Relativistic parametric instabilities in extended extragalactic radio sources

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Summary. A general discussion is presented of parametric instabilities of electromagnetic waves in cold plasmas. Previous results for $f = E\omega_e/m_e c \omega_0 > 1$ and $\ll 1$ are extended and the intermediate range $f \sim 1$, which could be relevant in some astrophysical applications, is analysed by numerical techniques. In the final section a model for particle acceleration and radiation emission by turbulent plasma modes excited in extended radiosources by parametric absorption of strong electromagnetic waves is tentatively discussed.

1 Introduction

A most crucial problem in theoretical astrophysics is the non-linear propagation of strong electromagnetic waves in plasmas when the ratio of their amplitude to frequency is high enough ($f = E\omega_e/m_e c \omega_0 \gtrsim 1$, where $m_e$ is the electron mass) to induce relativistic motions of electrons (and also ions in extreme cases for which $f > m_i/m_e$). Obviously high-energy processes occurring in astrophysical objects, such as pulsars, extragalactic radiosources, quasars, etc., differ from laboratory experiments (i.e. plasma heating by radiofrequencies and laser-induced thermonuclear fusion) mainly because of the high non-linearity of wave amplitude effects in relativistic regimes. For instance, it is a general theoretical and experimental (laboratory) result that large-amplitude electromagnetic waves can produce turbulence when interacting with a plasma. But only recently a number of investigations has been devoted to the analysis of conditions in relativistic regimes which would account for a full development of plasma turbulence and thence particle acceleration and radiation emission in active astrophysical objects.

In particular Max & Perkins (1972) have studied relativistic parametric interactions of circularly polarized waves in cold plasmas in connection with the propagation and absorption of the low-frequency waves of pulsars when crossing supernova shells.

Similarly a relativistic theory has been applied by Dobrowolny & Ferrari (1976) to the formation of ‘wisp’ structures in the Crab Nebula as produced by plasma turbulence arising from interaction of a linearly polarized, strong electromagnetic wave ($f \gtrsim 1$) emitted by the
pulsar NP0532 with a particle wind accelerated off its surface. In this same framework, Tsintsadze & Tsikarishvili (1976) and Drake, Lee & Tsintsadze (1976) have published results on the interaction of weakly relativistic waves \( f \ll 1 \) with plasmas at relativistic temperatures, that should support the existence of fully-developed plasma turbulence in supernova remnants, extragalactic radio sources or galactic nuclei.

In the present paper we report on the reciprocal situation in which parametric interactions arise when a linearly polarized wave with \( f \sim 1 \) irradiates a plasma of electrons and protons whose thermal energies are negligible as compared with that acquired in the wave field.

In Section 2 we discuss the most relevant results of the non-relativistic theory, and the general trend of relativistic effects is introduced in the limits \( f \ll 1 \) and \( f \gg 1 \) (this last case for pure electron plasmas). In Section 3 the linear equations for the fully relativistic regime are derived, and typical choices of electric and magnetic vectors and wavelengths of the external wave are discussed.

Results of the numerical study of these equations are presented in Section 4. Two kinds of longitudinal unstable modes are shown to overlap in the region \( f \sim 1 \): a typically relativistic mode related to electron charge oscillations and the relativistic limit of the classical ion–electron modes derived by Silin. This second kind tends to disappear for larger \( f \), while at the same time the electron mode survives only for frequencies which are marginal to the dispersion curve of non-linear waves in cold plasmas. In addition, electromagnetic unstable modes also exist, which might be relevant for radiation emission. Finally a tentative application of these results to a model for extended radio sources is presented in the last section.

2 Non-relativistic and weakly-relativistic instabilities

The non-relativistic theory of parametric interaction of electromagnetic waves irradiating cold plasmas has been thoroughly developed by Silin for the case of linearly polarized pumps (Pustovalov & Silin 1975, and references therein). The same process has also been treated by Nishikawa (1968) in the scheme of a three-wave interaction, in which one of the waves is the energy carrier at constant amplitude. A crucial assumption in these treatments is the so-called ‘dipole approximation’ for the driving wave, \( |k_0| \approx 0 \), which allows its Lorentz force to be neglected.

Relevant results of non-relativistic theory can be summarized as follows:

1. A linearly polarized electromagnetic wave drives purely electrostatic oscillations in cold plasma at two frequencies close to \( \omega_{pe} \) and \( \omega_{pi} \), the electron and ion Langmuir frequencies respectively. The high-frequency mode corresponds to electron oscillations dragging ions; the low-frequency mode, corresponding to ion oscillations dragging electrons, is a proper mode for hot plasmas (here it is excited by the external pump).

2. A parametric amplification of these two modes by three-wave resonant interaction with the driver occurs when the driver’s frequency is close to \( \omega_{pe} \). Growth rates depend on the parameter \( a = 2\pi r_e/\lambda \), the ratio between \( r_e \approx eE_0/m_e \omega_0^2 \) (the electron excursion in the driver’s field), and \( \lambda \) (the wavelength of high-frequency oscillations). They have a maximum \( (\Gamma_{\text{max}} \sim 0.02\omega_{pe}) \) for \( a \sim 1 \); for \( a > 1 \) and \( \omega_0/\omega_{pe} > 1 \), high- and low-frequency modes are decoupled.

