Black holes in radiation-dominated gas: an analogue of the Bondi accretion problem

Mitchell C. Begelman  
*Institute of Astronomy, Madingley Road, Cambridge CB3 0HA*

Received 1977 October 26

Summary. Black holes, unlike other compact objects, are able to accrete matter more rapidly than their Eddington rate, $\dot{M}_E = L_E/c^2$. Nevertheless, at such a high $\dot{M}$, radiation will probably be emitted by the in-falling gas in copious enough quantities to have a profound influence on the flow. To aid in understanding the nature of this influence, we study the steady flow, on to a stationary Schwarzschild black hole, of a uniform, non-relativistic gas in which radiation pressure swamps thermal pressure at infinity, and in which Thomson scattering provides the only radiation–gas couple. Asymptotic radiation pressure $p_\infty$ and matter density $\rho_\infty$ determine an asymptotic sound speed $c_\infty$, from which one can derive an accretion rate $\dot{M}_B$ corresponding to the adiabatic flow of a $\gamma = 4/3$ gas. The actual accretion rate depends on the optical depth $\tau_B$ of a column of unperturbed gas spanning the Bondi radius, $r_B = GM/c^2_\infty$. If $\tau_B > (\sqrt{2}/3)(c/c_\infty)$, then the flow is adiabatic, and $\dot{M} = \dot{M}_B$. For a somewhat smaller $\tau_B$, diffusion is efficient enough for the radiation to leak out of the gas as it moves towards the trans-sonic point. As a result, the sound speed decreases inwards in the subsonic region, while the density must increase steeply to maintain pressure balance. $\dot{M}$ may then exceed $\dot{M}_B$ by a factor of up to $(\sqrt{2}/3)(c/c_\infty)$, although this effect can be limited by thermal pressure. Finally, for small enough $\tau_B$ the diffusion approximation breaks down, and radiation drag limits an otherwise thermally-determined $\dot{M}$.

Our boundary conditions occur within super-massive ($\dot{M}/M_\odot \gtrsim 10^2$) stars, and in the pre- and post-recombination universe. If a super-massive star of $\dot{M}/M_\odot \lesssim 5 \times 10^5$ happens to have a small ($M_{BH} \lesssim M_\odot$) black hole passing through it on a bound orbit, it will capture and be swallowed by the hole, before it has a chance to ignite its nuclear fuel and blow itself apart. In any milieu where both small black holes and super-massive stars are likely to form (e.g. a dense star cluster), this may provide a natural mechanism for forming black holes in the mass range $10^2$–$10^5 M_\odot$.

1 Introduction

The luminosity reaching infinity from a steadily-accreting neutron star or white dwarf is always of order $\dot{M}(GM/r_*)$, where $\dot{M}$ is the accretion rate and $r_*$ is the stellar radius. If the
gas consists of Thomson scatterers (cross-section $\sigma_T$) and the flow is spherically symmetric, then the accretion rate may not exceed

$$\dot{M}_E \sim \left(\frac{GM}{r_*}\right)^{-1} L_E$$

where

$$L_E = \frac{4\pi GM c m_p}{\sigma_T} \sim 10^{38} \frac{M}{M_\odot} \text{ erg/s}$$

is the Eddington limit. Black holes, not having a hard surface, do not impose this constraint; any radiation which emerges must have been emitted by the gas during its flight. If the gas is optically thin, then the radiation will simply reduce the effective gravitational force felt by each particle, but if it is optically thick, an isotropic radiation pressure will alter the flow in more subtle ways. Kafka & Mészáros (1976), examining flows in which thermal pressure governs the accretion rate, but where radiation pressure eventually swamps all other kinds of internal energy and pressure, used what they thought was their prerogative to select the emergent luminosity arbitrarily. They found that, regardless of the assumed luminosity, all flows with $\dot{M} > \dot{M}_E$ become subsonic ‘settling’ solutions before reaching the event horizon. However, it can be shown (Michel 1972) that only supersonic flows can cross the horizon; therefore, the solutions described by Kafka & Mészáros do not really represent steady flows.

Some of the difficulty encountered by Kafka & Mészáros arises because they do not adequately consider the transformation of internal gas energy into radiation. In a subsequent paper, I shall argue that the snags may be avoided with a more careful accounting of energetics. But there is a problem, simpler than that of Kafka & Mészáros, in which their equations do yield a self-consistent solution. In the next three sections of this paper I discuss the basic problem and its solution, as well as some of the model-independent complications which may occur. Afterwards, I apply the theory to black holes inside super-massive stars.

2 The basic problem and its solution

Consider a non-rotating black hole situated in an asymptotically uniform, stationary medium in which there is no pressure or energy density other than that associated with radiation. The gas interacts with the radiation through Thomson scattering, and is optically thick on all relevant scales. Therefore, radiation is transported by diffusion and convection, and the radiation pressure is isotropic and related to the radiation density $U$ by $p = U/3$. The asymptotic boundary conditions are $p \rightarrow p_\infty$, $\rho \rightarrow \rho_\infty$, making this a radiation-dominated analogue of the problem considered by Bondi (1952). The equations for a steady accretion flow are exactly those used by Kafka & Mészáros.

$$4\pi\rho v^2 = \dot{M} = \text{constant} \quad (\dot{M}, v > 0 \text{ for inward flow})$$

$$\rho \frac{d}{dr} \left(\frac{v^2}{2}\right) + \frac{dp}{dr} + \frac{GM\rho}{r^2} = 0 \quad (2)$$

$$\dot{M} \frac{d}{dr} \left(\frac{v^2}{2} - \frac{GM}{r} + 4p\right) = -\frac{cm_p}{\sigma_T} \frac{d}{dr} \left(4\pi r^2 \frac{p}{\rho}\right) \quad (3)$$

A prime is an alternative symbol for differentiation with respect to $r$. The properties of equations (1)–(3) become clearer when they are written as an energy equation and a Parker wind-type equation, in $v^2$ and the Mach number $M = 3v^2\rho/4p$. 

