Cosmic-ray diffusion and the turbulent galactic magnetic field

J. C. Carvalho and D. ter Haar. Department of Theoretical Physics, 1 Keble Road, Oxford OX1 3NP

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Summary. Using Lerche’s turbulent kinematic-dynamo equations we calculate the mean square displacement of the fluctuating component of the galactic magnetic field and hence the cosmic-ray diffusion coefficient. We find that the radial variation of the diffusion coefficient thus calculated agrees well with that deduced from the γ-ray observations, provided we use a mean velocity field which includes a compressional component, which according to Burton is also responsible for the observed peaking of H II regions, CO, supernova remnants and γ-rays at about 6 kpc from the Galactic Centre.

1 Introduction

In an earlier paper (Carvalho & ter Haar 1977a) we have shown that in order to explain the galactic cosmic-ray density \( U \), as inferred from the observed diffuse γ-ray diffusion (Fichtel et al. 1975; see also Thompson et al. 1976) which has a maximum at about 5–6 kpc from the Galactic Centre, we must assume not only that \( U \) is proportional to the supernova remnant density (cosmic-ray source density) but also to the inverse ‘life-time’ \( \langle T \rangle \) which is given by the equation

\[
\langle T \rangle^{-1} = T_D^{-1} + T_d^{-1},
\]

where \( T_d \) is the characteristic time for nuclear destruction and \( T_D \) the leakage time. Moreover, we found that in order to explain the observational data it was also necessary that \( T_D \) is proportional to some positive power of the molecular density, with an index \( \alpha \) about equal to 1, which means that the reciprocal of the diffusion coefficient must show this density-dependence.

In the present paper we shall calculate the diffusion coefficient of 1–10 GeV cosmic rays, assuming that they are trapped on the magnetic lines of force and that these lines of force are performing a random walk at right angles to the average field. In our calculations we use Lerche’s (1971) turbulent kinematic-dynamo equation. In Section 2 we discuss our basic equations, which are an expression for the diffusion coefficient of the cosmic rays in terms of the fluctuating field amplitude and its correlation length, as well as the dynamo equations for the main averaged field strength and the fluctuating field component. In Section 3 we
simplify the dynamo equations by making suitable assumptions about the nature of the magnetic field and the velocity field. In Section 4 we discuss the solutions of our simplified equations in the two limiting cases of vanishing main velocity field component and of vanishing turbulent velocity component. The general case is discussed in Section 5 and we present there the results of some numerical studies of the equations. Our conclusions are presented in Section 6.

2 Basic equations

Jokipii (1973) has studied the effect of a random component of the galactic magnetic field on the diffusion of relatively low-energy charged particles across the average magnetic field. If such a particle moves a distance \( \Delta s \) along the mean field \( B_0 \), its mean square displacement \( \langle (\Delta z)^2 \rangle \) is according to Jokipii given by the equation

\[
\langle (\Delta z)^2 \rangle \approx eL\Delta s,
\]

where \( L \) is the correlation length of the fluctuating field \( b \), which we assume to be mainly at right angles to \( B_0 \),

\[
e = \langle b^2 \rangle / B_0^2,
\]

and the pointed brackets denote here and henceforth suitable ensemble averages. In deriving (2) it is assumed that the particles are tied to the field lines due to the fact that their Larmor radius is small compared to \( L \) so that they will random-walk with the field. Using different methods (see, e.g. Jones 1971 or Kaiser 1973) to calculate the mean square displacement of the field lines leads to essentially the same result as (2). From equation (2) we can now find the diffusion coefficient \( D \):

\[
D \propto \lambda \approx \frac{\langle (\Delta z)^2 \rangle}{\Delta s} \approx eL.
\]

If \( L \) may be taken to being essentially constant, the variation of \( D \) will be that of \( \langle b^2 \rangle / B_0^2 \). In deriving (4) we have used the fact that the diffusion coefficient is proportional to the mean free path \( \lambda \).

We shall see later that there is no reason to assume that the fluctuating component parallel to \( B_0 \) is small. If we drop that assumption, equation (2) still holds, but with \( \epsilon \) no longer given by (3). We show in Appendix A that we have

\[
e = -\alpha \exp \left[ \frac{1}{4} \frac{\sigma_x^2}{\sigma_z^2} \frac{\sigma_z}{\sqrt{2} \sigma_x} \right] \left( \frac{1}{2} \sigma_x^2 - \frac{B_0}{\sigma_z} \right),
\]

where \( \alpha \sim 0.6 \) and \( a \sim 1.6 \) are numerical coefficients, \( B_0 \) is the main field which is taken along the \( x \) direction in the galactic plane, and

\[
\sigma_x^2 = \langle b_x^2 \rangle, \quad \sigma_z^2 = \langle b_z^2 \rangle,
\]

where \( b_x \) and \( b_z \) are the fluctuating components of the field in and at right angles to the galactic plane.

As long as \( \Delta s \) is small compared to the scale of variation of \( B_0 \), we may put \( B_0 \) and \( b_x \) equal to the azimuthal components \( B_\phi \) and \( b_\phi \), in a cylindrical coordinate system \((r, \phi, z)\), centred at the Galactic Centre.

To find the magnetic field, and hence the diffusion coefficients \( D \), we use the turbulent
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kinematic-dynamo equations of Lerche (1971). To do this we split the magnetic field and the velocity field into a mean and a fluctuating component, \( B + b \) and \( V + v \), with

\[
\langle b \rangle = 0 \quad \text{and} \quad \langle v \rangle = 0.
\]

In a stationary state the dynamo equations can be written in the form

\[
-\eta V^2 B = \text{curl} [V \wedge B] + \text{curl} [\langle v \wedge b \rangle],
\]

\[
-\eta V^2 b = \text{curl} [V \wedge b] + \text{curl} [v \wedge B].
\]

Here \( \eta \) is the effective turbulent diffusion coefficient; it reflects the turbulent motion of interstellar clouds and is responsible for the dissipation of the magnetic field. Parker (1971a,b) has given the following estimate of \( \eta \):

\[
\eta = f(u^2)^{1/2} L_t,
\]

where \( L_t \) is the characteristic length of the turbulence which will be of the order of \( L \) and \( f \) a numerical constant of order 0.1.