3. Instabilities are characterized by a strict coupling between ion and electron modes, the ratio between absolute values of electron and ion mode amplitudes ranging from 5 to 20.

4. Instability is present, for linearly polarized pumps, only for wave-vectors \( k \) not perpendicular to the electric field of the wave; growth rates are maximal when \( k || E_0 \), and identically zero for \( k \perp E_0 \).
(5) For circularly polarized pumps the general picture is conserved, since growth rates are again determined by the component of $\mathbf{E}_0$ parallel to $\mathbf{k}$.

This instability is not modified as long as the wave-strength parameter is much less than unity, namely $f^2 < 1$; one only finds a progressive ‘widening’ of the region of unstable frequencies to the range $\omega_\text{pe}^* < \omega < \omega_\text{pe}$, where $\omega_\text{pe}^* = \omega_\text{pe}(1 - 3/16f^2)$ is the relativistic cut-off frequency (Kaw & Dawson 1970; Bourdier et al. 1975; Tsintsadze & Tsikarishvili 1976).

On the other hand, new parametric interactions may develop since typically relativistic effects arise if: (1) the electron inertia varies during oscillatory motions in the pump’s field; (2) the Lorentz force $(\mathbf{v}/c) \times \mathbf{B}$ of the perturbations becomes comparable with the electrostatic force (that of the pump was neglected in the dipole approximation). In the approximate model of a cold electron plasma and linearly polarized pump, Tsintsadze (1970) and Dobrowolny, Ferrari & Bosia (1976) have shown that uncoupled electrostatic and electromagnetic unstable modes, both with wave vectors $\mathbf{k} \parallel \mathbf{E}_0$ and $\mathbf{k} \perp \mathbf{E}_0$, are excited by electron-mass oscillations. In addition, when $\mathbf{k} \perp \mathbf{E}_0$, coupling of electrostatic and electromagnetic modes by current oscillations induced by the Lorentz force term occurs for $kc/\omega_0 > 1$. This last instability occurs also for circularly polarized pumps, that do not provide inertial mass oscillations (Max & Perkins 1972).

### 3 Basic equations and approximations for a fully relativistic theory

The interaction of an electromagnetic wave with cold plasma is described by Maxwell’s equations plus continuity and motion equations for the plasma components, electrons and ions. A linear analysis of the conditions for the generation of unstable modes is followed, assuming:

\[
\begin{align*}
\mathbf{E} &= \mathbf{E}_0 + \mathbf{E}_1 \\
\mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1 \\
\mathbf{v}_\alpha &= \mathbf{v}_{0\alpha} + \mathbf{v}_{1\alpha} \\
n &= n_0 + n_1
\end{align*}
\]

(1)

where $\alpha$ indicates the two plasma components; perturbed quantities $\mathbf{E}_1$, $\mathbf{B}_1$, $\mathbf{v}_1$ are assumed with spatial dependence of the form $\exp(ik \cdot r)$. The electromagnetic field of the pump is taken as purely sinusoidal in the dipole approximation

\[
\mathbf{E}_0(t) = \mathbf{E}_0 \sin \omega_0 t
\]

independent of space variations; then $|\mathbf{B}_0| = (ck_0/\omega_0)E_0 \approx 0$ also, in the limits we further discuss below.

The linearized system of fully relativistic equations describing the interaction:

\[
\begin{align*}
&i\mathbf{k} \times \mathbf{B}_1 = -\frac{1}{c} \frac{\partial \mathbf{E}_1}{\partial t} + \frac{4\pi}{c} \sum_\alpha e_\alpha (n_{0\alpha} \mathbf{v}_{1\alpha} + n_{1\alpha} \mathbf{v}_{0\alpha}) \\
&i\mathbf{k} \times \mathbf{E}_1 = -\frac{1}{c} \frac{\partial \mathbf{B}_1}{\partial t} \\
&i\mathbf{k} \cdot \mathbf{E}_1 = 4\pi \sum_\alpha e_\alpha n_{1\alpha}, \quad i\mathbf{k} \cdot \mathbf{B}_1 = 0 \\
&\frac{\partial n_{1\alpha}}{\partial t} = -i(k \cdot \mathbf{v}_{0\alpha} n_{1\alpha} + k \cdot \mathbf{v}_{1\alpha} n_{0\alpha}) \\
&\left[ \frac{\partial}{\partial t} + i(k \cdot \mathbf{v}_{0\alpha}) \right] \left[ \frac{\mathbf{v}_{1\alpha}}{(1 - u_{0\alpha}^2/c^2)^{3/2}} + \frac{\mathbf{v}_{0\alpha}}{c^2 (1 - u_{0\alpha}^2/c^2)^{3/2}} \right] = \frac{e_\alpha}{m_\alpha} \left[ \mathbf{E}_1 + \frac{1}{c} (\mathbf{v}_{0\alpha} \times \mathbf{E}_1) \right]
\end{align*}
\]

(3)
The velocities $v_{0\alpha}$ in the unperturbed state are obtained from the integration of the equations of motion at zero-order:

$$
\frac{v_{0\alpha}}{c} = \left[ \left[ \frac{m_e}{m_\alpha} \int f \cos \omega_0 t \right] \right] / \left[ 1 + \left( \frac{m_e}{m_\alpha} f \cos^2 \omega_0 t \right)^{1/2} \right] \hat{x}
$$

where $\hat{x}$ is the direction of the electric field of the driver $E_0$. Actually a fully self-consistent zero-order solution, including all Maxwell's equations, should be used, but this would introduce slight corrections only of the final growth rates. A self-consistent solution for $f^2 < 1$ has been derived by Bourdier et al. (1975).