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
2.1 THE ENERGY EQUATION

We may integrate (3) immediately, but first it is preferable to make the following substitution. From (2), we have

$$\frac{p'}{\rho} = -\frac{GM}{r^2} - \frac{d}{dr} \left(\frac{v^2}{2}\right).$$

Equation (4) implies that the rest frame luminosity

$$L = -4\pi r^2 \frac{c}{\sigma_T} \frac{p'}{\rho} - 16\pi r^2 p v$$

approaches $L_E - 3\dot{M}c^2_\infty$ as $r \to \infty$, where

$$c_\infty = \left(\frac{4 \rho}{3 \rho_\infty}\right)^{1/2}$$

is the asymptotic sound speed in the gas. Integrating (3) once, setting

$$\eta = \frac{2\pi c m_p}{3\sigma_T \dot{M}}$$

and introducing the Mach number, we obtain

$$\mathcal{M}^2 = \frac{\rho^2(1 + \mathcal{M}'^2/6)}{c_\infty^2 + (GM/3r) + \eta r^2 (v^2)'}.$$

To describe Kafka–Mészáros boundary conditions, $c_\infty^2$ in (7) would be replaced by $(L_E - L_\infty)/3M(\equiv 1 - \lambda c^2/3\mu)$ in their rationalized units.

2.2 THE WIND EQUATION

Equation (2) may be written in standard wind-equation form (see, e.g. Holzer & Axford 1970)

$$\left(\mathcal{M}^2 - \frac{4}{\mathcal{M}^2} \right) (\mathcal{M}'^2 - 1) = \left(\mathcal{M}'^2 - 1\right) \left(\frac{p'}{\rho} - \frac{4}{r} \frac{GM\rho}{2r^2} + \frac{3}{2} \left(\frac{\ln \rho}{\rho^{4/3}}\right) = 0.\right.$$}

The last term on the left-hand side of (8) represents the loss of specific entropy associated with radiation diffusion down the pressure gradient. After making substitutions from (2), (3) and (6), and performing some algebra, one obtains

$$\frac{(\mathcal{M}'^2 - 1)(\mathcal{M}'^2 - 1)}{\mathcal{M}^2(1 + \mathcal{M}'^2/6)} = \frac{4}{r} \frac{(7GM/3r^2) + (1 + 4\mathcal{M}'^2/3) \eta r^2 (v^2)'}{c_\infty^2 + GM3r + \eta r^2 (v^2)'}.$$

Critical points, where $\mathcal{M}^2 = 1$, $(\mathcal{M}'^2)'$ finite, occur at the zeros of the right-hand side of (9). When $\sigma_T \to \infty (\eta \to 0)$, the radiation is completely trapped, the gas behaves isentropically with $\gamma = 4/3$, and (9) reduces to the correct equation for adiabatic flow.

2.3 ANALYSIS OF THE SUBSONIC ASYMPTOTE

All solutions satisfying the boundary conditions have the asymptotic form

$$\mathcal{M}^2 = \left(\frac{\dot{M}}{4\pi \rho_\infty c_\infty r^2}\right)^2.$$

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
In this limit, $p/p \approx 3/c_\infty^2/4$ hence

$$\eta r^2(v^2)' = -\frac{\dot{M}_{\text{cm}}}{6\pi \rho^2 c_\infty^2 \sigma_T r^3}. \quad (10b)$$

Thus, at large enough $r$, the terms in (9) describing the diffusion of radiation become negligible compared with the other terms, and (9) tends toward the equation for isentropic accretion with $\gamma = 4/3$. In particular, if $|\eta r^2(v^2)'| \ll c_\infty^2$ as the flow crosses the Bondi radius $r_B = GM/c_\infty^2$, then the flow becomes supersonic very near to the critical radius for isentropic flow $r_{\text{cr},B} = r_B/4$. The condition for this to occur is

$$\frac{\dot{M}_{\text{cm}}}{6\pi \rho^2 c_\infty^2 \sigma_T r_B^3} \ll c_\infty^2,$$

which reduces to

$$\tau_B \equiv \frac{\rho \sigma_T r_B^2}{m_p} > \frac{c}{c_\infty}. \quad (11)$$

once we have substituted $\dot{M}_B = 2\sqrt{2\pi} \rho c_\infty^2 r_B^2$ given by Bondi theory (1952). $\tau_B$ is the optical depth characteristic of the length scale $r_B$, assuming $\rho(r_B) = \rho_\infty$. According to (5), $\tau_B(c_\infty/c)$ is roughly the ratio of convected to diffused luminosity at $r_B$, and physically, (11) means that radiation is being convected inward across $r_B$ faster than it can diffuse out. In fact convection overwhelms diffusion for all $r$ such that

$$\tau(r) \equiv \frac{\rho(r) \sigma_T r}{m_p} > \frac{\dot{M}_{\sigma_T}}{4\pi m_p c} > 1;$$

and to a stationary observer on the outside, $\tau(u/c) = 1$ marks the point of no return for radiation. We call this radius the 'trapping radius', and define

$$r_{\text{tr}} \equiv \frac{\dot{M}_{\sigma_T}}{4\pi m_p c} \quad [(6\eta)^{-1}]. \quad (12)$$

Note that $r_{\text{tr}}$ depends on the particulars of the problem only through $\dot{M}$; this fact is quite general and has important consequences for all optically thick accretion and wind flows.

Using definition (12), we see that (11) is equivalent to

$$r_{\text{tr},B} \gg r_B$$

where $r_{\text{tr},B}$ is the trapping radius calculated by assuming the Bondi accretion rate. If (11) is satisfied, then $v^2 \ll r^{-1}$ for all $r < r_B$, and $|\eta r^2(v^2)'| \ll GM/r$ for all $r$. Therefore, if diffusion does not become dominant in $r > r_B$, it will never become dominant, and the entire flow will be essentially adiabatic. All that diffusion can do in this case is to increase the critical radius to

$$r_{\text{cr}} \approx r_{\text{cr},B} \left[ 1 + \frac{128}{3} \frac{r_B}{r_{\text{tr},B}} \right] \quad (13)$$

with a corresponding increase of $\dot{M}$ over $\dot{M}_B$.

