of the variables \( r, E, J^2 \) and \( \text{sgn}(v_r) \), the last of which allows for asymmetry between the distributions of outward- and inward-bound particles. The Boltzmann equation reduces to

\[
u_r \left( E, J^2 \right) = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}
\]

| \begin{array}{l} v_r \left( E, J^2 \right) = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}} \\
\end{array}
\]

where \( v_r \) is constructed from the four variables. But

\[
\left( \frac{\partial f}{\partial r} \right)_{E, J^2} = \left( \frac{\partial (f_{\text{DC}} + \delta)}{\partial r} \right)_{E, J^2} = \left( \frac{\partial \delta}{\partial r} \right)_{E, J^2}
\]

therefore, as long as the collision term is dominated by \( f_{\text{DC}} \), which is symmetric (for vanishing loss-cone) about \( v_r = 0 \), \( \delta \) is antisymmetric about \( v_r = 0 \). (Physically, the binding of particles near the star gives to the remaining unbound distribution a net radial velocity away from the star.) In the bound distribution (henceforth denoted by \( f_b \)), however, there cannot be an overall asymmetry, because particle orbits are periodic with \( t_{\text{dyn}} < t_{\text{coll}} \); the asymmetric part of \( f_b \) can have a characteristic magnitude of no more than \( 0 \left[ (t_{\text{dyn}}/t_{\text{coll}}) f_b \right] \). Only when \( f_b \) becomes comparable to \( f_{\text{DC}} \), and unbound-bound and bound-bound collisions play an important role in the collision term, can \( f_b \) achieve a steady state. In other words, the number of bound particles increases until the unbinding rate is comparable to the binding rate.

2.2 A self-similar distribution function

While \( f_b \) should not depend quantitatively on \( r_s \) as the loss-cone vanishes, its form depends crucially on the fact that the central star can accrete. Saying that the star does not accrete
at all is not equivalent to saying that \( r_\ast \to 0 \), hence \( \dot{M} \to 0 \). Gas around a non-accreting star eventually reaches thermal equilibrium with the gas at \( r \to \infty \). As \( r \to 0 \), its density increases exponentially, and we must eventually reach a radius \( r_{\text{fl}} \) within which \( t_{\text{coll}} < t_{\text{dyn}} \). Suddenly turning the star into an accretor, we find that (1) the region within \( r_{\text{fl}} \) behaves like a fluid and is accreted within a time \( t_{\text{dyn}}(r_{\text{fl}}) \sim (GM/r_{\ast})^{1/2} \); and (2) this result is independent of \( r_{\ast} \), and remains valid as we shrink the star to a point. A similar argument holds for any isothermal Maxwellian of temperature \( \leq GM/r_{\ast} \). Since it would take the unbound distribution a time longer than \( t_{\text{dyn}}(r_{\text{fl}})/\rho_{\text{DC}}(r_{\text{fl}}) \sigma r_{\ast} > t_{\text{dyn}}(r_{\text{fl}}) \) [since \( \rho_{\text{DC}} \sim \rho_{\ast}(1 + r_{\ast}/r)^{1/2} \)] to repopulate the rapidly accreted part of the cusp, we conclude that (1) \( t_{\text{dyn}}/t_{\text{coll}} \sim \rho_{\ast} \) or \( \ll 1 \) everywhere in the cusp, and \( f_0 \) satisfies Jeans's theorem; and (2) an isothermal Maxwellian cannot describe the cusp in the limit \( r_{\ast} \to 0 \). For finite \( r_{\ast} \), an isothermal Maxwellian with temperature \( \geq GM/r_{\ast} \) is equally unacceptable because its particles would 'boil off' in the outer region of the cusp faster than they could be replenished by the cooler background distribution.

Non-Maxwellian distributions satisfying Jeans's theorem do not have a single temperature associated with them, but a temperature may be defined for particles of each \( E \) by \( 1/kT_{\ast}(E) = -\partial \ln f_0(E,J^2)/\partial E \) (Lynden-Bell 1975). We infer from the arguments about Maxwellians that \( T_{\ast}(E) \) cannot depend strongly on \( r_{\ast} \) or \( r_{\text{ac}} \), while in the region \( r_{\ast} < r < r_{\text{ac}} \), collisions should have wiped out any weak dependence associated with the details of \( \rho_{\text{DC}} \). We therefore conclude that \( kT_{\ast}(E) \) depends only on \( E, J^2, GM \), and pure numbers. The only constructible quantity having dimension \([kT]\) is \( E \), while the only dimensionless construct, \( J^2E/(GM)^2 \), is related to the eccentricity \( e = 1 + 2EJ^2/(GM)^2 \). Therefore \( kT_{\ast}(E) \propto E \) where the constant of proportionality is dimensionless, and \( f_0 \) must have the form

\[
K(e)|E|^p g(e) \quad E < 0
\]

where \( K(e) \) is a normalization whose dimensions depend on \( e \). If \( \sigma \propto r_{\ast}^2 \), it is reasonable to demand that the distribution of eccentricities be self-similar at radii \( r_{\ast} \ll r < r_{\text{ac}} \). Adding this constraint gives

\[
f_0 = C |E|^p g(e) \quad E < 0
\]

(4)

where \( p \) and \( C \) are constant, and \( g(e) \) is an arbitrary function of \( e \). Distributions of the form (4), yielding densities \( \rho_b \propto r^{-3/2} \), were proposed by Peebles (1972), and later used by Bahcall & Wolf (1976), Frank & Rees (1976), and others to study the behaviour of a stellar cusp around a massive collapsed object.

2.3 Dynamics and Structure of the Cusp

Particles distributed according to (4) collide with each other, as well as with unbound particles. Collisions between unbound particles continue to produce bound particles at a rate \( \sim \rho_{\text{DC}}/t_{\text{coll}} \sim \rho_{\text{DC}}^2 (GM/r)^{1/2} \propto r^{-3/2} \) for hard spheres. We cannot allow this to continue for very long in the absence of processes which limit \( \dot{\rho}_b \), since \( \dot{\rho}_b \propto r^{-3/2} \) would result in fluid-like behaviour at small \( r \), and rapid accretion of the inner cusp. To balance the growth of such a steep density gradient, the unbinding rate must vary at least as steeply as \( r^{-3/2} \), and \( \rho_b(r) \) can be no flatter than \( r^{-1/2} \).