In the relativistic equations (3) the Lorentz force is non-negligible ($v \approx c$); however, if we assume perturbations with $k || E_0$ which give the largest growth rates, it is exactly null in the electrostatic case because $B_1 = 0$ for longitudinal irrotational electric fields. Then decoupled equations can be easily obtained for longitudinal and transverse oscillations:

(a) longitudinal oscillations:

$$
\begin{align*}
\gamma_{0e} \nu''_e + (\gamma_{0e})' \nu'_e + \omega_{pe}^2 \left[ \nu_e + \nu_i \exp [ik \cdot \int (v_{0e} - v_{0i}) \, dt] \right] &= 0 \\
\gamma_{0i} \nu''_i + (\gamma_{0i})' \nu'_i + \omega_{pi}^2 \left[ \nu_i + \nu_e \exp [ik \cdot \int (v_{0i} - v_{0e}) \, dt] \right] &= 0
\end{align*}
$$

where $\nu_\alpha = n_{1\alpha} \exp (ik \cdot \int v_{0\alpha} \, dt)$ is a function of perturbed charge densities mathematically suitable for removing zero-order oscillations; $\gamma_{0a} = (1 - v_{0a}^2/c^2)^{-1/2}$ are Lorentz factors associated with motion in the pump field;

(b) transverse oscillations:

$$
B_1'' + \left( \frac{k^2 c^2 + \omega_{pe}^2}{\gamma_{0e}} + \frac{\omega_{pi}^2}{\gamma_{0i}} \right) B_1 = 0
$$

which involve purely electromagnetic perturbations.

The dipole approximation for the driving field implying a space independent zero-order state is exactly true at the cut-off frequency of the dispersion relation, so that our results will be strictly correct at the limit of transparency of the pump only. Outside the transparency curve the spatial dependence of the electric field should be considered. However, for perturbations developing over regions much smaller than the wavelength of the pump, i.e. $|k| \gg |k_0|$, the electric field is almost space-independent over distances $\lambda \approx 2\pi/k$, so that expression (2) for $E_0$ is still valid. Nevertheless it must be recalled that, $|k_0|$ being no longer zero, the pump magnetic field is also finite and therefore can give rise to a non-negligible Lorentz force $(v/c) \times B_0$ when $v \approx c$. This induces charge oscillations along the propagation vector $k_0$ (Akhiezer & Polovin 1956; Kaw & Dawson 1970) and a consistent electrostatic field is required to balance the charge density perturbations. Nevertheless this effect is negligible, i.e. the driver can be assumed 'quasi-transverse', if $f < (\omega_{pe}/\omega_0)^2 v_{ph}/c$ where $v_{ph}$ is the phase velocity. It can be easily seen that in our problem this condition is always satisfied. Concerning the instability analysis a power expansion of the equations in the smallness parameter $f(c/v_{ph})(\omega_0/\omega_{pe})^2$ shows that this longitudinal field would not affect growth rates by more than 1 per cent; the alternative possibility of new types of instability connected with this field is not examined in the present paper.

Another important effect in the relativistic theory is that linearly polarized strong waves show a sawtooth shape for $f > 1$ implying higher-order harmonics of decreasing amplitude.
This basic problem has been already discussed by Dobrowolny et al. (1976); higher harmonics do not essentially modify the form of equations (5)–(6) but slightly change their coefficients. This leads to the consequence that: (1) the growth rates of any instabilities at the fundamental harmonic are slightly modified; (2) instabilities at higher-order harmonics may appear. By the numerical method used in the solution of equations (5)–(6), we could verify that for \( f \approx 1 \) higher-order harmonics give variations of \( \sim 8 \) per cent at most, being proportional to the three-halves power of the product of the decreasing amplitudes of successive harmonics: then growth rates of instabilities at the fundamental frequency become influenced at a \( 2 \)–\( 3 \) per cent level. We did not analyse higher order instabilities, but on the same basis we may say that corresponding growth rates should always be smaller by factors which depend on numerical combinations of the amplitudes of harmonics.

Concerning the pump frequency \( \omega_0 \) it must always be \( \geq \omega_{pe}^* \) for penetration of the electromagnetic wave inside the plasma, where \( \omega_{pe}^* \) is the relativistically correct electron plasma frequency (Bourdier et al. 1975; Ferrari, Trussoni & Zaninetti 1975). Actually we shall examine the region close to \( \omega_{pe}^* \) only, since previous works (Pustovalov & Silin 1975; Ferrari et al. 1975) have already shown that no instability sets in for \( \omega_0 \) much larger than the cut-off frequency, and also partially extend the analysis to the region below \( \omega_{pe}^* \).