Next we consider the alternative case. If $r_{\text{tr},B} \ll r_B$, then at some point in the subsonic region, $|\eta r^2(v^2)'| > GM/3r$. Note that, according to equation (7), $|\eta r^2(v^2)'|$ cannot
exceed $c_s^2 + GM/3r$, since $\mathcal{M}^2$ and $\nu^2$ must be positive. The zeroth-order equations in this regime, with $\mathcal{M}^2 \ll 1$, are

\begin{equation}
(\ln \mathcal{M}^2)' = \left(\ln \left[c_s^2 + \eta r^2(\nu^2)\right]\right)' - \frac{4}{r}
\end{equation}

\begin{equation}
v^2 = \mathcal{M}^2 \left[c_s^2 + \eta r^2(\nu^2)\right]
\end{equation}

which may be combined and integrated once to give

\begin{equation}
\frac{4\pi \rho_\infty c_s^2}{M} \cdot \nu = \frac{c_s^2}{r^2} + \eta(\nu^2)'.
\end{equation}

When $|\eta r^2(\nu^2)| < c_s^2$, $\mathcal{M}^2$ and $\nu^2$ are given by (10). However, because $r_{tr,B} \ll r_B$, $|\eta r^2(\nu^2)|$ will catch up with $c_s^2$ while $\mathcal{M}^2 < 1$. This happens at the 'diffusion radius'

\begin{equation}
r_{\text{diff}} \approx \left(\frac{M \rho_\infty c_s^2}{6\pi c_m^2 \rho_\infty \sigma_T}\right)^{1/3} = \left(\frac{1}{3} \frac{r_{tr,B}^4}{M c_s^2}\right)^{1/3} \gg r_B.
\end{equation}

Inside $r_{\text{diff}}$, diffusion is so efficient that the gas 'deflates' as it moves inwards. Since $\mathcal{M}^2 < 1$, the radiation pressure remains nearly constant at $\sim p_\infty$, but the matter density increases steeply, resulting in the sound speed $c_S = [(4/3)(\rho/\rho)]^{1/2}$ decreasing with decreasing $r$. This has the effect of moving the critical radius (still given roughly by $r_{cr} = GM/c_s^2(\rho_\infty)$) outwards, as well as increasing the density at the critical radius, and results in an accretion rate $\sim 4\pi \rho(r_{cr}) c_s(r_{cr}) r_{\text{diff}}^2$, much larger than that obtained by applying Bondi theory to the asymptotic boundary conditions.

For $r < r_{\text{diff}}$, the leading behaviour of $\nu^2$ is

\begin{equation}
v^2 \approx \frac{3 \dot{M} \sigma_T}{2\pi c_m r} \approx \frac{6 c_s^2}{r} r_{tr}.
\end{equation}

Because $c_s^2 + \eta r^2(\nu^2) < c_s^2$, and $|\left(\ln \left[c_s^2 + \eta r^2(\nu^2)\right]\right)'| < 1/r$, it is necessary to consider first-order corrections to $\nu^2$ in order to solve (9) for $\mathcal{M}^2$. Neglecting the gravitational terms because $c_s^2 + \eta r^2(\nu^2) \gg GM/3r$, we have

\begin{equation}
v^2 \approx \mathcal{M}^2 \left[c_s^2 \left(1 - \frac{r_{tr}}{r}\right) + \eta r^2(\nu^2)\right]
\end{equation}

which, upon substitution into the approximate wind equation and integration once, yields

\begin{equation}
[c_s^2 + \eta r^2(\nu^2)'] \approx \frac{c_s^2}{2} \left[\frac{r_{tr}}{r} + \left(\frac{r_{tr}}{r}\right)^2 + \left(\frac{r}{r_{\text{diff}}}\right)^3\right]^{1/2}
\end{equation}

where we have fixed the constant of integration by taking $c_s^2 + \eta r^2(\nu^2)' = c_s^2/2$ at $r = r_{\text{diff}}$. Equation (19) is valid as long as both $c_s^2 + \eta r^2(\nu^2) > GM/3r$ and $\mathcal{M}^2 \ll 1$ are satisfied. Where the breakdown of the first condition is important, it occurs before the $\sim r_{tr}/r$ terms dominate, at a radius

\begin{equation}
r_{gr} \approx \frac{r_{tr}^{2/5} r_{\text{diff}}^{3/5}}{r_{tr}^{2/5} r_{\text{diff}}^{3/5}}
\end{equation}

while the possible critical point occurs at

\begin{equation}
r_{cr} \approx \frac{r_{tr}^{2/5} r_{\text{diff}}^{3/5}}{r_{tr}^{2/5} r_{\text{diff}}^{3/5}}
\end{equation}
The behaviour of the system is governed by the larger of the two radii. If \( r_{\text{tr}} \geq r_B \), we expect to find a critical point near \( r_{\text{cr}} \), and the problem becomes one of piecing on a supersonic solution at this point, our task in the next subsection. If \( r_{\text{tr}} < r_B \), then we still have \( \mathcal{M}^2 \ll 1 \) at \( r < r_{\text{gr}} \), and may proceed with our integration inward.

We must now include all gravitational terms in the approximate equations, obtaining

\[
(\ln v^2)' \approx \frac{(2GM/r^2) + 2[c_\infty^2 + \eta r^2(v^2)']}{(GM/3r) + [c_\infty^2 + \eta r^2(v^2)']} - \frac{4}{r}. \tag{21}
\]

As long as \( GM/r < c_\infty^2 \), hence \( r > r_B \), we must have \( c_\infty^2 + \eta r^2(v^2)' < c_\infty^2 \), and \( v^2 \) is still given by (17) to lowest order. Substituting \( v^2 \propto r^{-1} \) into the left-hand side of (21), we find

\[
c_\infty^2 + \eta r^2(v^2)' = \frac{GM}{5r}
\]

when \( r < r_{\text{gr}} \). We then have

\[
\mathcal{M}^2 \approx \frac{v^2}{(GM/3r) + [c_\infty^2 + \eta r^2(v^2)']} \approx \frac{45 r_{\text{tr}}}{4 r_B} = \text{constant} \ll 1. \tag{22}
\]