In this kinematic approach we take the velocity field to be given and equations (8) and (9) are then to be solved for \( B \) and \( b \).

3 The approximate equations

To be able to reach conclusions from equations (8) and (9) we must make further assumptions. First of all we assume that \( B \) and \( V \) vary only on a length scale of the order of \( L \) which we take to be of the order of a few kpc, while \( L \) is taken to be of the order of a few hundred parsec (Carvalho & ter Haar 1977b; Jokipii, Lerche & Schommer 1969), so that, indeed \( L \ll L \). In that case we can neglect the term on the left side of equation (8), except when the right side vanishes identically (vide infra). On the other hand, we assume \( b \) to vary both on the scale \( L \) and on the same scale, \( L \), as \( B \). Finally we assume the turbulent velocity to be uniform and to vary only on the scale \( L \), but not on the scale \( L \).

We can express the above assumptions through the equations \((i, j = r, \phi, z)\)

\[
\langle u_i(r') b_j(r) \rangle = G_{ij}(r) \Phi_i(|r - r'|),
\]

\[
\langle u_i(r') u_j(r) \rangle = \frac{1}{3} \langle u^2 \rangle \Phi_2(|r - r'|),
\]

\[
\langle b_i(r') b_j(r) \rangle = H_{ij}(r) \Phi_3(|r - r'|),
\]

where \( G_{ij}(r) \) and \( H_{ij}(r) \) are slowly varying functions of \( r \), changing on a scale \( L \), while the \( \Phi_i(x) \) are correlation functions with correlation lengths \( L_i \), which are normalized such that \( \Phi_i(0) = 1 \); by assumption the correlation lengths \( L_i \) are all of the same order of magnitude as \( L \).

We now write

\[
B \equiv (B_r, B_\phi, B_z), \quad V \equiv (U, V, 0), \quad b \equiv (b_r, b_\phi, b_z), \quad v \equiv (u_r, u_\phi, u_z),
\]

and we assume all quantities, including \( G_{ij} \) and \( H_{ij} \), to depend only on \( r \). In equations (14) \( U \) is the radial and \( V \) the rotational velocity of the interstellar gas, and we shall assume, through lack of better information, that the turbulent velocity field is uniform, so that

\[
\langle u_r^2 \rangle = \langle u_\phi^2 \rangle = \langle u_z^2 \rangle = \frac{1}{3} \langle v^2 \rangle.
\]
Equations (8) and (9) together with the Maxwell equation

\[
\text{div } (B + b) = 0
\]

now become

\[
\frac{d}{dr} \left( r \frac{dB_r}{dr} \right) = 0, \tag{17}
\]

\[
- \frac{d}{dr} (UB_\phi - VB_r) - \frac{d}{dr} (\nu b_\phi - vb_r) = 0 \tag{18}
\]

\[
- \frac{d}{dr} (rUB_z) + \frac{d}{dr} (r(\nu b_r - vb_\nu)) = 0, \tag{19}
\]

\[
\nabla^2 b_r = 0, \tag{20}
\]

\[
- \eta \nabla^2 b_\phi = - \frac{d}{dr} (\nu B_\phi - vb_\phi) - \frac{d}{dr} (UB_\phi - Vb_r), \tag{21}
\]

\[
- \eta \nabla^2 b_z = \frac{1}{r} \frac{d}{dr} [r(\nu B_r - vb_r)] - \frac{1}{r} \frac{d}{dr} (rUb_z), \tag{22}
\]

\[
\frac{dB_r}{dr} + \frac{B_r}{r} + \frac{db_r}{dr} + \frac{b_r}{r} = 0. \tag{23}
\]

Multiplying equations (20)–(23) by \( \nu \), and taking the average of the resulting equations, and using equations (11) and (12), as well as (B17) and (B20) from Appendix B, we find

\[
\frac{d}{dr} \left( r \frac{dB_r}{dr} \right) = 0, \tag{24} \quad \equiv (17)
\]

\[
- \frac{d}{dr} (UB_\phi - VB_r) - \frac{d}{dr} (F - R) = 0, \tag{25}
\]

\[
- \frac{d}{dr} (rUB_z) + \frac{d}{dr} [r(R - Z)] = 0, \tag{26}
\]

\[
\frac{d}{dr} \left( r \frac{dR}{dr} \right) = 0, \tag{27}
\]

\[
- \frac{1}{3} \langle \nu^2 \rangle \frac{d}{dr} (B_\phi - B_r) - \frac{d}{dr} (UF - VR) = \omega_0 F, \tag{28}
\]

\[
\frac{1}{3} \langle \nu^2 \rangle \frac{1}{r} \frac{d}{dr} [r(B_r - B_z)] - \frac{1}{2} \frac{d}{dr} (rUZ) = \omega_0 Z, \tag{29}
\]

\[
\frac{dR}{dr} + \frac{R}{r} = 0, \tag{30}
\]

where

\[
G_{rr} \equiv R, \quad G_{\phi\phi} \equiv F, \quad G_{zz} \equiv Z, \quad \omega_0 = \eta/L^2, \tag{31}
\]

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and where we have used the inequalities (cf. (B5) and (B18), and the fact that the $G_{ij}$ vary on the scale $L$)

$$\left| \left\langle \frac{db}{dr} \right\rangle \right| \ll \left| \frac{\langle vb \rangle}{L} \right|, \quad \left| \left\langle \frac{d^2 b}{dr^2} \right\rangle \right| \ll \frac{\langle vb \rangle}{L^2}. \quad (32)$$

For a given velocity field, equations (24)–(30) describe how the magnetic field is generated and dissipated as a result of the radial compression ($U$), differential rotation ($V$), and turbulent motion ($v$) of the gas. The dissipation terms are the ones on the right side of equations (28) and (29). The solution of equations (24)–(30) gives us the stationary magnetic field configuration after it has rearranged itself through interacting with the interstellar gas.