4 Relativistic instabilities of linearly polarized electromagnetic waves

Coupling, uncoupling and parametric amplification of cold plasma proper modes excited by external pumps are determined from the charge density perturbations \( \nu_e \) and \( \nu_i \) in equations (5) and (6) after suitable transformation to the observer’s frame. We developed a numerical method for deriving oscillation frequencies and thence growth rates; it was tested on a study of the instabilities already discussed in the classical and weakly relativistic limits. We shall divide the discussion of our results into four parts for different types of unstable modes: (1) longitudinal Silin-type modes with \( k \parallel E_0 \); (2) longitudinal electrostatic electron modes with \( k \parallel E_0 \); (3) transverse electromagnetic modes with \( k \parallel E_0 \); (4) extension of the theory to hot plasmas and modes with \( k \perp E_0 \).

4.1 Longitudinal Silin-Type Instabilities

We know from Silin that high-frequency electron modes and low-frequency ion modes, both electrostatic and with \( k \parallel E_0 \), can be coupled parametrically to the driver when the wavelength of the high-frequency perturbation is of the order of the electron excursion \( r_e \) in the driving field, i.e. \( a = 2 \pi r_e / \lambda \); the coupling is also led by the parameter \( f \). For \( f < 1 \), i.e. \( \gamma_0a \approx 1 \), the set of equations (5) is very similar to the one derived by Bourdier et al. (1975) using slightly different model approximations. Our numerical analysis showed that perturbations possess a general trend \( \nu_e \propto \exp[-i(\Gamma + \omega_0)t], \nu_i \propto \exp(-i\Gamma t) \), where the imaginary part of \( \Gamma \) is the instability growth rate; the real part, \( \Re \Gamma = \omega_L \approx \omega_{pi} \), gives a low-frequency oscillation component (coupled to the driving high-frequency \( \omega_0 \)) which is evidenced in \( \nu_i \). This low-frequency mode is the proper mode for ions in hot plasmas that we already discussed in Section 2; therefore the process is a three-wave interaction between the strong pump and the proper modes. The ion and electron mode coupling may be seen in that the amplitude ratio \( \nu_e/\nu_i \) is low, always less than \( \sim 20 \), but typically close to \( \sim 5 \). The instability growth rates depend on \( \omega_0 \), and \( a \) also appears in the coupling term \( \exp(-ia \sin \omega_0 t) \) as one easily obtains from (5). Maximum resonance is for \( \omega_0 \approx \omega_{pe}^* \) (so that electrons move with the pump and thence drag the ions) and for \( a \approx 1 \); in fact for \( a \ll 1 \) the exponential
term is essentially constant during one oscillation period \( \sim 2\pi/\omega_0 \) and thence poorly effective in exciting oscillations, while for \( a > 1 \) it is null on the average over the same scale \( 2\pi/\omega_0 \), so that the ion and electron modes are decoupled.

When \( f \) is increased above unity and the electron velocity approaches \( c \), we observe a progressive weakening of the resonant interaction. This is again due to the coupling terms:

\[
\frac{\nu_\alpha}{\gamma_{0\alpha}} \exp \left[ \pm ik \cdot \int \left( v_{0e} - v_{0i} \right) dt \right] = \frac{\nu_\alpha}{\gamma_{0\alpha}} \exp \left[ \pm i \frac{k_c}{\omega_0} \left[ \arcsin \frac{f \sin \omega_0 t}{(1 + f^2)^{1/2}} - \arcsin \frac{m_e f \sin \omega_0 t}{m_i \left( 1 + \frac{m_e^2}{m_i^2} f^2 \right)^{1/2}} \right] \right]
\]

(7)

which for \( 1 < f < m_i/m_e \) become:

\[
\frac{\nu_{\alpha e}}{\gamma_{0\alpha}} \exp \left[ \pm i a \left( \frac{\omega_0 t}{f} - \frac{m_e}{m_i} \sin \omega_0 t \right) \right] \approx \frac{\nu_\alpha}{\gamma_{0\alpha}} \exp \left[ \pm i a \frac{\omega_0 t}{f} \right].
\]

(7')

Then for \( a/f \gg 1 \) or \( \ll 1 \) the same arguments apply that were used above for \( a \); however, for \( a/f \sim 1 \), the coupling term in the electron equation is \( (\nu_1/\gamma_{0\alpha}) \cos \omega_0 t \) and goes to zero in the ion equation \( (\gamma_{0e} \gg 1) \): therefore no resonance is possible between the two phases, which oscillate at widely separated frequencies.

This mathematical result has a straightforward physical interpretation: mode coupling occurs when electrons 'feel' the high-frequency electrostatic perturbations as an almost constant field while moving along unperturbed orbits in the driver and then can exchange energy with them. This meant \( 2\pi r_e \sim \lambda, \) or \( a \sim 1 \), in the non-relativistic regime when \( r_e = eE_0/m_e \omega_0^2 \); in the relativistic range the excursion becomes \( r_e^* = E_0 f \), which tends to a large value \( c/(4\omega_0) \) for \( f > 1 \). Then electrons do not feel the high-frequency perturbations during their zero-order excursion, but simply average them to zero. The effect is quite clear in numerical results for the growth rates represented in Fig. 1; consistently one can see in Fig. 2 a shrinking of the range of unstable frequencies \( \omega_0 \). For \( f > 1 \) the instability (in the limit \( k_0 = 0 \)) eventually disappears.

Another effect to point out is that below the transparency limit \( \omega_0 \leq \omega_{pe}^* \) (although this range of frequencies has weak physical significance), ionic low-frequency oscillations at \( \omega_L \) completely disappear, as happened in Silin's theory, and ion charge density has an exponentially increasing pattern while electrons steadily oscillate at \( \sim \omega_{pe} \).