Equation (22) guarantees that we arrive at \( r_B \) subsonically, at which point the flow changes character again, to develop an asymptotic behaviour \( v^2 = b r^{-\beta} \) at \( r < r_B \). The only possibilities consistent with (21) are \( \beta = \pm 2 \). However, \( \beta = +2 \) corresponds to \( b < 0 \), which is unphysical, while \( \beta = -2 \) is a settling solution, which cannot enter the black hole. Therefore, we must conclude that \( r_{\text{tr}} < r_B \) is not consistent with a steady accretion flow.

\[ \text{2.4 INTEGRATION ACROSS THE CRITICAL POINT: DETERMINATION OF } \dot{M} \]

When \( r_{\text{tr}} \geq r_B \), we expect that \( r_{\text{cr}} \gg r_B \), and hence that there will be a region with \( \mathcal{M}^2 \geq 2/3 \). It follows from (7) that \( \gamma \equiv c_\infty^2 + GM/3r + \eta r^2(v^2)' < c_\infty^2 \) in this region, since \( \mathcal{M}^2 \) cannot exceed the free-fall speed. Writing (7) as

\[
y = \left( \frac{6}{\mathcal{M}^2} + 1 \right) c_\infty^2 \frac{r_{\text{tr}}}{r} \tag{23}
\]

we substitute into the full wind equation (9) to obtain

\[
2(\ln \mathcal{M}^2)' = \frac{1}{r} \left[ 4 - \mathcal{M}^2 \left( \frac{r_B}{3 r_{\text{tr}}} - 1 \right) \right]. \tag{24}
\]

which is easily integrated to give the result (taking into account the approximate boundary condition implied by (19) and (20b))

\[
\mathcal{M}^2 = \frac{1}{(4/15)(r_B/3 r_{\text{tr}} - 1) + (r/r_{\text{cr}})^{5/2}}. \tag{25}
\]

The qualitative behaviour of \( \mathcal{M}^2 \) depends on which of the two terms in the denominator of (25) is larger as \( r \to r_B \). If

\[
\left( \frac{r_B}{r_{\text{cr}}} \right)^{5/2} \leq \frac{4}{15} \left( \frac{r_B}{3 r_{\text{tr}}} - 1 \right) \tag{26}
\]
Black holes in radiation-dominated gas

\( M^2 \) levels off to \( \sim 15/4(r_B/3r_{\text{tr}} - 1) \) at \( r_t \sim r_{\text{cf}}(r_B/3r_{\text{tr}} - 1)^{2/5} \). Closer inspection of the approximate equations for \( M^2 \approx 1 \) (4\((r_B/3r_{\text{tr}} - 1)/15, L < 1 \)) in addition to satisfying (26) reveals that the \((r/r_{\text{tr}})^{5/2} \) term in (25) dominates corrections to \( M^2 \) only when \( r_t > r > r_{\text{tr}}^{1/7}r_{\text{tr}}^{2/7} \). As \( r \) falls below \( r_{\text{tr}}^{1/7}r_{\text{tr}}^{2/7} \), \( M^2 \) begins to decrease slowly, the fractional decrease approaching order unity as \( r \) approaches \( r_{\text{tr}} \). Although our perturbation analysis breaks down near \( r_{\text{tr}} \), it is apparent that the exact equations and boundary conditions are mathematically identical to those used in numerical integrations by Kafka & Mészáros, who consistently find a settling region at small \( r \), bounded by a diffusion-broadened shock in the vicinity of \( r_{\text{tr}} \) (Kafka & Mészáros 1976; Mészáros, private communication). We conclude that solutions satisfying (26) cannot enter the black hole.

On the other hand, when \( (r_B/r_{\text{tr}})^{5/2} > 4(r_B/3r_{\text{tr}} - 1)/15 \) the \( M^2 \approx r^{-5/2} \) behaviour persists until the flow reaches \( r \sim r_{\text{tr}} \). An examination of the approximate equations in this case indicates that near \( r_{\text{tr}} \), the behaviour of \( M^2 \) smoothly changes over to \( M^2 \approx r^{-1/2} \), the run of Mach number for a \( \gamma = 4/3 \) gas in adiabatic free fall, and that this behaviour extends into the region in which relativistic effects become important. Michel (1972) has shown that flows of this kind can cross the event horizon.

Thus, we have shown that there is a unique accretion rate corresponding to the asymptotic boundary conditions \( p_{\infty}, \rho_{\infty} \). The flow must be characterized by \( r_{\text{tr}} = r_B/3, \) to a fractional accuracy of \( \sim O[(r_B/r_{\text{tr}})^{5/2}] \). Using (12), (16) and (20b), we translate this result into the accretion rate

\[
\dot{M} = \frac{\sqrt{2}}{3} \frac{c}{\tau_B c_{\infty}} \dot{M}_B + O(\dot{M}_B).
\]

(27)

Comparison of (27) with our result for the case \( \tau_B > c/c_{\infty} \) suggests that a general formula for \( \dot{M} \), given that the diffusion approximation holds and thermal effects are unimportant, is

\[
\dot{M} \approx \dot{M}_B \left[ 1 + \frac{\sqrt{2}}{3} \frac{c}{\tau_B c_{\infty}} \right].
\]

(28)

3 Effects of thermal pressure and dissipation

A radiation-dominated gas in thermal equilibrium has a temperature

\[
T_{\infty} = 4.5 \times 10^3 (p_{\infty}/\text{erg cm}^{-3})^{1/4} \text{ K}
\]

(29)

and the associated thermal pressure \( n_{\infty}kT_{\infty}/\mu = \beta p_{\infty} \) is negligible compared with \( p_{\infty} \) if

\[
p_{\infty} > 5.2 \times 10^{-17} \left( \frac{n_{\infty}}{\mu} \right)^{4/3} \text{ erg cm}^{-3}
\]

(30)

where \( n_{\infty} \) is the baryon number density in cm\(^{-3}\) and \( \mu \) is the mean molecular weight per particle (\( \mu = \frac{1}{2}, n_{\infty} = \rho_{\infty}/m_p \) for pure hydrogen). Thermal equilibrium is maintained by a number of processes; here, we consider only the Compton process.