A local balance between binding and unbinding, with \( \dot{\rho}_b \propto r^{-1/2} \) exactly, could be set up only if the average energy redistribution due to collisions in the cusp were nil. This might be possible if all bound-bound collisions left both particles bound. However, a fraction of the bound-bound collisions unbinds one of the particles; the escape leaves the cusp in a time \( \sim t_{\text{dyn}} \), and the particles left behind become more rightly bound. Because unbound
particles remain in the cusp for so short a time, \( f_{DC} \) acts like a heat reservoir, and there is no reverse effect tending to loosen the orbits in the inner cusp. Thus, there is a systematic shrinkage of bound particle orbits, and a steepening of \( \rho_p \) past the \( \rho^{-1/2} \) threshold.

As a result, only bound-bound collisions are important for \( r < r_{ac} \), while newly bound particles are input mainly into the outer regions of the cusp. Tightly bound orbits in the inner region are kept populated by a cascade of particles inward from lower binding energies. This time-averaged ‘cascade’ should be distinguished from the diffusion believed to dominate the dynamics of stellar cusps around massive collapsed objects (Bahcall & Wolf 1976; Frank & Rees 1976; and others). Direct unbinding of cusp stars is ignored in treatments of the latter, hence energy may enter and exit only through the boundaries of the cusp. As \( r_* \rightarrow 0 \), \( \dot{M} \) vanishes, and the only way to maintain a steady state is to have a uniform radial ‘heat flux’ \( Q \propto (GM/r)(\rho r^3/t_R) \), where \( t_R \sim t_{coll}/\log \rho r^3 \) is the relaxation time. Simple scaling arguments yield the ‘zero flow’ value for \( p \) in equation (4). In the case we wish to consider, particles suffer large deflections during collisions, and while the net flux of particles vanishes as \( r_* \rightarrow 0 \), the fluxes of bound and unbound particles do not vanish separately. By following the evolution of the system subject to energy conservation and continuing self-similarity, we may deduce the \( r \) dependence of these fluxes.

In every \( t_{coll} \) (evaluated at \( r \sim GM/|E| \)), particles of energy \( E \) and eccentricity \( e \) exchange energy and angular momentum with other particles in the cusp. The details of these exchanges are extremely complicated, but if we require self-similarity, their net result must be that no energy or eccentricity redistribution occurs through internal exchanges. In addition, a fraction \( d[e, g(e)] \) of all particles \((E, e)\) become unbound. The binding energy gained by the cusp on average will be a fraction \( d[e, g(e)] \{1 + k[e, g(e)]\} \) of the total binding energy initially in \((E, e)\) particles, where \( k|E| > 0 \) is the mean positive energy of the particle which has become unbound. Self-similarity requires, on average, that this entire binding energy be absorbed by the remaining \((E, e)\) particles. It does not matter that the particles which absorb the binding energy may not be those originally characterized by \((E, e)\), but rather those which have been shifted into position by internal interactions—the self-similarity condition forces the outcome to be indistinguishable from the simple energy redistribution outlined here.

Since \( g(e) \) is to remain invariant during this process, we can further simplify the analysis by viewing the redistribution as a uniform fractional shift in the energy of each particle, keeping eccentricity fixed. Translation into mathematical terms is straightforward: let \( N(E) \) represent the number of particles with binding energy between \(|E|\) and \(|E + dE|\), and let \(|\dot{E}|\) be the average rate of change of energy, following particles of a given energy. In a steady state, the evolution of the flux across an energy surface is given by

\[
\frac{\partial}{\partial |E|} (N|\dot{E}|) = -\frac{\dot{d}N}{t_{coll}}
\]

where

\[
|\dot{E}| = \frac{\dot{d}}{t_{coll}} |E|(1 + \tilde{k}).
\]

Thus

\[
\frac{N(E)|E|}{t_{coll}(E)} \propto |E|^{-1/(1 + \tilde{k})}.
\]

Since \( t_{coll} \propto |E|^{p-2} \) for \( p = constant \), and \( N(E) \propto |E|^{p-5/2} \), \( p = -\frac{3}{4} - 1/2(1 + \tilde{k}) \). The cusp
density for hard spheres scales as
\[ \rho_b(r) \propto r^{-5/4 + 1/2(1+\kappa)}. \] (5)