4.2 LONGITUDINAL ELECTRONIC INSTABILITY

This type of instability, already discussed by Tsintsadze (1970) and Dobrowolny et al. (1976) in the approximation of fixed ions, is typically relativistic and sets in when electrons approach the speed of light during their oscillations in linearly polarized pumps. Then the value of their inertial mass oscillates also and this produces parametric coupling of the pump with electrostatic perturbations \( k \parallel E_0 \) when \( \omega_0 \approx \omega_{pe}^* \). The coupling is unstable in a narrow region along the transparency curve for \( \omega_0 > \omega_{pe}^* \) (0.4 < \( f < 1.1 \)) and is always present for \( \omega_0 \leq \omega_{pe}^* \).

In order to trace this instability in the numerical solutions one can obviously rely on the fact that it acts typically on electrons. Namely we expect that it is evidenced by the amplitude ratio \( \nu_e/\nu_1 \) becoming much larger than in the case of ion-electron instabilities, since ions cannot properly follow high-frequency perturbations as long as \( f < m_i/m_e \). Referring to the case for \( a \sim 1 \) of Fig. 1 corresponding to large growth rates, our results can be summarized as follows. For \( f = 10^{-6} \) the Silin-type instability is always prevailing: the growth
Figure 1. Growth rates of electrostatic unstable modes.

Figure 2. Regions of prevalence of the two types of unstable electrostatic modes (dotted region: purely electronic instability; dashed region: ion-electron Silin instability). The heavy line is the dispersion relation $\omega = \omega_{pe} (k_o = 0)$ as drawn from Dobrowolny et al. (1976).
rate shows two branches with a point of minimum at \( \omega_0 = \omega_{pe} \), in agreement with previous analytical results; in addition \( \nu_e/\nu_i < 20 \) for both branches, as we discussed already. Then from \( f \sim 0.3 \) the left branch (\( \omega_0/\omega_{pe}^* > 1 \)) starts decreasing, while the right branch becomes steeper, so that the minimum is slightly shifted towards the left: this right branch certainly includes the electronic instability since the ratio \( \nu_e/\nu_i \) suddenly increases. Eventually for \( f > 0.5 \) the electronic instability takes over the right branch definitely and \( \nu_e/\nu_i \sim 100 \). For \( f \sim 1 \), we already know that the Silin-type instability fades away, so that the growth rate in Fig. 1 contains only the contribution from the electronic instability. In this respect we could verify that the previous results by Dobrowolny et al. (1976) were not substantially influenced by ion motions, at least for \( f < m_i/m_e \). Finally the regions of prevalence of the two types of instability in the plane \( (\omega_0/\omega_{pe}^*, f) \) are shown in Fig. 2.

It is worthwhile to note that the relativistic electron instability is actually present also for very small values of \( f \), but only on the right side of the dispersion relation of Fig. 1, i.e. for \( \omega_0 < \omega_{pe}^* \sim \omega_{pe} \), so that no real instability can be claimed.

Although we have not explored regions of very high \( f \)'s, we expect that an ion instability of this electrostatic type arises around \( f \sim m_i/m_e \); it is also quite likely that it shows again a marginal character, so that only a very limited range of \( f \) is unstable.

4.3 Transverse Electromagnetic Instability

Electromagnetic unstable modes with \( \mathbf{k} \parallel \mathbf{E}_0 \) are found in the relativistic regime, which are again related with oscillations of inertial mass in relativistic pumps affecting the refractive index of propagation.

A most interesting point is that in linearly polarized pumps these transverse modes are completely decoupled from longitudinal oscillations. They were already derived for fixed ions; the presence of ion motions introduces an obvious change of the oscillation frequency (see equation (6)) to

\[
\left( \frac{\omega_{pe}^2}{\gamma_0 e} + \frac{\omega_{pl}^2}{\gamma_0 i} \right)^{1/2}.
\]

Since for \( 1 < f < m_i/m_e \), \( \gamma_0 e \gg 1 \) and \( \gamma_0 i \sim 1 \), the ion contribution is somewhat important in fixing the resonant frequencies. Growth rates are given in Fig. 3, taken from Dobrowolny et al. (1976).

4.4 Extension of Results to Transverse Propagation and Hot Plasmas

The above results refer to perturbations with propagation vector \( \mathbf{k} \) parallel to the electric vector \( \mathbf{E}_0 \). For different geometrical configurations the linearized system (3) cannot be reduced to the simple components (5)–(6). However, when \( \mathbf{k} \perp \mathbf{E}_0 \), we can again separate the system (2) in two independent components, describing respectively purely electromagnetic waves and mixed longitudinal–transverse oscillations. For the first ones the above considerations of case (3) are still applicable; their electric vector oscillates perpendicularly to the plane determined by \( \mathbf{k} \) and \( \mathbf{E}_0 \).

For the second solution we may say that the coupling between longitudinal and transverse modes is due to the Lorentz force exerted by the perturbed magnetic field \( \mathbf{B}_1 \) on charge density perturbations; however, for \( ke \ll \omega_{pe} \) this Lorentz term becomes negligible and the equation can be split into another transverse mode with electric vector oscillating along \( \mathbf{E}_0 \) plus a longitudinal mode. These two modes are purely electronic and hence unstable in relativistic regime only, i.e. for \( f > 0.3 \). We recall in this respect that, for circularly polarized
driving waves, coupling of longitudinal and transverse modes is alternatively always present, since no solution exists related to mass oscillations but only to current pinches.

