Validity of the linearized theory for complete viscous polytropes

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Summary. The linearized theory is shown to be self-consistent when viscosity and thermal conductivity are included in the study of complete polytropes.

1 Introduction

The polytropic models have been widely used (Lamb 1945; Skumanich 1955) in order to understand the effect of compressibility on the instabilities arising in the outer convection zone of stars. The complete polytropes where the density and temperature vanish at the top have been investigated for studying the convective (Spiegel & Unno 1962) and acoustic modes (Spiegel 1964; Jones 1976) in the framework of the linearized theory. In an earlier paper (Antia & Chitre 1978), hereinafter Paper 1 it is argued that for an inviscid thermally conducting polytropic fluid the linearized approximation is not valid in the region of vanishing temperature. This results from the behaviour of the Eulerian perturbations to the steady-state values of various physical quantities which become arbitrarily large compared to the steady-state values themselves in the limit of vanishing temperature. In this paper we extend our work to complete polytropes when both viscosity and thermal conductivity are taken into account and find that the linearized theory is self-consistent for viscous polytropes.

2 Mathematical statement of the problem

We consider a plane-parallel viscous fluid layer confined between the planes $z = 0$ and $z = d$ and stratified under constant gravity acting in the negative $z$ direction. The governing hydrodynamical equations in the usual notation are

$$\rho \left( \frac{\partial v}{\partial t} + (v \cdot \nabla)v \right) = -\nabla P + \rho g + \mu [ \nabla^2 v + \frac{1}{3} \nabla(\nabla \cdot v)]$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0$$

$$\rho C_v \left( \frac{\partial T}{\partial t} + v \cdot \nabla T \right) - RT \left( \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho \right) = \nabla \cdot (K \nabla T) + \Phi$$

$$P = R \rho T.$$
Here $\Phi$ is the viscous dissipation term and we have assumed the coefficient of viscosity $\mu$, the gas constant $R$, specific heat at constant volume $C_v$, and the radiative conductivity $K$ to be constant over the layer. We shall follow the procedure of Paper I to linearize and nondimensionalize these equations. All the equations that follow are in dimensionless form. Furthermore, for the polytropic fluid with index

$$\Gamma = \frac{d \ln P_0}{d \ln \rho_0} = \frac{m + 1}{m}$$

we have the basic state given by,

$$T_0 = 1 - \beta z, \quad \rho_0 = T_0^m, \quad P_0 = T_0^{m+1},$$

where

$$\beta = \frac{\Gamma - 1}{\Gamma} = \frac{1}{m + 1}.$$

Assuming the fluid to be optically thick the linearized equations take the following form:

$$G_\mu \left( \beta^2 \frac{d^2}{dT_0^2} - k_\perp^2 \right) - \omega \rho_0 \left( \frac{W}{\Gamma T_0} + \beta \frac{dW}{dT_0} + \frac{\omega}{T_0} \theta - \frac{\omega P_1}{\rho_0 T_0} \right)$$

$$+ k_\perp^2 \left( 1 + \frac{\omega G_\mu}{3\rho_0 T_0} \right) P_1 - \frac{\omega G_\mu}{3 T_0} k_\perp^2 \theta - \frac{G_\mu k_\perp^2}{3 \Gamma T_0} W = 0$$

$$G_\mu \left( \beta^2 \frac{d^2}{dT_0^2} - k_\perp^2 \right) - \omega \rho_0 \left( W + \beta \frac{dP_1}{dT_0} - \frac{P_1}{T_0} + \frac{\rho_0}{T_0} \theta - \beta \frac{G_\mu}{3} \frac{d}{dT_0} \left( \frac{\omega}{T_0} - \frac{\omega P_1}{\rho_0 T_0} + \frac{W}{\Gamma T_0} \right) = 0 \right)$$

$$G_k \beta^2 \frac{d^2 \theta}{dT_0^2} - (G_k k_\perp^2 + \omega \gamma \rho_0) \theta + \left( 1 - \frac{\gamma}{\Gamma} \right) \rho_0 W + \omega (\gamma - 1) P_1 = 0$$

(2)

$$ik_\perp u = \frac{W}{\Gamma T_0} + \beta \frac{dW}{dT_0} + \frac{\omega}{T_0} \theta - \frac{\omega P_1}{\rho_0 T_0}.$$

Here $G_k = K_\beta / \| \rho_0(0) C_v(\gamma RT_0(0))^{3/2} \|$, $G_\mu = \mu_0 g / \| \rho_0(0)(\gamma RT_0(0))^{3/2} \|$, $\gamma = C_p / C_v$ and $P_1, \theta, W$ and $u$ are respectively the Eulerian perturbations to the pressure, temperature and the vertical and horizontal components of velocity. We shall consider the rigid boundary conditions (cf. Gough et al. 1976) at $T_0 = 0$, namely

$$\rho_0 W = 0, \quad \theta = 0, \quad \frac{d u}{d T_0} = 0.$$

(3)

The elimination of $\theta$ and $P_1$ from equation (2) yields a sixth-order differential equation in $W$ and we look for series solution about $T_0 = 0$. The resulting indicial equation has the roots $s = 0, 1, 2, 3, m + 1$ and $m + 2$. The behaviour of perturbed eigenfunctions in the neighbour-

*The quantities on the right side of these definitions of parameters $G_k, G_\mu$ and $\gamma$ are in standard units.
hood of $T_0 = 0$ are as follows:

$s = 0$: \[ W \sim T_0^0, \quad \theta \sim T_0^{m+2}, \quad P_1 \sim T_0^m \]

$s = 1$: \[ W \sim T_0^1, \quad \theta \sim T_0^{m+3}, \quad P_1 \sim T_0^{m+3} \]