If a radiation-dominated gas, initially in equilibrium at \( T \), is shifted out of equilibrium by an amount \( \Delta T/T \sim O(1) \), the Compton equilibration timescale is the time it takes an electron to be struck by photons \( \sim m_e c^2/kT \) times. The rate of such collisions per electron is \( \sim (U/kT) c \sigma_T \), where \( U \) is the energy density of radiation, and the equilibration time is therefore \( t_c \sim m_e c / (U \sigma_T) \). Within the accretion flow, a sufficient (but not necessary, since free–free absorption, etc, promote equilibrium as well) condition for equilibrium at each

© Royal Astronomical Society • Provided by the NASA Astrophysics Data System
Condition (31) is automatically satisfied at all \( r \) when \( \tau_B > c/c_\infty \), hence thermal pressure remains insignificant everywhere, and in the absence of turbulent or magnetic pressures, \( \dot{M} \sim M_B \). For \( \tau_B < c/c_\infty \), (31) may or may not be satisfied, in which case \( T_{\text{gas}} \) may exhibit a variety of behaviours. If cooling is efficient, \( T_{\text{gas}} \) may drop below \( T_{\text{rad}} \), or be forced into equilibrium by processes other than Compton scattering. If cooling is very inefficient, then \( p_{\text{th}} \propto r^{-5/2} \) for \( r \ll r_{\text{diff}} \). This latter behaviour may also characterize \( \gamma > 5/3 \) energy reservoirs such as turbulence and tangled magnetic field, if there is sufficient dissipation. By \( \gamma > 5/3 \), we mean that the effective pressure \( p_{\text{eff}} \) associated with a particular form of energy is related to the corresponding energy density by \( \epsilon = p_{\text{eff}}/(\gamma - 1) \), with \( \gamma > 5/3 \). Bondi (1952) showed that a gas dominated by such pressures cannot be accreted if it behaves adiabatically. However, Mészáros (1975a, b) argued that dissipation would probably lead to \( p_{\text{eff}} \sim \xi p_{\text{fr}}^2 \propto r^{-5/2} \) with \( \xi < 1 \), allowing accretion to take place.

If equilibrium is maintained at \( r < r_{\text{diff}} \), then \( T_{\text{gas}} \approx T_{\infty} \), and thermal pressure overtakes radiation pressure at

\[
 r_{\text{th}} \sim \beta^{2/3} r_{\text{diff}} \tag{32}
\]

but can influence \( \dot{M} \) only if \( r_{\text{th}} > r_{\text{cr}} \sim r_{\text{fr}}^{2/5} r_{\text{diff}}^{3/5} \) (equation (20b)). Therefore, the condition for thermal determination of \( \dot{M} \) is

\[
 r_{\text{diff}} > \beta^{-5/3} r_{\text{fr}} \tag{33}
\]

or equivalently

\[
 \frac{\tau_B c_\infty}{c} < \beta^{5/2}. \tag{33}
\]

When (33) is satisfied, the density levels off to \( \sim \beta^{-1} \rho_\infty \) at \( r_{\text{th}} \), and the velocity becomes

\[
 v^2 \approx 6 c_\infty^2 (r_{\text{fr}}/r_{\text{th}})^4. \]

The accretion rate, obtained by demanding a critical point near the thermal accretion radius, is

\[
 \dot{M} = \beta^{-5/2} \dot{M}_B \tag{34}
\]

where, as before, \( \dot{M}_B \) refers to the \( \gamma = 4/3 \) Bondi accretion rate calculated for the asymptotic radiation-dominated gas. We may combine (28), (33) and (34) to construct a formula for \( \dot{M} \) incorporating the thermal pressure

\[
 \dot{M} = \dot{M}_B \left[ 1 + \frac{\sqrt{2}}{3} \frac{c}{\tau_B c_\infty} \left( 1 + \frac{\sqrt{2}}{3} \frac{c}{\tau_B c_\infty} \beta^{5/2} \right)^{-1} \right]. \tag{35}
\]

We next consider the case \( p_{\text{eff}} \sim 6 \xi \rho c_\infty^2 (r_{\text{fr}}/r) \). For \( r < r_{\text{diff}} \), \( p_{\text{eff}} \propto r^{-5/2} \) and overtakes radiation pressure at

\[
 r_{\text{eff}} = (8 \xi)^{2/5} (r_{\text{fr}}^{2/5} r_{\text{diff}}^{3/5} \leq r_{\text{cr}}. \tag{36}
\]

Therefore, the presence of turbulence or a tangled magnetic field should not change \( \dot{M} \) by more than a factor \( \sim O(1) \), even if dissipation is only marginally adequate.
4 Effects of finite mean free path

Up to now, we have assumed that the diffusion approximation accurately describes radiative transfer on all length scales relevant to the problem. This is not true if

$$\tau_{\text{diff}} \sim n_\infty \sigma_T r_{\text{diff}} < 1.$$  (37)

Using (16) and (35), we find that (37) never applies for $\tau_B > c/c_\infty$, and when $\tau_B < c/c_\infty$, it applies for

$$\tau_B < \frac{9}{4} \left( \frac{c_\infty}{c} \right)^2 \left[ 1 + \sqrt{1 + \frac{8\sqrt{2}}{27} \left( \frac{c}{c_\infty} \right)^3 \beta^{3/2} } \right].$$  (38)

Where (38) applies, the radiation has uniform energy density $U_\infty$, and opposes the motion of matter through it by exerting a 'radiation drag' force $\sim U_\infty (v/c) \sigma_T$ per particle. The net effect is to decrease the central mass 'seen' by the gas outside the critical radius, to

$$M_{\text{eff}} \sim M \left( 1 - \frac{U_\infty \sigma_T \dot{M}}{4\pi GM \rho_\infty} \right) = M \left( 1 - \frac{9}{4} \frac{M c^2}{L_E} \right) > 0.$$  (39)

Since the radiation pressure is decoupled from the matter, the accretion rate is determined by thermal pressure alone, and is given by

$$\dot{M} \sim 4\pi \rho_\infty (GM_{\text{eff}}) \left( \frac{kT_\infty}{\mu m_p} \right)^{3/2} \sim \dot{M}_{\text{th}} \left( 1 - \frac{9}{4} \frac{M c^2}{L_E} \right)^2$$  (40)

where $\dot{M}_{\text{th}} \sim \beta^{-3/2} \dot{M}_B$ is the accretion rate given by Bondi theory in the absence of radiation. Solving (40), we obtain

$$\dot{M} \approx \begin{cases} \dot{M}_{\text{th}} & \dot{M}_{\text{th}} \leq \frac{4}{9} \frac{L_E}{c^2} \\ \frac{4}{9} \frac{L_E}{c^2} = \frac{4}{9} M_E \left( \frac{c}{c_\infty} \right)^2 & \dot{M}_{\text{th}} > \frac{4}{9} \frac{L_E}{c^2} \end{cases}$$  (41)

where $M_E$ is the Eddington accretion rate for a black hole.