We show in Appendix B that it is possible in our model to express $\langle b_z^2 \rangle$ and $\langle b_\phi^2 \rangle$ in terms of $Z$ and $F$ as follows:

$$\langle b_z^2 \rangle \approx 3 Z^2 \langle v^2 \rangle, \quad \langle b_\phi^2 \rangle \approx 3 F^2 \langle v^2 \rangle, \quad (33)$$

and we get then from equations (4) and (5)

$$D(r) \propto \exp \left[ \frac{a^2 P^2}{4 Z^2} - \frac{a B \langle v^2 \rangle^{1/2}}{\sqrt{6} |Z|} \right] \operatorname{erfc} \left[ \frac{a}{2} \frac{F}{Z} - \frac{B \langle v^2 \rangle^{1/2}}{\sqrt{6} |F|} \right], \quad (34)$$

provided $\langle v^2 \rangle \neq 0$. If $\langle v^2 \rangle = 0$, one must use a slightly different expression (vide infra).

4 Two limiting cases

We now turn to the task of finding a solution of equations (24)–(30). We first of all notice that from equations (27) and (30) it follows that

$$R = 0, \quad (35)$$

while equation (24) leads to

$$B_r = B_{r0} + \beta \ln \frac{r}{r_0}, \quad (36)$$

where $\beta$ and $B_{r0}$ are integration constants.

Before looking at the general case, we consider two limiting cases:

case 1: $V = 0, \quad \langle v^2 \rangle \neq 0$;

case 2: $\langle v^2 \rangle = 0, \quad V \neq 0$.

In case 1 there is no overall motion of the gas and the magnetic gas is generated by the turbulent motion of the gas. Equations (25), (26), (28) and (29) now become

$$dF/dr = 0, \quad (37)$$

$$\frac{1}{3} \langle v^2 \rangle \frac{dr}{d} (B_\phi - B_r) = - \omega_0 F, \quad (39)$$

$$\frac{1}{3} \langle v^2 \rangle \frac{1}{r} \frac{dr}{d} [r(B_r - B_z)] = \omega_0 Z. \quad (40)$$
Once again we shall take $B_z = 0$ which is in accordance with the observational data (Heiles 1976) that the galactic field lies in the plane of the Galaxy and we shall here and henceforth assume that $\beta$ in (36) vanishes, that is,

$$B_r = \text{constant} \equiv B_{r_0}. \tag{41}$$

Using equations (5), (35) and (41) we find from equations (37) to (40)

$$F = F_0, \tag{42}$$

$$Z(r) = Z_0 \frac{r_0}{r} \tag{43}$$

$$B_\phi(r) = B_{\phi 0} - \frac{3 \omega_0}{\langle u^2 \rangle} F_0 (r - r_0). \tag{44}$$

The value of $D$ now follows from equation (34) and in the next section we present numerical results following from equations (42) to (44). The quantities with subscript 0 indicate values at $r = r_0$.

In case 2 we consider merely the effect of the mean (radial and rotational) velocity on the magnetic field. We now multiply equation (22) by $b_z$ and take the average. From equations (15), (B19) and (B21) we then get the following equations for $B_\phi$ and $b_z$:

$$\frac{d}{dr} (UB_\phi - VB_r) = 0, \tag{45}$$

$$\omega_0 M = - \frac{M}{r} \frac{d}{dr} (rU) - \frac{1}{2} U \frac{dM}{dr} \tag{46}$$

where we have put $G_{ij} \equiv 0$ and where

$$M \equiv H_{zz}. \tag{47}$$

As in deriving equation (8) we have already assumed that $b_\phi$ and $b_r$ are independent, we can put

$$\langle b_\phi b_r \rangle = 0, \tag{48}$$

and we can get an equation for $N$, defined by

$$N \equiv H_{\phi \phi}, \tag{49}$$

through multiplying equation (21) by $b_\phi$ and averaging. The result is

$$\omega_0 N = - N \frac{dU}{dr} - \frac{1}{2} U \frac{dN}{dr}, \tag{50}$$

where we have used equations (B19) and (B21).

If we assume that $U = U_0 = \text{constant}$, we find the solution

$$N(r) = N_0 \exp \left[ - \frac{2 \omega_0}{U_0} (r - r_0) \right] \tag{51}$$

$$B_\phi(r) = B_{\phi 0} + \frac{B_{r 0}}{U_0} [V(r) - V_0], \tag{52}$$

$$M(r) = M_0 \left( \frac{r_0}{r} \right)^2 \exp \left[ - \frac{2 \omega_0}{U_0} (r - r_0) \right]. \tag{53}$$
We shall take $V(r)$ from the Schmidt model (Schmidt 1957). As far as $U$ is concerned, Yuan (1969) concluded from the asymmetry of the velocity profile of the thermal emission in the Perseus arm that there is a motion towards the Galactic Centre. In the inner region of the Galaxy (l$^m = 0^\circ$) the same behaviour is observed for negative latitude, but for positive latitude the distribution is essentially centred at zero velocity. Kerr (1969) has pointed out, however, that these observations may have been affected by turbulent motions. In the framework of Lin’s density-wave theory (Lin, Yuan & Shu 1969) one would expect a radial velocity of the order of $10 \text{ km/s}$ towards the Galactic Centre. As a first approximation we feel justified in assuming $U$ to be constant and to have a value between 0 and $-15 \text{ km/s}$.