The lack of a precise definition for \( t_{\text{coll}} \) does not affect our result, since \( \bar{k}|E| \) is well defined as the mean positive energy of those particles which become unbound in collisions involving a particle of energy \( E \), and \( \bar{\kappa} \) as the mean fraction of collisions which unbind a particle. Given \( \bar{k} \) and self-similarity, \( p \) is determined. However, the value of \( \bar{k} \) depends on the details of the collision process, and on \( g(e) \). For example, if all particles are on circular orbits or all on radial orbits, then \( \bar{k} = 0 \), and the system exhibits 'zero flow' (\( \rho_b \propto r^{-3/4} \) for hard spheres). In general, \( g(e) \) is not likely to be highly skewed, nor will it correspond to velocity isotropy (\( g = 1 \)). Collisions between a loosely bound (\( e \to 1 \)) particle and one on a nearly-circular orbit are most likely to result in an unbinding. Because particles with energy \( E \) may be found at \( r_{\text{coll}} \ll GM/|E| \), some of these will be unbound with energies \( \sim GM/r_{\text{coll}} \gg |E| \); these collisions will raise \( \bar{k} \) by an amount that is easily estimated. When a particle of energy \( -GM/r_1 \) becomes unbound at \( r < r_1 \), its post-collision (positive) energy will be, on average, \( \eta GM/r \), where \( \eta \approx 1 \) varies slowly with \( r \). If the frequency of unbinding collisions scales with \( r_{\text{coll}} \), then \( \bar{k} \) is approximately
\[ \bar{k} = \eta \left[ \int_{r=r_*}^{r_1} r_{\text{coll}}^{-1}(r) d \left[ \frac{t_{\text{dyn}}(r)}{t_{\text{coll}}(r)} \right] \right] / \left[ \int_{r=r_*}^{r_1} r_{\text{coll}}^{-1}(r) d \left[ \frac{t_{\text{dyn}}(r)}{r_{r_1}} \right] \right] \]
where \( \mathcal{F}(r/r_1) \) is the fraction of particles with energy \( -GM/r_1 \), whose orbits carry them to radii \( <r \), and \( t_{\text{dyn}}(r)/t_{\text{dyn}}(r_1) \) is the fraction of time spent at radii \( <r \) by such a particle. In the self-consistent problem, \( \mathcal{F}(r/r_1) \) depends on the eccentricity distribution \( g(e) \), but two limiting cases may be computed by assuming that all orbits are radial (\( \mathcal{F} = 1 \)) or that the velocity distribution is isotropic (\( \mathcal{F} = r/r_1 \)). Then, for hard spheres,
\[ \bar{k} \approx \eta \times \begin{cases} \frac{1}{r^2} \int_{r=r_*}^{r_1} \frac{dr}{r^3} & \text{radial} \\ \frac{1}{r} \int_{r=r_*}^{r_1} \frac{dr}{r} & \text{isotropic} \end{cases} \]
where \( \bar{k} \) is found self-consistently by substituting \( p = -\frac{1}{4} - 1/2(1 + \bar{k}) \). The result is
\[ \bar{k} \approx \eta \times \begin{cases} \frac{1}{r^2} \int_{r=r_*}^{r_1} \frac{dr}{r^3} & \bar{k} > 1 \\ \frac{1}{r} \int_{r=r_*}^{r_1} \frac{dr}{r} & \bar{k} = 1 \\ \frac{-p - \frac{1}{2} \frac{r_{r_1}}{r_*}^{p+3/2}}{p + \frac{1}{2}} & \bar{k} < 1 \end{cases} \]
radial:
\[ \bar{k} \approx \eta \times \begin{cases} \frac{1}{r^2} \int_{r=r_*}^{r_1} \frac{dr}{r^3} & \bar{k} > 1 \\ \frac{1}{r} \int_{r=r_*}^{r_1} \frac{dr}{r} & \bar{k} = 1 \\ \frac{-p - \frac{1}{2} \frac{r_{r_1}}{r_*}^{p+3/2}}{p + \frac{1}{2}} & \bar{k} < 1 \end{cases} \]
isotropic.
Self-consistent solutions with reasonable values of \( \eta (\eta > 0.04) \) are possible only when \( \tilde{k} \) is, effectively, infinite. However, the generalization of (5) for any \( \sigma \propto v_{\text{rel}}^{-q} \), \( q > 0 \), gives \( t_{\text{dyn}} > t_{\text{coll}} \) when \( r < r_{\text{f1}} \) for \( \tilde{k} > 1 \). This is unacceptable for a cusp embedded in a collisionless ambient medium, for reasons cited earlier. We are uncertain about the time-averaged structure of a \( \tilde{k} > 1 \) cusp, but it probably conforms to our model outside \( r_{\text{f1}} \). Thus, in the outer regions of the cusp, at least, we expect \( \rho_b(r) \propto r^{-5/4} \), which is the hard sphere analogue of the Peebles (1972) \( \rho \propto r^{-9/4} \) stellar cusp. Within \( r_{\text{f1}} \), the density may flatten out to maintain \( t_{\text{dyn}} \approx t_{\text{coll}} \). Estimates of \( \tilde{k} \) will be lower for particles with \( q > 0 \); for example, we expect \( \tilde{k}_{\text{max}} \approx 2 \) for particles with the Coulomb-like cross-section \( \sigma \propto v_{\text{rel}}^{-4} \).

### 2.4 Effects on the accretion process

Accretion fluxes from the cusp and unbound distributions are additive. A cusp particle may find itself on a loss-cone orbit following a collision, if (1) its velocity vector lies within the solid angle \( \Omega = 2 \pi (GM/r^2)(r_{\ast}/r) \sim r_{\ast}/r \) for a typical collision] centred about the radial direction; or (2) \( v^2 < 2GM/r(r_{\ast}/r) \). For \( f_{\text{b}} \) roughly isotropic, \( (r/r_{\ast})^{1/2} \) times as many collision outcomes satisfy (1) as (2), and it is possible to show that the accretion rate due to the cusp is

\[
\dot{M}_b \approx \dot{M}_{\text{DC}} \left( \rho_{\omega \sigma}(r_{\text{ac}}) r_{\text{ac}} \right) \left( \frac{r_{\text{ac}}}{r_{\ast}} \right)^{\tilde{k}/(1+\tilde{k})}
\]  

(6)

for \( \rho_{\omega \sigma}(r_{\text{ac}}) r_{\text{ac}} < 1 \), \( \tilde{k} < 1 \), where \( \sigma(r_{\text{ac}}) \) is the cross-section \( \propto v_{\text{rel}}^{-q} \) corresponding to \( v_{\text{rel}} \sim (GM/r_{\text{ac}})^{1/2} \). As \( \rho_b \) tends towards zero flow (\( \tilde{k} \to 0 \)), the enhancement of the accretion rate over \( \dot{M}_{\text{DC}} \) becomes marginal. For \( \tilde{k} \to 0(1) \), the main enhancement comes from the regions near the star; therefore, the maximum enhancement can never exceed that corresponding to \( \tilde{k} = 1 \). Then \( \dot{M}_b > \dot{M}_{\text{DC}} \) if \( \rho_{\omega \sigma}(r_{\text{ac}}) r_{\text{ac}} > (r_{\ast}/r_{\text{ac}})^{1/2} \). The theory outlined here may be modified to treat the accretion of interstellar grains (which collide inelastically), and has relevance to the study of stellar cusps around massive collapsed objects.*

### 3 The ambient distribution

When \( \sigma \) is finite, \( f_{\text{DC}} \) ceases to represent the ambient distribution (the part of the distribution that asymptotically describes the ambient gas), since the region in which \( t_{\text{dyn}} < t_{\text{coll}} \) no longer extends to infinity. Instead, a self-consistent model must contain both fluid-like and collisionless regions. We attempt to simulate this situation by assuming that there is some radius \( r_0 \), outside of which we may treat the gas as a fluid, but within which particles travel on the orbits they were given at \( r_0 \). To choose the best value for \( r_0 \), we need a fluidity criterion which scales logarithmically in a physically meaningful way, and tells us to expect fluid-like behaviour when \( t_{\text{dyn}} > t_{\text{coll}} \). \( r_0 \) is adequately defined by \( \rho(r_0) \sigma(r_0) r_0 = 1 \) when the

* \( \rho_b(r) \) is independent of \( \tilde{k} \) for a gas whose particles suffer only hard (large deflection) collisions, because there is no way to dispose of the excess binding energy other than to distribute it among the remaining particles. In gases of 'soft' spheres, such as stars, diffusion of excess binding energy to the edges of the cusp will occur more rapidly than its internal redistribution through hard collisions. It is no longer valid to require that the energy \( \propto 1 + \tilde{k} \) \( E \) be absorbed by particles of energy \( E \). Instead, the particles absorb a fraction \( t_{\text{diff}}/t_{\text{coll}} \sim 1/(\log p r^3) \) of this energy, where \( r = (GM/E)^{1/2} \), by diffusion. The resulting density run is roughly \( \rho_b(r) \approx r^{-7/4} - \frac{d(1+\tilde{k})/2 \log p r^3}{2r^{7/4}} \). which \( r^{-7/4} \) corresponds to zero flow. It is apparent that when energy transport occurs mainly through diffusion, the excess steepening is not nearly as dramatic as in the purely hard collision case. (A brief derivation of the above result is presented in an Appendix.)
gas at $r \geq r_0$ is streaming subsonically, and the corresponding fluidity criterion is $\rho(r) a(r) r > 1$.