5 Application: black holes within super-massive stars

It has been suggested that super-massive stars (SMS) may form in the nuclei of galaxies (Hoyle & Fowler 1963) or as a late phase in the evolution of other types of dense star clusters (Begelman & Rees, in preparation). An SMS embedded in the remnant of a dense cluster is likely to have a small number of stellar remnants, namely neutron stars and $1-10 M_\odot$ black holes, orbiting it, and possibly penetrating it. A neutron star would probably have little effect on the evolution of the SMS, since it cannot accrete SMS material faster than $\sim 0.1 (L_E/c^2)$, and therefore requires a time much longer than the collapse time of a non-rotating, non-magnetic SMS, even to double its mass. (If rotation or magnetic fields slow the SMS evolution sufficiently, neutron stars might have an effect, but only in extreme circumstances.) A black hole, however, may accrete at a rate far exceeding $L_E/c^2$ and, as we shall show in this section, has a strong chance of being captured by an SMS of $\lesssim 5 \times 10^5 M_\odot$, and of influencing its evolution. To demonstrate this, we first discuss the conditions present
inside a non-rotating, non-magnetic SMS which is collapsing quasi-statically, and then apply the accretion theory of Sections 2–4 to these conditions.

5.1 SUMMARY OF NORMAL QUASI-STATIC EVOLUTION

A mass of gas \( m \geq 50 \) (where lower-case \( m \) denotes mass in solar units), radiating freely and collapsing in free fall, becomes dominated by radiation pressure before it reaches a quasi-stationary state. When quasi-static equilibrium is reached, the binding energy is smaller than the gravitational potential energy \( \sim GM^2/R \) by a factor \( \sim m^{-1/2} \), and the rate of change of binding energy in free fall is comparable with \( L_E \). We assume that for the conditions of interest to us, Thomson scattering is the dominant opacity; this will turn out to be the case. The radius at which free fall ceases is

\[
R_{ff} = 7 \times 10^{13} m^{2/5} \text{cm} = 2 \times 10^8 m^{-3/5} R_S
\]

where \( R_S = 2GM/c^2 = 3 \times 10^5 m \text{ cm} \) is the Schwarzschild radius of the SMS. During its quasi-static phase, the SMS radiates at \( L_E \), and evolves (decreasing in radius \( R \) by a factor \( \sim 2 \)) on a timescale

\[
t_{ev} \approx 4.3 \times 10^8 m^{-1/2} \left( \frac{R_S}{R} \right) \text{yr.}
\]

However, this phase does not generally last until the final collapse or explosion (caused by nuclear ignition) of the SMS. As Fowler (1964) pointed out, general relativistic corrections to the binding energy result in an instability when the SMS reaches

\[
R_I \approx m^{1/2} R_S.
\]

For \( R > R_{ff} \) and \( R < R_I \), evolution will probably occur on the free-fall timescale, which is too short for a black hole with \( M_{BH} \ll m \) to have much effect. Therefore, we focus attention on the region \( R_I < R < R_{ff} \).

In the language of our accretion theory, the ambient conditions are

\[
p_\infty = 3.9 \times 10^{36} m^{-2} \left( \frac{R_S}{R} \right)^4 \text{ erg cm}^{-3}
\]

\[
\rho_\infty = 1.8 \times 10^{16} m^{-2} \left( \frac{R_S}{R} \right)^3 \text{ g cm}^{-3}
\]

and for a black hole of mass \( M_{BH} \), the derived quantities are

\[
n_\infty = 1.1 \times 10^{40} m^{-2} \left( \frac{R_S}{R} \right)^3 \text{ cm}^{-3}
\]

\[
c_\infty = 1.7 \times 10^{10} \left( \frac{R_S}{R} \right)^{1/2} \text{ cm/s} \quad \left[ = \frac{c}{\sqrt{3}} \left( \frac{R_S}{R} \right)^{1/2} \right]
\]

\[
T_\infty = 8.2 \times 10^{12} m^{-1/2} \left( \frac{R_S}{R} \right) K
\]

\[
r_B = 4.6 \times 10^5 m_{BH} \left( \frac{R}{R_S} \right) \text{ cm}
\]

\[
\tau_B = 3.4 \times 10^{21} m^{-2} m_{BH} \left( \frac{R_S}{R} \right)^2
\]
where we have assumed that the gas is in thermal equilibrium with the radiation. Note that \( r_B \approx (M_{BH}/M) R \).

5.2 RESULTS FROM ACCRETION THEORY

Using our theoretical results, we may determine the parameter ranges for which various effects dominate.

1. The flow is adiabatic (diffusion effects unimportant — see Section 2) for

\[
\frac{R}{R_S} < 4.4 \times 10^8 m^{-4/5} m_{BH}^{2/5}.
\]  

(47)

2. \( \dot{M} \) is determined by thermal pressure (Section 3) if

\[
\frac{R}{R_S} > 10^8 m^{-3/10} m_{BH}^{2/5}.
\]  

(48)

3. Finite mean free-path effects (Section 4) are important if

\[
\frac{R}{R_S} > 3.3 \times 10^{21} m^{-2} m_{BH} \quad \text{and} \quad \frac{R}{R_S} > 0.08 m^{5/6}
\]  

(49a)

or

\[
\frac{R}{R_S} > 8.3 \times 10^{11} m^{-11/14} m_{BH}^{4/7} \quad \text{and} \quad \frac{R}{R_S} < 0.08 m^{5/6}.
\]  

(49b)

In Fig. 1 we show the regions defined by (47)–(49) for \( m_{BH} \approx 1 \), as well as \( R_H \) and \( R_I \). Finite mean free path effects may be important only if \( m_{BH} \leq 10^{-2} \), which is unreasonably low for a stellar remnant. The lower boundary of the thermally-dominated region is sufficiently insensitive to \( m_{BH} \) (\( \approx m_{BH}^{2/3} \)), that it will lie above \( R_H \) for all reasonable values of \( M_{BH} \).