![Graph](https://example.com/graph1.png)

**Figure 1.** The function $Q(r) = \left[ \frac{\rho_H(r)}{\rho_H(0)} \right]^{-\alpha}$ for various values of $\alpha$. The hydrogen density given by Gordon & Burton (1976) in the form of a histogram has been smoothed out.

![Graph](https://example.com/graph2.png)

**Figure 2.** The quantities $\langle b_z^2 \rangle^{1/2}$, $B_\phi(r)$ and $D(r)$ (curves A and B) as functions of distance from the Galactic Centre in the limiting case 1: $V = 0$, $\langle v^2 \rangle \neq 0$. The diffusion coefficient is normalized to 1 at $r = 10 \text{ kpc}$. The value of the various constants and initial conditions are in Table 1.
5 The general case: numerical results

We now solve equations (24)–(30) for the general case when both \( \langle u^2 \rangle \) and \( V \) are non-vanishing, but where equations (35) and (41) still hold and also the equation \( B_z = 0 \). We introduce the quantities

\[
S(r) = \frac{\langle u^2 \rangle}{\langle u^2 \rangle - 3U_0^2} B_{r0} [V(r) - V_0].
\]

\[
\lambda = \frac{3 \omega_0 U_0}{\langle u^2 \rangle - 3U_0^2},
\]

(54)

(55)

Figure 3. The quantities \( \langle b_z^2 \rangle^{1/2}, \langle b_\phi^2 \rangle^{1/2}, B_\phi(r) \) and \( D(r) \) as in Fig. 2 in the limiting case 2: \( V = 0, \langle u^2 \rangle = 0 \).

Figure 4. The quantity \( D(r) \) as in Fig. 3 for different initial conditions.
and the solutions for $Z$, $F$ and $B_\phi$ then are

$$Z(r) = Z_0 \frac{r_0}{r},$$ (56)

$$F(r) = F_0 + S(r) - S_0 - \lambda \exp(\lambda r) \int_r^{r_0} S(r') \exp(-\lambda r') \, dr',$$ (57)

$$B_\phi(r) = B_{\phi 0} - \frac{\lambda B_0}{\omega_0} [V(r) - V_0] + \frac{\lambda}{U_0} \int_r^{r_0} F(r') \, dr'.$$ (58)

Figure 5. The quantities $\langle b_\phi^2 \rangle^{1/2}$, $B_\phi(r)$ and $D(r)$ in the general case for $\omega_0 = 0.8$ km/(s kpc), $U_0 = -12$ km/s and $\langle v^3 \rangle^{1/3} = 10$ km/s.

Figure 6. The quantities $\langle b_\phi^2 \rangle^{1/2}$, $B_\phi(r)$ and $D(r)$ in the general case for $\langle v^3 \rangle^{1/3} = 8$ km/s, $\omega_0 = 1.8$ km/(s kpc) (curve A) and $\omega_0 = 1.9$ km/(s kpc) (curve B).
From equation (34) we see that equations (56)–(58) determine $D$. In our numerical work we shall take $r_0 = 10$ kpc. The turbulent diffusion coefficient is taken from Parker's work (1971a,b) to lie between $10^{25}$ and $10^{26}$ cm$^2$ s$^{-1}$ and $L$ from our earlier work (Carvalho & ter Haar 1977b) to be about 300 pc. In that case $\omega_0$ lies between 0.37 and 3.7 km/(skpc). We also found earlier from considering the rotation measure of extragalactic radio sources that the magnitude of the fluctuating component of the magnetic field in the Galaxy is quite large: $\langle b^2 \rangle / B_0^2$ seems likely to be of the order of 0.8 (Carvalho & ter Haar 1977b). We shall

![Figure 7](image1)

Figure 7. The quantity $D(r)$ in the general case for $U_0 = -15$ km/s, $\langle \sqrt{b^2} \rangle^{1/2} = 10$ km/s, $\omega_0 = 2.5$ km/(skpc), $F > 0$ (curve A) and $F < 0$ (curve B).

![Figure 8](image2)

Figure 8. The quantity $D(r)$ in the general case for $U_0 = -15$ km/s, $\langle \sqrt{b^2} \rangle^{1/2} = 8$ km/s, $\omega_0 = 3$ km/(skpc) (curve A) and $\omega_0 = 1.5$ km/(skpc) (curve B).
therefore choose $M_0$ and $N_0$ or $F_0$ and $B_{r_0}$ such that $(\langle b_x^2 \rangle + \langle b_y^2 \rangle + \langle b_z^2 \rangle)/[B_{r_0}^2 + B_{\phi_0}^2] \sim 0.8$. We take the rms value of a single component of the fluctuating velocity $\langle v^2 \rangle^{1/2}/\sqrt{3}$, to be 8–10 km/s and express all magnetic field strengths in units of $B_{\phi_0}$ which we take to be about 3 $\mu$gauss in accordance with our earlier results.

The values of $D$ which we calculate must then be compared with the suggested behaviour as deduced from the $\gamma$-ray flux from the galactic plane. As we mentioned in the Introduction, we deduced in our earlier paper (Carvalho & ter Haar 1977a) that $D$ should be proportional to some negative power of the gas density. We have therefore plotted in Fig. 1 the function $Q(r)$ given by the equation

$$Q(r) = \left[ \frac{n_H(r)}{n_H(r_0)} \right]^\alpha,$$

(59)

for various values of $\alpha$, using the hydrogen density distribution given by Gordon & Burton (1976).

The numerical results obtained in cases 1 and 2 from equations (42) to (44) and (51) to (53) are presented in Figs 2, 3 and 4 ($r_0 = 10$ kpc).