Since the ambient distribution $f_{am}$ is Maxwellian at $r > r_0$, some of the particles entering the collisionless region will be bound, but none will have a binding energy greater than $GM/r_0$. Within $r_0$, there will be a bound cusp,

$$\rho_b(r) \sim \rho_{am}(\min[r_ac, r_0]) \left( \frac{r}{\min[r_ac, r_0]} \right)^{-3/2}$$

for $r < \min[r_ac, r_0]$. Because a fluid-like cusp cannot survive in a collisionless ambient gas, the self-consistency of the collisionless region must depend on the slope of $\rho_{am}$ near $r_0$, which in turn may depend on the total accretion rate. Since the cusp may contribute substantially to $\dot{M}$, it can play an important role in determining the global nature of the flow.

### 3.1 Derivation of $f_{am}(r < r_0)$

At $r = r_0$,

$$(f_{am})_0 = \rho_0 \left( \frac{m}{2\pi kT_0} \right)^{3/2} h_{r_0}(v) = \rho_0 \left( \frac{a}{\pi} \right)^{3/2} \exp \left[ -a(v - v_0)^2 \right]$$

where $a = m/2kT_0$ and $0 < |v| < \infty$. Specializing to spherical symmetry and choosing $v_0 = -v_0 \hat{r}$, we have, according to Liouville's theorem,

$$h_t(E, J^2) = h_{r_0}(E, J^2) = \exp \left[ -a \left( E + \frac{GM}{r_0} + v_0^2 - 2v_0 \sqrt{2 \left( E + \frac{GM}{r_0} - \frac{J^2}{r_0^2} \right)} \right) \right]$$

$$= h(r, v_t^2, v_\perp^2)$$

$$= \exp \left[ -a \left( v_t^2 + v_\perp^2 - 2GM \left( \frac{1}{r} - \frac{1}{r_0} \right) + v_0^2 \right) \right.$$ 

$$\left. - 2v_0 \sqrt{v_t^2 - 2GM \left( \frac{1}{r} - \frac{1}{r_0} \right) + v_\perp^2 \left[ 1 - \left( \frac{r}{r_0} \right)^2 \right]} \right]$$

for allowed values of $v_t, v_\perp$; and $h_t = 0$ otherwise. When $v_t < 0$, $v_r$ and $v_\perp$ are constrained by the dynamical condition

$$v_\perp^2 > \left[ 2GM(1/r - 1/r_0) - v_t^2 \right]/[1 - (r/r_0)^2].$$

(9)

Outward-bound particles ($v_t > 0$) must also satisfy the condition that they not be on loss-cone orbits. The joint constraint reads

$$\max \left[ \left( \frac{r}{r_0} \right)^2 v_t^2 + 2GM(1/r - 1/r_0) - v_t^2 \right] / (1 - (r/r_0)^2), \quad 2GM(1/r - 1/r_0) - v_t^2 / (1 - (r/r_0)^2) < v_\perp^2 < \infty.$$  

(10)

The space density is then given by

$$\rho_{am}(r) = \rho_0 \left( \frac{a}{\pi} \right)^{3/2} \int h(r, v_t^2, v_\perp^2) dv_r d(v_\perp^2)$$

(11)

where $v_r$ and $v_\perp^2$ satisfy constraints (9) and (10).
3.2 The Accretion Rate

It is possible to obtain an exact expression for the accretion rate due to distribution (8), by integrating

\[ \dot{M}_{\text{amb}} = 4 \pi r_0^2 \rho_0 \left( \frac{a}{\pi} \right)^{3/2} \int_{v_t < 0} h(r_*, v_t, v_t^2) |v_t| d|v_t| d(v_t^2) \]  

(12)

subject to restriction (9) on \( v_t^2 \). When \( r_* \ll r_0 \) and \( (r_*/r_0)(GMm/r_0kT_0) \ll 1 \), this reduces to

\[ \dot{M}_{\text{amb}} = 2(2\pi)^{1/2} r_0GM \sqrt{\frac{m}{kT_0}} \rho_0 \left\{ 2 - \exp(-av_0^2) + \sqrt{\pi av_0^2} \left[ 1 + \Phi(\sqrt{av_0^2}) \right] \right\} \]  

(13)

where

\[ \Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) \, dt \]

is the error function. In the limit \( r_0 \to \infty, \rho_0, T_0 \to \rho_\infty, T_\infty, v_0 \to 0 \) and (13) reduces exactly to \( \dot{M}_{\text{DC}} \).

Given \( v_0 \) and \( r_0 \), one can determine \( \rho_0 \) and \( T_0 \) in terms of \( \rho_\infty \) and \( T_\infty \) by applying the fluid equations in \( r > r_0, r_0 \) is related to \( \rho_0 \) through \( \rho_0(r_0) r_0 = 1 \), while \( v_0 \) must satisfy the self-consistency condition

\[ \dot{M} = \dot{M}_{\text{amb}} + \dot{M}_b = 4 \pi r_0^2 \rho_0 v_0. \]  

(14)

\( \dot{M} \) does not depend strongly on \( v_0 \) unless \( av_0^2 \gtrsim 1 \), which can occur only when \( r_0 < r_{\text{ac}} \). We can adapt equation (6) to this case, with the result

\[ \dot{M}_b \approx \dot{M}_{\text{amb}} \times \begin{cases} \left( \frac{r_0}{r_*} \right)^{k/(1+k)} & \bar{k} < 1 \\ \left( \frac{r_0}{r_*} \right)^{1/2} & \bar{k} \geq \dot{M}_{\text{amb}} \end{cases} \]  

(15)

Substituting into (14), we find that \( v_0 \) is negligible when

\[ \left( \frac{r_*}{r_0} \right)^{1/(1+k)} \left( \frac{r_\text{ac}}{r_0} \frac{T_\infty}{T_0} \right) \ll 1. \]

When \( av_0^2 \gtrsim 1, r_* \ll r_0 \), equation (12) gives

\[ \dot{M}_{\text{amb}} \approx 2\pi(2\pi)^{1/2} \rho_0 r_0^2 \sqrt{\frac{kT_0}{m}} \left\{ 2 - \exp(-av_0^2) + \sqrt{\pi av_0^2} \left[ 1 + \Phi(\sqrt{av_0^2}) \right] \right\} \]  

(16)

where \( 1 - e^{-1} < \alpha < 1 \). Any self-consistent steady state corresponding to equation (16) must relax to a fluid-like flow pattern.