Finally, the effects of temperature on the above parametric instabilities should be commented on; all relevant results may directly derived from two papers by Drake et al. (1976) and Tsintsadze & Tsikarishvili (1976). First, as long as the electron temperature is low, i.e. $kT_e \ll m_e c^2$, a warm plasma behaves as a cold one in relativistic conditions; an extension of all results is algebraically straightforward. However, the relativistic electron instability exists above the transparency limit also for $f \gg 1$ (Drake et al. 1976).

Conversely, when $kT_e \gg m_e c^2$, excitation of unstable Silin-type modes is favoured, as low-frequency ion modes are already present, and their coupling with high-frequency electron modes is possible independently of the value of the ratio $\alpha$, and for $f$ as large as $\sim (kT_e/m_e c^2)$. However, the minimum wavelength allowed is $\lambda = \lambda_{De} = (kT_e/4\pi e^2 n_e)$, and only values of $\alpha \gg (\pi/2)\omega_{pe}^*/\omega_0$ are consistent with the plasma theory ($\omega_{pe}^*$ is here the electron plasma frequency corrected for the relativistic temperature) in the sense that too small wavelength instabilities would be rapidly damped. Concerning the relativistic electron instability, no substantial effect arises even though the inertial mass of electrons already increased by thermal motions requires larger $f$’s for them to start oscillating; growth rates are then lower than in the cold plasma case. Again a Debye length effect must be considered, which allows only perturbations $\lambda > v_{th,e}/\omega_{pe}^*$ to be excited.

5 Astrophysical applications

Results of our relativistic theory lead to the general conclusion that a large-amplitude electromagnetic wave, when crossing a plasma above transparency cut-off, is subject to strong parametric instabilities, some of which involve ions and electrons to a similar extent, while others mainly electrons. At relatively low amplitudes ($f < 1$) the two kinds of instabilities coexist; for large amplitudes ($f > 1$) high-frequency electron modes only are excited in hot plasmas, although ions are also dragged to some extent and eventually may become electrostatically unstable when $f > m_i/m_e$. This has interest for the astrophysical problem of generation of turbulence, and particle acceleration by strong waves in radio galaxies and supernova remnants.
A. Ferrari, E. Trussoni and L. Zaninetti

Concerning the generation of plasma or hydromagnetic turbulence, we recall that in the case of the Crab Nebula, Dobrowolny & Ferrari (1976) have shown how a parametric interaction between the strong low-frequency wave and the particle wind, both emitted by the pulsar in the scheme of the oblique rotator model, could be the initial step for the formation of turbulent bright structures at the inner edges of the nebula, where for standard models $f \sim 1$. We can now add to previous results that, owing to Silin-type instability (in regions where $f < 1$) and electrostatic instability (where $f > m_i/m_e$), ions also could absorb energy directly from the pump, although at a lower rate than electrons (see for instance non-linear simulations by De Groot & Katz 1973; Thomson et al. 1974); nevertheless this could lead to hydromagnetic activity by various wave couplings and thence to direct high-energy particle generation.

A similar pattern may be tentatively applied to electromagnetic continuous-supply theories for the extended blobs of extragalactic radio sources (for a general review see Rees 1976), again showing bright spots and turbulent activity. Current models for these objects suggest that a central region, observed in the visible as a bright galaxy or a QSO, continuously emits jets of relativistic particles and possibly low-frequency electromagnetic waves: this flow and its ejection depend on the model of the central active region, which has been proposed to be a single massive object or, alternatively, a galactic nucleus powered by recurrent supernova explosions and pulsar emission. Recently Ozernoy & Usov (1977) have discussed the regular variability of quasars and galactic nuclei, suggesting that they behave like compact rotating magnetoids: their intrinsic structure could be either a black hole with an accreting optically thick magnetized corona or a many-pulsar system with collective properties. However, as far as our interpretation is concerned, these magnetoids would emit large-amplitude electromagnetic waves and generate recurrent outflows of relativistic matter: both rotation and matter ejection would become apparent from their optical variability. In a scheme of this kind, where particles and waves are present in jets expelled from parent galaxy, parametric instabilities can be discussed as the basis of a mechanism of formation of the radio blobs, and their radiation by turbulent activity.

We start by noticing that a cold plasma model, as used in previous Sections, is legitimate, since thermal speeds are likely not to be relevant in jets which are dominated by flow velocities. In addition we are also allowed to examine in principle one-dimensional models, since wave-plasma streams appear to be mainly stable in relativistic regimes, both from a hydrodynamical point of view against Kelvin–Helmholtz and Rayleigh–Taylor instabilities (Turland & Scheuer 1976; Blandford & Pringle 1976; Ferrari, Trussoni & Zaninetti, in preparation) and from a non-linear electromagnetic point of view against self-modulation and self-focusing (Max, Arons & Langdon 1974; Dobrowolny, Ferrari & Massaglia, in preparation).