We can see that in both cases 1 and 2, the diffusion coefficient tends to decrease in the inner region of the Galaxy as $Q(r)$ does, having a minimum at about 5–6 kpc from the centre. Thus, it seems that this kind of behaviour is independent of whether it is the turbulent or the overall motion of the gas that dominates.

In the general case, when both $\langle u^2 \rangle$ and $\mathbf{V}$ are non-zero, the diffusion coefficient calculated with the solutions given by equations (56)–(58) has a similar behaviour, as we can see in Figs 5–8. In some cases the curves of $D(r)$ do not quite follow those of $Q(r)$ as they tend to decrease much faster at distances of about 10–8 kpc. The values of the various parameters used in all figures are shown in Table 1.

Table 1.

<table>
<thead>
<tr>
<th>Figure</th>
<th>$B_{r_0}$ (km/s kpc)</th>
<th>$\omega_0$ (km/s)</th>
<th>$U_0$ (km/s)</th>
<th>$\langle \frac{1}{3} v^2 \rangle^{1/2}$ (km/s)</th>
<th>Notes</th>
</tr>
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<tbody>
<tr>
<td>2 curve A</td>
<td>-0.40</td>
<td>2.1</td>
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<td>10</td>
<td>$\langle b_{\phi_0}^2 \rangle^{1/2} = 0.944$</td>
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<td>$\langle b_{\phi_0}^2 \rangle^{1/2} = 0.7$</td>
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<td>-0.11</td>
<td>2.5</td>
<td>-10</td>
<td>0</td>
<td>$\langle b_{\phi_0}^2 \rangle^{1/2} = 0.45$</td>
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<tr>
<td>4 curve B</td>
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<td>-10</td>
<td>0</td>
<td>$\langle b_{\phi_0}^2 \rangle^{1/2} = 0.6$</td>
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<td>6 curve A</td>
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<td>-12</td>
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<td>(B): $F &lt; 0$, $Z &lt; 0$</td>
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<td>$F &gt; 0$, $Z &lt; 0$</td>
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<td>-15</td>
<td>8</td>
<td>$F &gt; 0$, $Z &lt; 0$</td>
</tr>
</tbody>
</table>

Note: In the last column we have given for some cases the value of $\langle b_{\phi_0}^2 \rangle^{1/2}$ and for others the sign of $F$ and $Z$.

The value of $B_{r_0}$ in all cases satisfies the relation $|B_{r_0}/b_{\phi_0}| < 1$ (except in Fig. 2 where we have $|B_{r_0}/b_{\phi_0}| \sim 0.40–0.45$) and this is in agreement with the observational fact that, at least locally, the field is almost parallel to the spiral arms (Helles 1976).
6 Conclusions

From our calculations we can come to the conclusion that within a relatively wide range of parameter values the compressional velocity and differential rotation of the interstellar gas together with its fluctuating component, is able to generate a magnetic field configuration such that the mean square displacement of the field lines has a radial variation which behave just like a negative power of the gas density and this leads to an increase in the cosmic-ray confinement time in the region \( r \sim 5\text{--}6\text{ kpc} \) which is sufficient to explain the observed \( \gamma \)-ray distribution. The values of \( \langle \tau_{\gamma} \rangle^{1/2} \) and \( U_0 \) in the range \( 8\text{--}10\text{ km/s} \) and \( -10 \text{ to } -15\text{ km/s} \), respectively, give the best results for the diffusion coefficient, as well as being in good agreement with observations.

We may also draw attention to the fact that Burton (1976) has pointed out that the similarity between the radial distributions in the inner parts of the Galaxy of H II regions CO (and indirectly \( \text{H}_2 \)), supernova remnants, pulsars and \( \gamma \)-rays all apparently peaking a \( r \sim 5\text{--}6\text{ kpc} \), may be connected with a compression produced by galactic density waves. This compression is strong when the velocity component at right angles to the spiral arm (which corresponds to our \( U_0 \)) exceeds the ambient acoustic speed (which would be of the order of \( \langle u^2 \rangle^{1/2} \) so that we feel that it is not altogether fortuitous that the agreement between \( D(r) \) and \( \mathcal{Q}(z) \) is best when \( |U_0| > \langle u^2 \rangle^{1/2} \), even though there is already some agreement of other parameter values.

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References


Appendix A

In this appendix we shall derive equation (5) from Section 2. Let \( B_0 \) be the mean field component in the galactic plane, which we take to be in the \( x \) direction, \( b_x \) the fluctuation c
the field in this direction and \( B_z \) and \( b_z \) the mean and fluctuating components at right angles to the galactic plane (in the \( z \) direction). In order to simplify the discussion we have neglected the fluctuating component in the \( y \) direction. The lines of force are described by the solutions of the equation

\[
\frac{dz}{B_z + b_z} = \frac{dx}{B_0 + b_x},
\]

(A1)

which gives us the normal displacement \( \Delta z \) when we travel a distance \( \Delta x \) in the \( x \) direction:

\[
\Delta z = \int_0^{\Delta x} \frac{B_z + b_z}{B_0 + b_x} \, dx.
\]

(A2)

If we put \( B_z = 0 \) (see discussion in the main text) the mean square displacement \( \langle (\Delta z)^2 \rangle \) will be given by the equation

\[
\langle (\Delta z)^2 \rangle = \int_0^{\Delta x} dx \int_0^{\Delta x'} dx' \left\langle \frac{b_z(x) b_z(x')}{[B_0 + b_x(x)][B_0 + b_x(x')]} \right\rangle.
\]

(A3)