Of course, the collisionless accretion rates presented above are valid only if the assumption of collisionlessness can be justified. For hard spheres, this means \( d \log \rho/d \log r > -1 \) as \( r \to r_0 \). A discussion of self-consistency in the hard sphere case forms the core of the next section, and from it we make inferences about the nature of the collisionless-to-fluid-like transition of the flow pattern.
4 Collisionless-to-fluid transition for hard spheres

For $a v_{0}^{2} < 1$, integral (11) describing $\rho_{\text{amb}}(r)$ is tractable, and immediately reveals that $\rho_{\text{amb}}$ approaches an $r^{-1/2}$ dependence as $r \ll r_{0}$. This means that the system becomes more collisionless with decreasing $r$ if one approaches the star from far enough in. But in order to ensure self-consistency for the collisionless region, it is essential that $(d/dr)(\rho_{\text{sf}}) > 0$ in the region immediately interior to $r_{0}$ as well.

Direct integration of (11) yields, in the limit $\nu_{0} \to 0$,

$$
\rho_{\text{amb}}(r) = \rho_{0} \left\{ \exp \left( \frac{G M_{m}}{r k T_{0}} \right) \left[ 1 - \Phi \left( \frac{G M_{m}}{r k T_{0}} \right) \right] + \frac{2}{\sqrt{\pi}} \frac{G M_{m}}{r k T_{0}} \exp \left[ - \frac{G M_{m}}{r k T_{0}} \right] \right\} \left[ \log \frac{r}{r_{0}} \right] \left\{ \frac{G M_{m}}{r_{0} k T_{0}} \left[ \frac{r}{r_{0} k T_{0}} \right]^{2} \right\}.
$$

(17)

Setting $G M_{m}/k T_{0} = R_{ac} = \gamma(T_{m}/T_{0}) r_{ac}$ and approximating the integral we have

$$
\rho_{\text{amb}}(r) = \rho_{0} \left\{ \exp \left( \frac{R_{ac}}{r} \right) \left[ 1 - \Phi \left( \frac{R_{ac}}{r} \right) \right] + \frac{2}{\alpha \sqrt{\pi}} \frac{r_{0}^{2}}{r^{3/2} R_{ac}^{1/2}} \left[ 1 - \exp \left( - \alpha \frac{R_{ac} r}{r_{0}^{2}} \right) \right] \right\}
$$

(18)

where $1 < \alpha < 2$. The model of Danby & Camm is characterized by $r_{0} \to \infty$, and in this limit only two typical behaviours are possible: $\rho(r) \propto$ constant for $r > R_{ac}$, and $\rho(r) \propto r^{-1/2}$ for $r < R_{ac}$. $d \log \rho/dr \log r$ never falls below $-\gamma$, and the system becomes more collisionless as $r$ decreases. When we relax one constraint on their model by assigning to $r_{0}$ a finite value, an $r^{3/2}$ dependence appears in the region $r_{0}(r_{0}/R_{ac}) < r < r_{0}$, when $r_{0} < R_{ac}$.

This section of $\rho \propto r^{-3/2}$ violates our assumption of no collisions within $r_{0}$. The density enhancement is certainly physical, and corresponds to the gravitational focusing of cool particles onto radial orbits in the outer region of the collisionless zone. Could this be a 'mechanism' for the 'jelling' of a non-fluid into a fluid accretion pattern at $r_{0} \to R_{ac}$? Before jumping to conclusions, we should examine the fluid-like gas configurations which might be sitting outside our supposed collisionless region. Since $\sqrt{\nu T_{0}^{2}/2 k T_{0}} < 1$, the density run in the fluid-like region must be very close to that of a static atmosphere. For a gas of adiabatic index $\gamma$,

$$
\rho(r) = \rho_{\infty} \left[ 1 + (\gamma - 1) \frac{r_{ac}}{r} \right]^{1/(\gamma - 1)}.
$$

(19)

For $\gamma < \frac{3}{2}$, this density run becomes steeper than $r^{-1}$ at some point close to $r_{ac}$. If $r_{0}$ lies within this point, then there exists an $r > r_{0}$, such that $\rho(r) \sigma r < 1$. There is a region outside $r_{0}$ which does not satisfy the fluidity criterion, and therefore $r_{0}$ cannot be a good choice for the boundary between fluid and collisionless regions. In fact, the minimum allowable $r_{0}$ must lie near the radius at which $d \log \rho/dr \log r = -1$, as specified by equation (19). When the fluidity criterion produces an $r_{0}$ smaller than this radius, no self-consistent collisionless region can exist. In this case we may conclude that the system has been forced to 'jell' into a completely fluid-like state through the behaviour of the external, fluid-like region.

This mechanism, and the 'self-jelling' mechanism suggested by the $\rho \propto r^{-3/2}$ dependence of the collisionless region when $r_{0} \leq R_{ac}$, both result in sharp transitions from the collisionless accretion rate $M_{DC} + \dot{M}_{b}$ to $M_{BH}$, with no stable intermediate flows. It could be interesting to learn whether one or the other of these mechanisms dominates the transition. Model calculations suggest that marginally lower values of $\rho_{\infty} \sigma r_{ac}$ are required to trigger the
self-similar mechanism than are needed for the externally-driven transition to occur, but the predicted differences are so small that they are likely to be strongly model-dependent. The results are reassuring in that, if $\rho \sigma r = c$ is our fluidity criterion, then $\rho_\infty \sigma r_{\text{ac}} \approx c/2.5$ makes a good transition point, regardless of $\gamma$. This behaviour is depicted schematically in Fig. 1.

![Figure 1](image_url)

**Figure 1.** Variation of $\dot{M}_{\text{amb}}$ with $\rho_\infty \sigma r_{\text{ac}}$ for hard sphere gas (Section 4). $\dot{M}_{\text{amb}}$ approximates total accretion rate when $\xi = 0$. (a) Variation with $\rho_\infty$, $T_\infty$ fixed. (b) Variation with $T_\infty$, $\rho_\infty$ fixed.