Then we propose that beams emerge from the central galaxy with an almost monochromatic distribution of high-energy particles and large-amplitude plane waves in steady configuration. A definition of the values of the parameters for these beams is rather difficult due to lack of complete observational data. We shall assume a standard radio source as having a parent galactic nucleus of dimensions $R_g \approx 10$ pc and radio regions of larger sizes $R_b \approx 100$ pc at a distance $L \approx 100$ kpc; it should however be borne in mind that radio regions are not uniform in brightness, but hot spots are generally evident of size $\approx R_g$, surrounded by more diffuse and weaker distributions. These hot spots could eventually be the site where beam-energy dissipation takes place. Concerning the magnetic field intensity, equipartition arguments suggest $B_b \approx 10^{-5}$–$10^{-6}$ G in blobs; higher values, up to $B_g \approx 10^4$ G, in the central galaxy have been proposed by Morrison (1969) and Ozernoy & Usov (1973) for a compact magnetoid. Predictions on particle densities are rather uncertain, as all
observations of radiation refer to relativistic particles; however, Faraday rotation measures indicate thermal particle densities in radio blobs \( \approx 10^{-5} \text{--} 10^{-3} \text{cm}^{-3} \left( n_p \sim n_e \right) \). The central galaxy is much thicker, but the density at the edges where beams emerge into intergalactic gas should be in the same range \( \lesssim 10^{-3} \text{cm}^{-3} \). No definite data exist so far for physical parameters along the jets. We can now define the parameters for our wave plus wind system, using observational limits in connection with possible models of the formation of jets from the parent galaxy. One can use two main scenarios: (i) the wave is emitted with the relativistic particle wind from the edge of the galaxy and the field in the blobs is essentially due to the wave itself (electromagnetic wind model); (ii) the wave digs its way from the galaxy pushing out the matter as a whole, which then reaches the radio blobs whose fields are connected with the frozen fields at the moment of ejection (hydromagnetic model). Since typical frequencies of the wave emitted from the magnetoid are \( \omega_{\text{MP}} \approx 10^2 \text{rad/s} \) for a many pulsar system and \( \omega_{\text{SR}} \approx 10^{-7} \text{rad/s} \) for a single massive rotator, we can derive the field strength factor along the streaming jets:

\[
f = \frac{eE}{m_e c \omega} = f_b \left( \frac{R_b}{R_g} \right)^\alpha,
\]

where \( \alpha = 0 \) for completely self-focused waves (no energy density decrease for one-dimensional flux) and \( \alpha > 0 \) for different propagation conditions where flux conservation gives an amplitude decrease; however, we again remark that \( f \) is not uniquely determined by the blobs’ magnetic field estimate, \( B_b \), since this could be completely uncorrelated with the wave fields, as discussed above.

5.1 ELECTROMAGNETIC WIND MODEL

For this case we assume that particles in jets are continuously energized by the wave itself in the inner regions of the central galaxy where fields are stronger; we remark, however, that in this picture we do not consider the initial stages of expulsion of the jets but a subsequent stationary structure. Thence physical parameters may be quite different from those required for the jet formation. Namely we assume that a relativistic particle (ions + electrons) wind exists and the wave may be considered as a proper mode in the rest frame of this plasma:

\[
\omega' > \omega_{\text{pe}} f^{1/2},
\]

where \( \omega' = \frac{\omega}{\gamma_w} \) takes into account the Doppler shift by means of the Lorentz factor \( \gamma_w \) of the wind (\( f \) is obviously Lorentz invariant). This means that the density of the wind (in its rest frame) is limited by

\[
n_w \lesssim 10^{-9} \frac{\omega^2 f}{\gamma_w^2}.
\]

It is rather uncertain how to fix a value for \( \gamma_w \), as this depends on the model of the galactic nucleus, but a reasonable limit could be \( \gamma_w = \min \left[ \frac{(m_e/m_i) f}{\left\| I \right\|} \right] \), where \( m_i \sim m_p \) = proton mass, which is suitable for a near-field structure of magnetic rotators and excludes further heavy-particle acceleration in the wind (Ferrari & Trussoni 1974). Then, for \( B = 10^{-5} \text{G} \), equation (9) gives wave strength factors along the beam \( f_{\text{MP}} \approx (R_b/R)^6 \) in the many-pulsar system and \( f_{\text{SR}} \approx 10^9 (R_b/R)^6 \) in the single-rotator model. The many-pulsar system can thence be shown to require wind densities near the central galaxy which are below \( \approx 10^{-5} \text{cm}^{-3} \), although this value can be lowered or increased for different \( \alpha \) and \( \gamma_w \); this wind would be
somewhat lighter than the matter in the radio blobs and central galaxy, and probably with densities of the order of the intergalactic matter density. Similarly one derives, for single rotators, upper limits of particle density $\rho \approx 10^{-23}$ cm$^{-3}$ which seem to be unreasonably low even for a relativistic wind. In fact one should remember that the wind is also essential for electromagnetic containment of the wave and cannot be too thin. Therefore this purely electromagnetic scenario seems to be precluded for single rotators. Coming back to the many-pulsar system, the mildly relativistic wind and strong wave propagate through intergalactic matter without energy dissipation (no emission is observed in general). However, condition (10) may be violated in two cases: (a) when $n_w$ is constant along the beam, but $\alpha > 0$ because electromagnetic dissipation is present; (b) when $\alpha = 0$, because self-focusing is complete, but $n_w$ suddenly increases at the head of jets in connection with original inhomogeneities at expulsion. Then, as we have discussed in the previous sections, conditions are met for parametric instabilities $\Omega^f = \Omega_{pe}^f / f^{1/2}$; since the many-pulsar system would suggest $f_0 \sim 1$, both Silin-type and relativistic electron instabilities are allowed in the streaming plasma, which can dissipate the electromagnetic wave in plasma turbulent modes.