The accretion theory developed in Sections 2–4 is valid only for a black hole at rest with respect to the ambient gas; if the black hole is moving at a speed \( v_\infty \) through the gas, the accretion rate cannot exceed \( \dot{M} \approx (M_{BH}/c_\infty)^3 \). Obviously, accretion is most efficient if the black hole is initially stationary at the centre of the SMS, but this is unlikely in the context of dense cluster evolution. It is more plausible that any black hole destined to interact strongly with the SMS initially would be on a bound orbit, periodically passing through the interior of the SMS at speeds comparable with \( c_\infty \). The time-averaged accretion rate is initially \( \dot{M} \approx M_B \approx 10^{-8} (M_{BH}/M)^2 (R_S/R)^{3/2} \) g/s, while the time required to double the mass of the black hole is

\[
t_{\text{doub}} \approx \frac{M_{BH}}{\dot{M}} = 2 \times 10^{-5} \frac{m^2}{m_{BH}} (\frac{R}{R_S})^{3/2} \text{ s}.
\]

During free fall, the dynamical timescale is

\[
t_{\text{dyn}} \approx \frac{R}{v_{\text{ff}}} \approx 5 \times 10^{-6} m (\frac{R}{R_S})^{3/2} \text{ s},
\]

so accretion effects are not important for \( R > R_H \), as expected. For \( R_I < R < R_H \), 'capture' of the black hole, and its rapid spiral towards a stationary position at the centre of the SMS,
1978MNRAS.184...53B

Figure 1. Accretion theory of Sections 2—4 applied to a $1M_\odot$ black hole on a bound orbit passing inside a super-massive star (SMS) (Section 5). Characteristics of the accretion process are determined by the mass and instantaneous radius (in units of the Schwarzschild radius) of the SMS, and regions of the $R-M$ plane dominated by different effects are separated by dashed lines. Outside the region enclosed by $R_1$ and $R_{ff}$, evolution of the SMS is too rapid for accretion to be important. When radius of SMS shrinks to $R_{doub}$, black hole is captured and swallows SMS before further evolution. SMS of $m \lesssim 4 \times 10^8 m_{BH}^{2/5}$ can thus be accreted by a seed black hole before it has a chance to ignite its nuclear fuel and blow itself apart.

occurs as soon as the black hole can double its mass. The epoch of SMS evolution at which this happens is given by the condition

$$t_{doub} \approx t_{ev}$$

which yields the radius

$$R_{doub} \approx 2 \times 10^8 \frac{m_{BH}^{2/5}}{m} R_S.$$  

We see from Fig. 1 that for $m_{BH} = 1$, $R_{doub}$ lies in the region where accretion is adiabatic, and therefore the post-capture accretion rate is $\dot{M}_B$. Since $R_{doub}$ and the radius separating the adiabatic and diffusive regions both vary $\propto M_{BH}^{2/5}$, this observation is in fact independent of $M_{BH}$.

Once $R_{doub}$ is reached, the growth of the black hole is catastrophic, and if no processes impede the accretion (e.g. nuclear reactions) the black hole will swallow most of the SMS before it can evolve further. The timescale for swallowing the SMS is roughly

$$t_{doub}(R_{doub}) \approx 2 m^{1/2} m_{BH}^{2/5} \text{ yr},$$  

much shorter than the minimum time for a non-rotating, non-magnetic SMS to evolve to $R_1$.

$$t_{ev}(R_1) \approx 4.3 \times 10^8 \text{ m}^{-1} \text{ yr}.$$
However, accretion will not play a major role if $R_1 > R_{\text{doub}}$. Therefore, we expect that only an SMS of $m \leq 4 \times 10^5 m_{\text{BH}}^{1/5}$ can have its evolution cut short by a black hole on a bound orbit which passes through it.

The next stage of the argument is to check for self-consistency by asking whether nuclear reactions can inhibit the accretion process. To find out, we must first determine the thermal properties of the accreting gas.

5.3 THERMAL BEHAVIOUR OF THE ACCRETING GAS

Since $\tau_B > c/\omega$, the Compton effect is able to maintain thermal equilibrium between gas and radiation within the accretion flow (see discussion in Section 3, following condition (31)). The radiation energy density within the flow varies $\propto T^4 \propto \rho^{4/3} \propto r^{-2}$, and we have

$$T \approx T_\infty \left( \frac{r_\text{cr}}{r} \right)^{1/2} = 7 \times 10^8 m_{\text{BH}}^{-1/5} \left( \frac{r_S}{r} \right)^{1/2} \text{K}$$

where $r_S = 2GM_{\text{BH}}/c^2$. The density run is

$$\rho \approx 10^4 m^{-1/2} m_{\text{BH}}^{-3/5} \left( \frac{r_S}{r} \right)^{3/2} \text{g cm}^{-3}.$$  

For comparison, the corresponding values of $T_\infty$ and $\rho_\infty$ are

$$T_\infty(R_{\text{doub}}) \approx 4 \times 10^4 m^{1/2} m_{\text{BH}}^{-2/5} \text{K}.$$  

(For interesting values of $M$ and $M_{\text{BH}}$, this is hot enough to keep the gas collisionally ionized, and justify our use of Thomson scattering as the main opacity.)

$$\rho_\infty(R_{\text{doub}}) \approx 2 \times 10^{-9} m m_{\text{BH}}^{-6/5} \text{g cm}^{-3}.$$  

5.4 NUCLEAR BURNING AND OTHER IMPEDIMENTS

Nuclear burning is important in a normal quasi-static SMS if it yields a luminosity exceeding $L_E$. We are worried about it if it is likely to have made a difference for $R > R_{\text{doub}}$. Because the rate of total nuclear-energy generation increases with $\rho_\infty$ and $T_\infty$, we only need ask whether it is important at $R_{\text{doub}}$. Candidates for the dominant reaction are the proton–proton reaction, the carbon cycle, and the triple-alpha process, and using the reaction rates given in Schwarzschild (1958) with abundances $X = 0.9$, $X_{\text{CN}} = 0.003$ and $Y = 0.1$, we ascertain that nuclear reactions cannot disrupt the SMS before efficient accretion begins.