Of course, we do not claim that there actually is a discontinuous jump in $\dot{M}$ as $\rho_\infty \sigma r_{\text{ac}}$ exceeds some magic number. Viscous effects will modify perfect fluid behaviour near $r_{\text{ac}}$ when $\rho_\infty \sigma r_{\text{ac}} \gtrsim 1$, lowering the accretion rate below $\dot{M}_{\text{BHL}}$ and smoothing the transition somewhat. However, our self-consistency analysis indicates that in any steady-state flow, $\dot{M}$ must vary much more steeply with $\rho_\infty \sigma r_{\text{ac}}$ when $\rho_\infty \sigma r_{\text{ac}} \approx 1$ than when $\rho_\infty \sigma r_{\text{ac}}$ lies far from this value, if it is true that the flow in most regions is adequately described by either a fluid-like or a collisionless model. Could a smoother transition occur via a flow which is mainly in some intermediate state? Such a flow is implausible, because $\rho(r)$ would have to vary as $r^{-1}$ over the bulk of the region. Otherwise, the flow would become more or less collisional with decreasing $r$. There is no obvious reason why such a delicate equilibrium should be preferred to a relatively sudden switch between two fully self-consistent types of flow.
5 Collisionless-to-fluid transition for particles of Coulomb cross-section

Unfortunately, it is merely the special form of $\sigma$ that causes the hard sphere case to separate so neatly into collisionless and fluid-like regimes. We demonstrate this by discussing the collisionless-to-fluid transition for a gas whose particles suffer hard (large deflection) collisions through a Coulomb potential.

The main contribution to fluid-like behaviour in a non-magnetized, quiescent plasma, comes from close encounters between protons. To a very crude approximation, the average cross-section varies as $\sigma(r) \sim \sigma_\infty [T_\omega / T(r)]^2$ in the fluid-like region, and $\sim \sigma_\infty (6 k T_\omega / m v_{\text{rel}}(r))^2$ in the collisionless region. In the previous section, we pointed out that $\rho_{\text{amb}}$ in the collisionless region may slope as steeply as $r^3$ only when $R_{\text{ac}} > r_0$, but never more steeply than $r^{3/2}$, while the cusp must remain collisionless throughout. $R_{\text{ac}} > r_0$, given small $v_\infty$, is just the condition under which $v_{\text{rel}}^2(r) \sim GM/r$. Thus, not only is $\sigma(r)$ markedly lower than $\sigma_0$ just inside $r_0$, due to the initial acceleration of the cool particles, but it also decreases $\propto r^2$, giving $\rho \sigma r \propto \rho(r) r^3$ for $r < r_0$. We infer that for particles with a Coulomb cross-section, the collisionless region is always internally self-consistent.

Apparently, only the fluid-like region external to $r_0$ comes into play in determining when a partially collisionless flow is stable. The relevant cross-section is $\sigma_\infty [T_\omega / T(r)]^2$, and we might ask whether there is a point, in the fluid-like region, where $d \log \rho \sigma r / d \log r < 0$. Such a point would indicate the same sort of phenomenon that drives the hard sphere gas into a fluid-like state throughout. The critical radius is given by $r_{\text{cr}} = (4 - 3 \gamma) r_{\text{ac}}$ for a static atmosphere with adiabatic index $\gamma$. For $\gamma > \frac{4}{3}$, $r_{\text{cr}}$ does not exist, and we expect to find a smooth transition between $M_{\text{DC}}$ and $M_{\text{BHL}}$, while for $1 < \gamma < \frac{4}{3}$, a sudden jump might occur, possibly at an $r_0$ somewhat smaller than $r_{\text{ac}}$. Of course, for $\rho_\infty \sigma_\infty r_{\text{ac}} \ll 1$, $r_0 \gg r_{\text{ac}}$, and

$$\dot{M} \approx M_{\text{DC}} = 2(2\pi)^{1/2} r_* G M \frac{m}{k T_\infty} \rho_\infty.$$

Similarly, $r_0 \ll r_{\text{ac}}$ when $\rho_\infty \sigma_\infty r_{\text{ac}} > 1$. Then $\dot{M}_b > \dot{M}_{\text{amb}}$, and

$$\frac{\dot{M}}{M_{\text{DC}}} \approx \left[ \frac{\rho_\infty (T_\infty)}{T_\omega} \right]^{1/2} \left( \frac{r_0}{r_*} \right)^{\tilde{k}/(1+\tilde{k})} \approx \left[ \frac{r_{\text{ac}}}{r_0} \right]^{(3-\gamma)/2(\gamma-1)} \left( \frac{r_0}{r_*} \right)^{\tilde{k}/(1+\tilde{k})}$$

$$\tilde{k} < 1 \text{ (replace } \tilde{k} \text{ by } 1 \text{ for } \tilde{k} > 1), \text{ is valid as long as } (r_0/r_*)^{(1+\tilde{k})} (G M m / r_0 k T_\omega) < 1, \text{ and the external flow may be described as static. Given a static flow, this condition is satisfied when } r_0 \geq (\gamma - 1)^{(1+\tilde{k})} r_* \text{, but an essentially static configuration is possible only when }$$

$$\dot{M} \sim \dot{M}_{\text{BHL}} \approx (r_{\text{ac}}/r_*) \dot{M}_{\text{DC}}.$$

This holds when

$$r_0 > r_{\text{ac}}^{(1+\tilde{k})(5-3\gamma)/(3-\gamma)+(5-3\gamma)\tilde{k}} \left\{ r_*^{2(\gamma-1)/(3-\gamma)+(5-3\gamma)\tilde{k}} \right\} \gg r_*.$$

(21)

For simplicity, we shall assume $\tilde{k} = 0$ in what follows. Parallel arguments can easily be constructed for any $0 < \tilde{k} < 1$, and qualitative trends follow from the fact that $\dot{M}$ increases more steeply with $\rho_\infty \sigma_\infty r_{\text{ac}}$ as $\tilde{k}$ increases.