In order to calculate the average inside the integral, we note that \( b_x \) and \( b_z \) are stationary Gaussian processes and, therefore, a Gaussian probability distribution function may be used. It turns out that at points of zero field, where \( b_x = -B_0 \), the integrand has a quite intractable singularity. One may find a crude approximation if the perturbation of the field in the \( x \) direction \( \langle b_x^2 \rangle^{1/2}/B_0 \) is small. Expanding the denominator in a power series and taking \( b_z \) and \( b_x \) to be independent, we obtain up to fourth order

\[
\langle (\Delta z)^2 \rangle \approx \int_0^{\Delta x} dx \int_0^{\Delta x'} dx' \left\langle \frac{b_z b_z'}{B_0^2} \left[ 1 + 3 \frac{\langle b_z^2 \rangle}{B_0^2} + 15 \frac{\langle b_z^2 \rangle^2}{B_0^2} \right] \right\rangle,
\]

(A4)

if \( b_x(x) \) and \( b_x(x') \) are totally correlated, and

\[
\langle (\Delta z)^2 \rangle \approx \int_0^{\Delta x} dx \int_0^{\Delta x'} dx' \left\langle \frac{b_z b_z'}{B_0^2} \left[ 1 + 2 \frac{\langle b_z^2 \rangle}{B_0^2} + 7 \frac{\langle b_z^2 \rangle^2}{B_0^2} \right] \right\rangle,
\]

(A5)

if \( b_x(x) \) and \( b_x(x') \) are uncorrelated.

In the case where \( \langle b_x^2 \rangle^{1/2}/B_0 \) is not small, we must go back to equation (A1) and integrate it along the force line instead of over \( x \) so as to avoid turning points. Calling \( ds \) the element of length along the field line and \( \theta \) the angle between it and the \( x \) direction, one has

\[
\langle (\Delta z)^2 \rangle = \int_0^{\Delta s} ds \int_0^{\Delta s'} ds' \langle \sin \theta \sin \theta' \rangle,
\]

(A6)

where

\[
\sin \theta = \frac{b_z}{\sqrt{(B_0 + b_x)^2 + b_z^2}}.
\]

(A7)

We expect that \( \langle \sin \theta \sin \theta' \rangle \ll 1 \), when \( |s - s'| > L \), being the correlation length, whereas for \( |s - s'| \rightarrow 0 \), \( \langle \sin \theta \sin \theta' \rangle \rightarrow \langle \sin^2 \theta \rangle \). Therefore we may assume that \( \langle \sin \theta \sin \theta' \rangle \) has a behaviour identical to that of the correlation function \( \Phi_z \), for example, of the \( b_z \) field, which satisfies the relations

\[
\Phi_z(x = 0) = 1, \quad \Phi_z(x \gg L) \ll 1,
\]

(A8)

where \( x = s - s' \).
If the turbulence is homogeneous and if we consider path elements $\Delta s$ smaller than the characteristic scale of variation of $B_0$, we can write

$$\langle (\Delta z)^2 \rangle \approx \lim_{\Phi_z \to 1} \langle \sin \theta' \sin \theta' \rangle \int_0^{\Delta s} ds' \int_0^{\Delta s} ds \Phi_z(s - s').$$  \hspace{1cm} (A9)

Since $\Phi_z(\xi) = \Phi_z(-\xi)$, integrating by part gives

$$\langle (\Delta z)^2 \rangle = \eta(\Delta s) S(\sigma_z, \sigma_x, B_0),$$  \hspace{1cm} (A10)

where

$$\eta(\Delta s) = 2 \int_0^{\Delta s} (\Delta s - \xi) \Phi_z(\xi) d\xi,$$  \hspace{1cm} (A11)

$$\sigma_z = \langle b_z^2 \rangle, \quad \sigma_x = \langle b_x^2 \rangle,$$  \hspace{1cm} (A12)

and

$$S = \langle \sin \theta' \sin \theta' \rangle.$$  \hspace{1cm} (A13)

We now proceed to evaluate $S$. The bivariate Gaussian probability distribution $\Psi_z(b_z, b_x')$ has the form

$$\Psi_z(b_z, b_x') = \frac{1}{2\pi \langle b_z^2 \rangle(1 - \Phi_z^2)^{1/2}} \exp \left[ -\frac{b_z^2 - 2\Phi_z b_z b_x' + b_x'^2}{2\langle b_z^2 \rangle(1 - \Phi_z^2)} \right].$$  \hspace{1cm} (A14)

An identical expression is taken for $\Psi_x(b_x, b_x')$. In order to simplify the notation, we write

$$b_z = Z, \quad b_x = X, \quad B_0 = B,$$  \hspace{1cm} (A15)

and

$$C^{-1}_{z, x} = \sqrt{2} \sigma_z \sqrt{1 - \Phi_z}.$$  \hspace{1cm} (A16)

From equation (A7) we have

$$\sin \theta = \frac{Z}{\sqrt{(B + X)^2 + Z^2}}.$$  \hspace{1cm} (A17)

and

$$\langle \sin \theta' \sin \theta' \rangle = \int \int \int_{-\infty}^{+\infty} dZ' dZ dX dX' dX.$$  \hspace{1cm} (A18)

Changing from the variable $Z'$ to $Y = C_z(Z' - \Phi_z Z)$ we get

$$\frac{1}{\sqrt{\pi a_z}} \int \int \int_{-\infty}^{+\infty} dX' dX dZ \Psi_x \frac{Z \exp (-Z^2/2a_z^2)}{\sqrt{(B + X)^2 + Z^2}} \int_{-\infty}^{+\infty} dY \frac{C_z^{-1} Y + \Phi_z Z \exp (-Y^2)}{\sqrt{(B + X')^2 + (C_z^{-1} Y + \Phi_z Z)^2}}.$$  \hspace{1cm} (A19)