$s = 2$: \[ W \sim T_0^2, \quad \theta \sim T_0^{m+4}, \quad P_1 \sim T_0^{m+2} \]

$s = 3$: \[ W \sim T_0^3, \quad \theta \sim T_0^{m+3}, \quad P_1 \sim T_0^{m+1} \]

$s = m + 1$: \[ W \sim T_0^{m+1}, \quad \theta \sim T_0^0, \quad P_1 \sim T_0^m \]

$s = m + 2$: \[ W \sim T_0^{m+2}, \quad \theta \sim T_0^0, \quad P_1 \sim T_0^{m+1} \]

It can be seen that solutions given by $s = 1, 2, 3$ and $m + 2$ are consistent with the linear theory, while the solutions corresponding to $s = 0$ and $m + 1$ are eliminated by the boundary conditions (3). The boundary conditions also eliminate the solution corresponding to $s = 2$. Thus it is clear that the linearized theory would be self-consistent for any set of boundary conditions which eliminate solutions corresponding to $s = 0$ and $m + 1$. The noteworthy feature is that the introduction of vanishingly small viscosity in the problem validates the linearized theory.

### 3 Discussion and conclusions

We have shown that the linearized theory is self-consistent in the case of a viscous polytrope even when the temperature vanishes at one of the boundaries. Thus it appears that the viscous forces damp the perturbations in such a way as to make them small as compared to the corresponding steady-state values, in the region of vanishing temperature. This is in contrast to the case of inviscid fluid where the perturbations can be arbitrarily large compared to the corresponding steady-state values. It should be stressed that the indicial equation is completely different for the viscous fluid layer since all the highest-order terms and hence the behaviour of perturbed quantities in the neighbourhood of the boundary at $T_0 = 0$ are controlled by the viscous terms in the equations of motion. This is expected from the boundary layer theory in which the behaviour of perturbations in the boundary layer turns out to be significantly different from the corresponding inviscid solutions. Consequently it is not altogether surprising to find that the viscous forces can lead to solutions which are consistent with the basic assumptions of the linear theory.

It should be noted that we have used Eulerian perturbations in this work as well as in Paper I, and it would be interesting to examine the behaviour of Lagrangian perturbations in the neighbourhood of $T_0 = 0$. It can be easily seen that in the viscous case Lagrangian perturbations will not make any difference since the linear theory is self-consistent in any case. For the inviscid case considered in Paper I it can be seen that the Lagrangian perturbation in pressure $\Delta P_1$ is given by:

\[
\Delta P_1 = P_1 + \xi \cdot \nabla P_0 = P_1 - \frac{\rho_0 W}{\omega}
\]

where $\xi$ is the displacement. It then follows that the leading term in $P_1$ is cancelled by the second term for both optically thick and thin cases and so the relative Lagrangian perturbation $\Delta P_1/P_0$ remains finite at $T_0 = 0$, which is consistent with the assumptions of the linear
theory. However, the behaviour of Lagrangian perturbation of the density is not so straight-forward,

\[ \Delta \rho_1 = \rho_1 + \xi \cdot \nabla \rho_0 = \rho_1 - \frac{\rho_0 W}{\Gamma T_0 \omega} \]

\[ = \frac{a_0 T_0^{m-1}}{\omega (\omega \gamma + q)} \quad \text{for the optically thin case} \quad [s = m + q/(\omega \gamma + q)] \]

\[ = \frac{a_0}{\omega} T_0^m \ln T_0 \quad \text{for the optically thick case.} \quad (4) \]

Here we have assumed \( q \) to be constant, while actually the assumption of a polytropic steady state implies \( q = a_0 T_0^{-m} \) and so for \( m \neq 6 \) the indicial equation would be different. For \( m < 6 \) the indicial equation would have roots \( s = 0, m \) and the solutions corresponding to \( s = m \) are

\[ \rho_0 W \approx a_0 T_0^m, \quad P_1 \approx \frac{a_0 T_0^m}{\omega}, \quad \rho_1 \approx \frac{a_0 T_0^{m-1}}{\omega \Gamma}, \quad \rho_0 \theta \approx \frac{\alpha_0}{\omega} T_0^m. \]

Thus, although all the relative Eulerian perturbations \( P_1/P_0, \rho_1/\rho_0, \theta/T_0 \sim 1/T_0 \), the Lagrangian perturbations \( \Delta P_1/P_0, \Delta \rho_1/\rho_0, \Delta \theta/T_0 \) all remain finite at \( T_0 = 0 \). Thus the linear theory would be self-consistent for Lagrangian perturbations. For \( m > 6 \) the indicial equation has roots \( s = 0, m + 1 \) and the solutions corresponding to \( s = m + 1 \) are

\[ \rho_0 W \approx a_0 T_0^{m+1}, \quad P_1 \approx \frac{a_0 T_0^{m+1}}{\omega}, \quad \rho_1 \approx \frac{a_0 T_0^m}{\omega}, \quad \rho_0 \theta \approx \frac{\gamma \alpha_0}{\Omega_0} T_0^{2m-5} \]

and it can be seen that \( P_1/P_0, \theta/T_0, \rho_1/\rho_0, \Delta P_1/P_0, \Delta \theta/T_0, \Delta \rho_1/\rho_0 \), all are finite at \( T_0 = 0 \) thus validating the linear theory.

For the optically thick case no such adjustment is possible since the assumption of a polytropic steady state implies \( G_k \) is constant. However, in this case it should be noted that the solution corresponding to \( S = m \), which is eliminated by the boundary condition \( \theta = 0 \), will be consistent with the linear theory if Lagrangian perturbations are considered. Thus by choosing a suitable boundary condition it may be possible to make the linear theory consistent even for the optically thick case.

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References