When $\gamma > \frac{4}{3}$, $r_0$ may assume all values down to the minimum given by (21), and the corresponding variation of $\dot{M}$ is

$$\dot{M} \approx \dot{M}_{\text{DC}} \left[ (\gamma - 1) (\rho_\infty \sigma_\infty r_{\text{ac}}) \right]^{(3-\gamma)/2(3\gamma-4)} \rho_\infty^{5(\gamma-1)/2(3\gamma-4)} T_\infty^{-5/2(3\gamma-4)}.$$

(22)
The maximum $\rho \sigma_\infty r_{ac}$ for which the static atmosphere approximation holds is

$$[\rho \sigma_\infty r_{ac}]_{\text{max}} \approx \left(\frac{r_{ac}}{r_\bullet}\right)^{2(3\gamma - 4)/(3 - \gamma)}$$

(23)

When $\rho \sigma_\infty r_{ac}$ exceeds this value, the predicted collisionless accretion rate exceeds $\dot{M}_{\text{BHL}}$ for the fluid at $r \rightarrow \infty$ — this situation cannot be maintained indefinitely. In fact, if $\rho \sigma_\infty r_{ac}$ is increasing slowly enough, we expect the static atmosphere gradually to become less steep, and the flow at $r > r_\bullet$ to resemble ordinary fluid accretion. However, we cannot be certain of this until we have checked the self-consistency of the proposed BHL flow.

Once a static atmosphere turns into an accretion flow, fluidity loses ground within the previously fluid region because the density run becomes less steep, but simultaneously gains ground in a system dominated by Coulomb collisions, because the temperature run also becomes shallower. However, this latter effect is partly offset by the fact that $\rho \sigma r > 1$ is no longer a good enough fluidity criterion where $r < r_{ac}$. Previously, we were concerned with systems where the thermal speed exceeded the streaming speed, and we were able to use $v_{th} \sim \sqrt{kT/m}$ to calculate both $t_{\text{coll}}$ and $t_{\text{dyn}}$. In an unimpeded accretion flow, collisions are still governed by $v_{th}$, but $\tilde{\sigma} \sim \sqrt{G\dot{M}/r}$ is generally supersonic when $r < r_{ac}$, and therefore governs $t_{\text{dyn}}$. If our fluidity criterion is to reflect $t_{\text{coll}} < t_{\text{dyn}}$, we should impose the stronger condition

$$\rho \sigma r > \frac{\tilde{\sigma}}{v_{th}} \sim \left(\frac{r_{ac}}{r}\right)^{(5 - 3\gamma)/2}$$

(24)

when $r < r_{ac}$. Again using $\rho \propto r^{-3/2}$, $T \propto \rho^{1/2}$ for BHL flow, we conclude that

$$[\rho \sigma_\infty r_{ac}]_{\text{max}} \approx \left(\frac{r_{ac}}{r_\bullet}\right)^{(3\gamma - 2)/2}$$

(25)

is a necessary condition for the fluid-like region to have a self-consistent accretion flow. In general, $\rho \sigma_\infty r_{ac} > [\rho \sigma_\infty r_{ac}]_{\text{max}}$ does not guarantee that (25) is satisfied, unless $r_\bullet > r_\bullet$. Thus, a self-consistent fluid-like region accreting at $\dot{M}_{\text{BHL}}$ is not possible unless (1) there is a large collisionless region as well; or (2) $\rho \sigma_\infty r_{ac} > [\rho \sigma_\infty r_{ac}]_{\text{max}}$. But we have seen (Section 3) that a collisionless zone is compatible with $\dot{M} \sim \dot{M}_{\text{BHL}}$ only if

$$\left(\frac{r_\bullet}{r_\bullet}\right) \frac{G M m}{r_\bullet k T_0} \geq 1$$

a condition which is inconsistent with $r_\bullet > r_\bullet$. Therefore, we are forced to conclude that for some large range of $\rho \sigma_\infty r_{ac} > [\rho \sigma_\infty r_{ac}]_{\text{max}}$ there is either a very special flow which can be characterized neither as collisionless nor as fluid-like, or there is no steady flow at all. These results are depicted in Fig. 2.

The $\gamma < 4/3$ case is equally problematic. Although model calculations suggest an externally-induced 'jelling' when $\rho \sigma_\infty r_{ac} \geq 1$, there is again no apparent state for the system to 'jell' to. As in the $\gamma > 4/3$ case, self-consistent BHL flow does not seem to be possible until $\rho \sigma_\infty r_{ac} > \left(\frac{r_{ac}}{r_\bullet}\right)^{(5 - 3\gamma)/2}$.

6 Astrophysical implications

Real gas near a star, either ionized or about to be ionized, will exhibit a plasma behaviour much more complicated than the simple Coulomb collision picture presented here. We have
Figure 2. Behaviour of weakly collisional gas with $\sigma \equiv \nu_{\text{rel}}^{-1}$ (Section 5). (a) Schematic density run for $\gamma > 4/3$, $1 < \rho_{\infty} \sigma_{\infty} r_{\infty} < (r_{\infty}/r_{\#})^{2(3\gamma-4)/(3-\gamma)}$, $r_{\infty} > R_{\infty} > r_{\#}$. Accretion rate intermediate between $\dot{M}_{\text{DC}}$ and $\dot{M}_{\text{BHL}}$. (b) Maximum $\rho_{\infty} \sigma_{\infty} r_{\infty}$ at which static atmosphere approximation holds (hence $\dot{M} \ll \dot{M}_{\text{BHL}}$), as function of $\gamma$, for various values of $(r_{\infty}/r_{\#})$ (equation 23). (c) Variation of $\dot{M}$ with $\rho_{\infty}$, $T_{\infty}$ fixed, $k = 0$, for various $\gamma$. Dashed line indicates that self-consistent steady flow pattern has not been found. Dot-dash line represents $\dot{M}_{\text{BHL}}$, which does not necessarily correspond to self-consistent flow. (d) Variation of $M$ with $T_{\infty}$, $\rho_{\infty}$ fixed, $k = 0$, for various $\gamma$. (e) Variation of $M/\dot{M}_{\text{DC}}$ with $\rho_{\infty} \sigma_{\infty} r_{\infty}$ for various $\gamma$. 
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not attempted to model a realistic system, but rather to highlight some of the dangers in assuming that any quasi-collisional behaviour, although it might seem to occur on small enough scales in the unperturbed medium, is capable of maintaining a fluid-like flow pattern all the way to the stellar surface. Our results indicate that Coulomb collisions will often be inadequate to guarantee fluid-like behaviour, and some other fluidizing agent, such as turbulence or tangled magnetic fields, must be invoked. Consider two often-used arguments for justifying the assumption of fluidity in an accreting plasma. A star travelling supersonically through interstellar plasma focuses a sheath of material into a conical region in its wake. Electrostatic instabilities prevent the interpenetration of plasma streams even when the plasma is collisionless, and the angular momenta of the converging streams cancel. It is then argued that the newly thermalized plasma will accrete onto the star at the rate predicted by the BHL theory. But no mechanism is given for keeping the plasma within the ‘accretion column’ fluid-like as it accretes, and we have seen that Coulomb collisions may be insufficient, even if the plasma is still Maxwellian at a radius well inside $r_{ac}$. Another argument invokes magnetic fields tangled on scales smaller than $r_{ac}$. A mean free path is provided by the scale length of field reversal. But unless there is some means of shearing the gas on progressively smaller scales as it accretes, it is not clear that the field reversal scale will remain smaller than $r$.