We see clearly that if $\Phi_z \to 0$, $C_z^{-1} \to \sqrt{2} \sigma_z$ and the integral vanish. Putting $\Phi_z = 1$, the integral over $Y$ is $\sqrt{\pi}$. Using now

$$\xi = C_z(X - \Phi_z X'),$$  \hspace{1cm} (A20)

one has (for the case $\Phi_z = 1$)

$$\frac{1}{2\pi^{3/2} \sigma_x \sigma_z} \int \int dX' dZ \frac{Z^2 \exp (-Z^2/2a_x^2 - X'^2/2a_z^2)}{\sqrt{(B + X')^2 + Z^2}} \int_{-\infty}^{+\infty} d\xi \frac{\exp (-\xi^2)}{\sqrt{(B + C_z^{-1} \xi + \Phi_z X')^2 + Z^2}}.$$  \hspace{1cm} (A21)
Cosmic-ray diffusion

There are two cases to be considered: $\Phi_x = 1$ and $\Phi_x = 0$. In the latter case we have, after some algebraic manipulation,

$$\frac{2\sigma_x^2}{\pi^{3/2}} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{\exp(-r^2)}{\sqrt{(B + \sqrt{2\sigma_x} r)^2 + 2\sigma_x^2 t^2}} dr \cdot (\int_{-\infty}^{\infty} \frac{\exp(-r^2)}{\sqrt{(B + \sqrt{2\sigma_x} r)^2 + 2\sigma_x^2 t^2}} dr)^2. \quad (A22)$$

Because of the complexity of this integral we shall only estimate it in the limit $\sigma_x = 0$, namely

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{\exp(-t^2)}{t^2 + B^2/2\sigma_x^2}. \quad (A23)$$

We see that if $\sigma_x/B \ll 1$, to second order, we get the usual result $2\sigma_x^2/B^2$. In general we have

$$1 - \sqrt{\pi} \frac{B}{\sqrt{2}\sigma_x} \exp(-B^2/2\sigma_x^2) \text{erfc}\left(\frac{B}{\sqrt{2}\sigma_x}\right), \quad (A24)$$

where erfc$(x)$ is the complementary error function. In a reasonable range of values of $B/\sqrt{2}\sigma_x$, the above expression may be represented by

$$\exp(-aB/\sqrt{2}\sigma_x), \quad a \sim 1.4 \text{ to } 1.6, \quad (A25)$$

which, as we show below, is the same limit as the case $\Phi_x = 1$, for $\sigma_x \to 0$.

We now take $\Phi_x = 1$, in equation (A21), and after integrating over $\xi$ we get

$$\frac{1}{\sqrt{2}\pi \sigma_x} \int_{-\infty}^{\infty} dX' \frac{\exp(-X'^2/2\sigma_x^2)}{\sqrt{2\sigma_x}} \int_{-\infty}^{\infty} \frac{r^2 \exp(-r^2)}{r^2 + (B + X')^2/2\sigma_x^2}. \quad (A26)$$

In order to obtain a simple answer we have to perform the integral in $X'$ from $-B$ to $+\infty$. Because the Gaussian in $X'$ is centred at zero, we think that this is a good estimate. Integrating over $r$ and putting

$$y = (B + X')/\sqrt{2}\sigma_x \quad (A27)$$

we get

$$\frac{1}{\sqrt{\pi} \sigma_x} \int_{0}^{+\infty} dy \exp\left(\frac{1}{2\sigma_x^2} (\sqrt{2} y - B)^2\right) \left[1 - \sqrt{\pi} y \exp(y^2) \text{erfc}(y)\right]. \quad (A28)$$

Using the approximated form (A25) of the function in the bracket, we finally obtain

$$S(\sigma_x, \sigma_z, B) = \frac{1}{2} \exp\left[\frac{a^2}{4} \frac{\sigma_x^2}{\sigma_x^2} - \frac{a B}{\sqrt{2} \sigma_x^2}\right] \text{erfc}\left(\frac{a \sigma_x}{2 \sigma_x} - \frac{B}{\sqrt{2} \sigma_x}\right). \quad (A29)$$

The following limits can be found:

$$\sigma_z = 0, \quad \sigma_x = 0: S = 0, \quad (A30)$$

$$\sigma_x \neq 0, \quad \sigma_x \to 0: S \approx \exp(-aB/\sqrt{2}) \sigma_z, \quad (A31)$$

$$\sigma_z \to 0, \quad \sigma_x \neq 0: S \approx \frac{2}{\sqrt{\pi} \sigma_x} \exp(-B^2/2\sigma_x^2) \left[1 + \frac{\sqrt{2} B \sigma_x}{a \sigma_x^2}\right]. \quad (A32)$$

The limit (A31) is not the same as found in equations (A4) and (A5) because there the integral is over $x$ and here it is to be performed over $s$ (equation (A6)). On the other hand it is the same as in (A25).
In Fig. A.1, we plot the function \( S(\sigma_x, \sigma_z, B) \) for various values of \( \sigma_x/B \) and \( \sigma_z/B \).

In order to complete the proof of equation (5) we must evaluate the function \( \eta(\Delta s) \) and show that it is proportional to \( \Delta s \). We shall not give this proof, which involves using the correlation function given by Kaiser (1973). One of us has given elsewhere (Carvalho 1978) a discussion of the assumptions implied in putting \( \eta(\Delta s) \propto \Delta s \).