Deep within the accretion flow, timescales are so short that any calculation using equilibrium reaction rates must yield an upper bound on the rate of nuclear-energy generation. To stop the infall, we require

$$\frac{\epsilon_{\text{nuc}}}{\rho} > \frac{\nu_{\text{H}}^3}{r}$$  

where $\epsilon_{\text{nuc}}$ is the rate of nuclear-energy generation per unit volume. Since $\epsilon_{\text{nuc}}$ is exponential in $T$, the left-hand side of (54) increases inwards more rapidly than the right-hand side. Therefore, we consider (54) in the limit $r \rightarrow r_S$. If the carbon cycle dominates, (54) becomes

$$2 \times 10^{28} m^{-1/2} m_{\text{BH}}^{-2/15} \exp \left( -17.4 m_{\text{BH}}^{1/15} \right) > 9 \times 10^{25} m_{\text{BH}}^{-1}$$

(55)
for which a necessary condition, given $m_{\text{BH}} > 1$, is

$$6 \times 10^{20} m^{-1/2} > 9 \times 10^{25} m_{\text{BH}}^{8/15}.$$  \tag{56}

Inequality (56) is never satisfied for the masses in question. If the triple-alpha process dominates, (54) becomes

$$7 \times 10^{13} m^{-1} m_{\text{BH}}^{-3/5} \exp (-7 m_{\text{BH}}^{1/5}) > 9 \times 10^{25} m_{\text{BH}}^{-1}$$  \tag{57}

which is never satisfied for $m_{\text{BH}} \geq 1$. Therefore, we conclude that nuclear reactions cannot impede the accretion flow.

Angular momentum, mechanical turbulence and magnetic fields may also decrease $\dot{M}$. We can only appeal to uncertain dissipation mechanisms to overcome them. However, it is worth noting that these effects will also slow down the evolution of the SMS as a whole. In the higher density environment near the black hole it is conceivable that dissipative processes governing $\dot{M}$ will be relatively more efficient than in the bulk of the SMS, in the sense that they might allow accretion timescales which are still shorter than the lengthened timescale for SMS evolution. Furthermore, the slowing down of SMS evolution through the epoch when accretion is diffusion-dominated may result in the black hole being able to devour the SMS at an $R$ much larger than $R_{\text{doub}}$.

6 Discussion

Our analysis of optically thick radiation-dominated accretion underlines a very important difference between the ways in which radiation pressure and thermal pressure operate in dynamic processes. Whereas thermal pressure is an intrinsic property of the gas, radiation pressure may be thought of as a gas property only in so far as photons interact with the gas particles. Since the pressure of a radiation-dominated gas has no fixed relationship with its density, radiation diffusion may lead to the 'deflation' of the gas at constant pressure, and consequently to an enhanced accretion rate, if the gas is squeezed slowly enough at large $r$. Indeed, even when there is enough thermal pressure to stymie this process, diffusion, coupled with the resulting effort of the gas to minimize pressure gradients, may yield an accretion rate as much as $\sim (c/c_\infty)^2$ times higher than the Bondi accretion rate computed for thermal pressure alone. In other words, a radiation-dominated gas can often respond to a gravitational field at distances much larger than what one would ordinarily regard as the accretion radius.

Another interesting result is the apparently unique selection of the accretion rate $\dot{M}_B[(\sqrt{2}/3)(c/\tau_B c_\infty)]$ in the limit $(\sqrt{2}/3)(c/\tau_B c_\infty) \gg 1$, when thermal pressure is neglected. Even a slightly lower $\dot{M}$ results in the flow encountering a region where radiation pressure must grow as $r^{-5/2}$, followed at smaller $r$ by the forbidden settling solution. Since adiabatic flow yields $p \propto r^{-2}$, the $r^{-5/2}$ dependence implies that kinetic energy of infall is being converted to radiation energy. But virtually all of the kinetic energy goes into the gas particles, not into the photons, and therefore it is essential that there be some reservoir of internal energy in the gas, to mediate the conversion of kinetic energy into radiation. This is the point that Kafka & Mészáros missed. It appears that $P_{\text{rad}} \propto r^{-5/2}$ and $P_{\text{rad}} > P_{\text{eff}}$, where $P_{\text{eff}}$ represents all pressures intrinsically associated with the gas (thermal, magnetic, turbulent, etc), are incompatible statements, and therefore that Kafka & Mészáros should have included $P_{\text{eff}}$ in their equations. I will discuss this point in a forthcoming paper.

On the applied aspect of the problem, we note that the large swallowing rates inferred for small black holes in super-massive stars enable us to circumvent the commonly accepted
lower mass limit on black holes formed from SMS's. Von Hoerner & Saslaw (1976) and others have argued that an SMS of \( m \leq 3 \times 10^6 \) would ignite its nuclear fuel and blow itself apart before reaching its Schwarzschild radius. Our results indicate that an SMS of \( m \leq 5 \times 10^5 \) has a chance to save itself from dispersal if it can persuade a stellar-mass black hole to become bound to it and penetrate its interior.

Another application of the theory, accretion on to black holes in the pre- and post-recombination universe, is complicated by the presence of the Hubble flow, and therefore a detailed treatment has been omitted from this paper. In many cases of interest the flow is optically thin, and hence affected by radiation drag. These effects have been considered independently by Carr (in preparation).

Acknowledgments

It is a pleasure to thank Bernard Carr, Martin Rees, Jim Pringle and Peter Mészáros for stimulating discussions. I acknowledge the support of a National Science Foundation Graduate Fellowship, and of an Isaac Newton Studentship, during the course of this work.

References