The strong dependence on $\gamma$ of the qualitative features of Coulomb gas accretion, suggests the possibility of a secular instability. We have seen that accretion flows greatly enhanced over $\dot{M}_{DC}$ but still basically collisionless, are possible only when $\gamma > \frac{2}{3}$. While one of these flows is occurring, there is a static atmosphere ‘trapped’ between $r_0$ and $r_{ac}$, ‘waiting’ to be accreted. If heat can conduct across this distance more rapidly than an average particle can travel from $r_{ac}$ to $r_0$ as part of the slow accretion flow, then we may expect a $\gamma > \frac{2}{3}$ system to tend towards a $\gamma < \frac{2}{3}$ configuration on the thermal conduction timescale. Once $\gamma = \frac{2}{3}$, the system may become unstable, and dump most of the trapped material onto the star at once. Rid of its isothermal component, the gas would revert to $\gamma > \frac{2}{3}$ and the process would begin anew. In addition, the apparent absence of self-consistent fluid-like or partially collisionless flows for any $\gamma$ over a wide range in $\rho_{\infty} \sigma_{\infty} r_{ac}$, makes plausible flows which are unsteady on dynamical timescales.

Finally, we cite the deliberate omission of any discussion pertaining to the reverse transition: how does a BHL flow revert to a collisionless flow when $\rho_{\infty} \sigma_{\infty} r_{ac}$ decreases? The treatment is complicated by the presence of large streaming velocities in the initial state. Since the density run is everywhere less steep than in the corresponding static atmosphere, it is
conceivable that the reverse transition may not occur under exactly the same conditions as
the forward transition. In the hard sphere case, it is even possible that the 'self-jelling'
behaviour of the collisionless distribution may play a much more decisive role in postponing
the reverse transition than it did in triggering the forward one.

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Appendix: Effect of close encounters in a stellar cusp

This discussion is the product of a collaboration between the author and J. Frank.

Suppose there is a slow inward drift of bound stars, characterized at each \( r \) by a time-
scale \( t_{\text{diff}}(r) \). This flux \( \sim \rho r^3/t_{\text{diff}} \) varies with \( r \) due to the unbinding of stars in occasional
close encounters. Let the timescale for these hard collisions be \( t_{\text{coll}}(r) \), and let the fraction of
stars unbound in each \( t_{\text{coll}} \) be \( \tilde{d} < 1 \). Then, mass conservation gives

\[
\frac{d}{dr} \left( \frac{\rho r^3}{t_{\text{diff}}} \right) = \frac{\rho r^2 \tilde{d}}{t_{\text{coll}}}. \tag{A1}
\]

While particles flow at a rate \( t_{\text{diff}}^{-1} \), energy flows at a rate given by the relaxation time,
\( t_R \sim t_{\text{coll}} \log \rho r^3 \), where \( \rho r^3 \) is approximately the number of stars contained within \( r \). Since
\( t_R < t_{\text{coll}} \), energy is transported primarily by diffusion, and the energy flux \( (\rho r^3/t_R)(GM/r) \)
must satisfy

\[
\frac{r}{t_R} \left( \frac{\rho r^3 GM}{r} \right) = -\frac{\rho r^2 \tilde{d}}{t_{\text{coll}}} \frac{GM}{r} \left( 1 + \tilde{k} \right) \frac{GM}{r} \tag{A2}
\]

where \( \tilde{k}(GM/r) \) is the average positive energy carried away by an escaper from radius \( r \).

Because we know a priori the relation between \( t_R \) and \( t_{\text{coll}} \), we may solve equation (A2)

\[
\frac{d}{dr} \left( \frac{\rho r^2}{t_R} \right) = -\left( \frac{\rho r^2}{t_R} \right) \frac{\tilde{d}(1 + \tilde{k})}{r \log \rho r^3} \tag{A3}
\]

by ignoring variations in \( \log \rho r^3 \), to obtain

\[
\frac{\rho r^2}{t_R} \propto \tilde{d}(1 + \tilde{k})/\log \rho r^3. \tag{A4}
\]
In this approximation, $t_{R} \propto t_{\text{coll}} \propto 1/\rho(r) \sigma(r) v(r) \propto r^{-3/2} \rho^{-1}(r)$ where we have used the large-angle scattering Coulomb cross-section $\propto v_{\text{rel}}^{-4} \propto r^2$. Substitution into (A4) yields

$$\rho(r) \propto r^{-7/4} - d(1 + \tilde{k})/2 \log \rho r^3.$$  \hfill (A5)

The effect of energetic unbindings on the maximum value of $\tilde{k}$ may be estimated as discussed in the text. For an isotropic velocity distribution,

$$\tilde{k} = \eta \frac{5 - 2\rho}{3 - 2\rho}$$

where $\eta \leq 1$. Substituting $-2\rho = -\frac{1}{2} - d(1 + \tilde{k})/\log \rho r^3$, we obtain

$$\tilde{k} = \eta \left( \frac{9 - 2d(1 + \tilde{k})/\log \rho r^3}{5 - 2d(1 + \tilde{k})/\log \rho r^3} \right).$$

Formally, we find two solutions for $\tilde{k}$. One, with $\tilde{k} \sim 0(\log \rho r^3/d) > 1$, may be dismissed as unphysical because it gives $\rho(r) \propto r^{-3}$, and hence a non-conservative, divergent mass flux. The physically reasonable solution, when $\eta d/\log \rho r^3 \ll 1$, is $\tilde{k} \approx \eta d/5$. When $\eta < 1$ or $d \ll 1/2$, the correction due to hard collisions may be small over a wide range in $\log \rho r^3$. However, if $\eta, d \sim 0(1)$, the additional steepening will be large if there are too few stars in the cusp. In particular, if $d(1 + \tilde{k})/2 \log \rho r^3 \rightarrow 0(1)$, the diffusion equations are no longer a good enough first approximation, and the system may be better treated to lowest order as a gas of ‘Coulomb billiard balls’. In this case, the density run may be as steep as the Peebles solution, $\rho(r) \propto r^{-9/4}$. Serious difficulties may be encountered with the diffusion model even when there are as many as 50–100 stars in the cusp.