Appendix B

A simple, but relatively crude, way to express the fact that the components \( b_l \) of the fluctuating field vary on both a scale \( L \) and a scale \( \mathcal{L} \) is to write

\[
b_l(r) = \sum' b_{\lambda l}(r) \cos (k_{\lambda} r - \phi_{\lambda l}),
\]

where the summation is over wavenumbers \( k_{\lambda} \) which lie in a restricted range

\[
k_0 - \Delta k < k_{\lambda} < k_0 + \Delta k,
\]

indicated by the prime on the summation sign, and where

\[
k_0 \sim 1/L, \quad \Delta k \sim 1/\mathcal{L},
\]

while the \( b_{\lambda l}(r) \) vary on a scale \( \mathcal{L} \). The \( \phi_{\lambda l} \) are random phases, uniformly distributed between 0 and \( 2\pi \).
Similarly we can write for the fluctuating velocity components

\[ u_i(r) = \sum' v_{\lambda i} \cos (k_\lambda r - \psi_{\lambda i}) , \]  

(B4)

where again the \( \psi_{\lambda i} \) are random phases, but where now the \( v_{\lambda i} \) are constants.

The assumption about the scale of variation of the \( b_{\lambda i} \) can be expressed by the relation

\[ \left| b_{\lambda i}(r) \frac{db_{\lambda i}(r)}{dr} \right| \approx L \Rightarrow L. \]  

(B5)

The ensemble averages which we have encountered in the main body of the text are now averages over the phases, and we easily find

\[ \langle b_i(r) \rangle = \langle u_j(r) \rangle = 0. \]  

(B6)

In order to simplify the notation, we shall consider the corresponding one-dimensional case, that is, drop all indices \( i \) and \( j \) so that we have just one component, \( u(r) \), of the fluctuating velocity, and one, \( b(r) \) of the fluctuating field.

Consider now the average \( \langle u(r') u(r) \rangle \). For this expression we get from equation (C4):

\[ \langle u(r') u(r) \rangle = \left( \sum' \sum_{\mu} v_{\mu} v_{\lambda} \cos (k_\lambda r' - \psi_{\lambda}) \cos (k_\mu r - \psi_{\mu}) \right), \]  

(B7)

\[ = \left( \sum_{\lambda} u_{\lambda}^2 \cos (k_\lambda r' - \psi_{\lambda}) \cos (k_\lambda r - \psi_{\lambda}) \right) \]

\[ = \sum_{\lambda} u_{\lambda}^2 \cos [k_\lambda (r - r')] \]

\[ = \langle u^2 \rangle \Phi_2 (r - r'), \]  

(B8)

where we have used the randomness and independence of the phases; equation (B8) is essentially the same as equation (12).

Similarly, we find

\[ \langle b(r') b(r) \rangle = \sum_{\lambda} b_{\lambda} (r) b_{\lambda} (r') \cos [k_\lambda (r - r')], \]  

(B9)

which we can write in the form

\[ \langle b(r') b(r) \rangle = H(r) \Phi_3 (r - r'), \]  

(B10)

where

\[ H(r) = \sum_{\lambda} b_{\lambda}^2 (r) = \langle b(r) b(r) \rangle, \]  

(B11)

if we normalize \( \Phi_3 \) such that \( \Phi_3 (0) = 1 \).

To find \( \langle u(r') b(r) \rangle \) we must move with care. If the \( \psi_{\lambda} \) and \( \phi_{\lambda} \) are independent, we would find zero for this average: the field and the velocity are uncorrelated in this case. However, in the interstellar gas one would expect the two to be correlated and therefore we shall assume that we may put \( \phi_{\lambda} = \psi_{\lambda} \) in which case

\[ \langle u(r') b(r) \rangle = \sum_{\lambda} v_{\lambda} b_{\lambda} (r) \cos [k_\lambda (r' - r)] \]  

(B12)

which leads to

\[ \langle u(r') b(r) \rangle = G(r) \Phi_1 (r - r'), \]  

(B13)
with

\[ G(r) = \sum_{\lambda} \nu_{\lambda} b_{\lambda}(r) = \langle v(r) b(r) \rangle. \] (B14)

From equation (B8) it follows that

\[ \left. \frac{d}{d\xi} \Phi_{2}(\xi) \right|_{\xi=0} = 0, \] (B15)

and in as far as the \( b_{\lambda} \) change slowly as compared to the periodic factors, we shall assume that similarly

\[ \left. \frac{d}{d\xi} \Phi_{i}(\xi) \right|_{\xi=0} = 0, \quad i = 1, 3. \] (B16)

In that case we get from equation (B13)

\[ \langle \nu \frac{db}{dr} \rangle = \frac{dG}{dr}. \] (B17)

As far as the magnitude of the right side of equation (B17) is concerned, we expect that

\[ \frac{dG}{dr} \sim \frac{d^2 G}{dr^2} \sim \frac{G}{r^2} \ll \frac{G}{L^2}. \] (B18)

From equation (B10) we find, using equation (B16),

\[ \langle b(r) \frac{db(r)}{dr} \rangle = \frac{1}{2} \frac{dH}{dr}. \] (B19)

We also need the averages \( \langle \nu \nabla^2 b \rangle \) and \( \langle b \nabla^2 b \rangle \). Here we need \( \left. \left( d^2/d\xi^2 \right) \Phi_{i}(\xi) \right|_{\xi=0} \), and those quantities will be of the order of \(-k_0^2 \Phi_{i}\) so that we get as to order of magnitude

\[ \langle \nu \nabla^2 b \rangle \sim \frac{G(r)}{L^2}, \] (B20)

and

\[ \langle b \nabla^2 b \rangle \sim \frac{H(r)}{L^2}. \] (B21)

We finally need the averages \( \langle b^2 \rangle \) and \( \langle b_{\lambda}^2 \rangle \). For our one-dimensional case this mean \( \langle b^2 \rangle \). We could, of course, use equation (B11). A different procedure was used in derivin equations (33). To do that we note that \( \langle bv \rangle \sim \langle b^2 \rangle \langle v^2 \rangle \) from which equations (33) follow. This equality follows because the spectra of \( b_{\lambda} \) and of \( \nu_{\lambda} \) are very narrow, centred a \( k_{\lambda}^{-1} \approx k_0^{-1} \approx L \). Thus no 'cross-term' appears.