Two simple considerations should be added. The first one concerns the increase of $\omega_{pe}^f / f^{1/2}$ towards $\omega^f$ when the density of the head of the beam increases; one should recall from Max & Perkins (1971) that electromagnetic flux conservation would require also $f$ to increase as the density if the gradient scale $L$ is larger than the wavelength $\lambda = 2\pi/k_0$; more precisely if $[\Omega_{pe}^f / \Omega^f]^2 \ll k_0 L$. However, we are now considering a situation in which $k_0 \rightarrow 0$, and moreover the density increase is sudden. Then flux conservation is no longer applicable straightforwardly and instability conditions are more valid criteria.

The second consideration is quite tentative; the two ways discussed above in which the wave plus wind system could meet the instability conditions, could represent in fact two phases of the evolution of the jets. Initially the wave would not find any density increase at the head of the wind, being consequently very poorly confined: then $\omega_{pe}^f / f^{1/2}$ would decrease owing to increase of $f$. At a later stage, after ignition of the parametric process, density inhomogeneities would certainly form by external pressure effects acting on the unstable front of the wind, and therefore parametric absorption would continue, at constant $f$ eventually.

### 5.2 HYDROMAGNETIC MODEL

In this second picture we assume that some hydrodynamic process has ejected the radio blobs in times of strong galactic activity. This matter has moved into the intergalactic medium without diffusion, owing either to ram-pressure containment, or magnetic confinement by the galactic field stretched along the jets, or even gravitational self-confinement, if one thinks of compact objects, perhaps black holes. Strong waves produced by the central galaxy have then found favourable physical conditions to propagate in the wakes of these bodies, maintaining low-density cavities by means of their pressure and being confined by self-focusing. A relativistic wind may be also present in this picture, but it could be required only for confinement: it could be eventually a simple collection of particles and not a plasma, because of its low density.

The wave-strength factor is not related to the magnetic field in radio blobs, since this is the remnant field after ejection; actually the wave-field amplitude could be much higher than $B_0$. In agreement with observations the Lorentz factor of radio blobs $\gamma_b$ is likely to be not very high, although it can be maintained above unity by the wave pressure itself.

In this picture, parametric instabilities could arise where the strong wave moving in the cavity reaches the radio blobs where the density is suddenly higher, such as to stop propaga-
tion even for relativistic non-linear waves. A limit to this density is given by a relation similar to (11):

$$n_b \approx 10^{-9} \omega^2 \frac{f}{\gamma_b^2}. \tag{12}$$

Then for a many-pulsar system $n_b \approx 10^{-5} f/\gamma_b^2$, and for a single rotator $n_b \approx 10^{-3} f/\gamma_b^2$. Since now $\gamma_b$ and $f$ are uncorrelated, the observed value $n_b \gamma_b \lesssim 10^{-3}$ can be met for $(f/\gamma_b)_{MP} \lesssim 10^2$ and $(f/\gamma_b)_{SR} \lesssim 10^{20}$, i.e. in principle in a single-rotator model also, since for $B_\gamma \approx 10^4$ G the strength factor $f_\gamma$ would be as high as $10^{25}$.

The parametric instability then dissipates the wave energy and, if this is very strong, its pressure contributes to the outward motion of the radio blobs. Anyhow, although this process is not in this picture directly responsible for the original expulsion of the blobs, it would be the energetic and dynamic support over long timescales.

The general conclusion of our discussion is that, for either of the two proposed models, conditions exist for energy supply to the radio blobs by parametric instabilities of strong electromagnetic waves emitted from galactic nuclei. Actually the wave energy is converted into turbulent motions not only of electrons which are responsible for radiation emission, but also of the ions themselves. At the same time the interaction should affect a well-defined region of the radio component, where this is directly struck by the self-collimated electromagnetic wave. Thence the energy is diffused into the surrounding gas by turbulent processes involving a full cascade of plasma modes. It seems appealing to link these preferential regions with the so-called ‘hot-spot’ components (Readhead & Hewish 1976). A more detailed analysis of this aspect of the model, referring also to radio spectra, will be pursued in a forthcoming paper. Here we simply recall that generation of turbulent modes diffusing into large-scale tails of extended radiosources would allow stochastic acceleration of high-energy electrons in regions far from the hot-spots themselves; this seems to agree with observations that the spectral index of radio emission does not sensibly increase farther from the centres of these spots, as would be expected on the basis of a pure diffusion theory of high-energy electrons.

Finally the same stochastic acceleration of ions in radio blobs could be responsible for generation of high-energy extragalactic cosmic rays; and as this general physical picture has also been applied to supernova shells which, as the Crab, include a pulsar, a galactic cosmic ray source could be envisaged in plasma and MHD turbulence excited by parametric instabilities of the pulsar large-amplitude wave in remnant shells; this turbulence would be the basis for stochastic processes or Fermi-type mechanisms as proposed by Chevalier (1977).

References