On the equilibrium structures of self-gravitating masses of gas containing axisymmetric magnetic fields

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Summary. We give the general equations describing the equilibrium shapes of self-gravitating gas clouds containing axisymmetric magnetic fields. The general equations admit of a large class of solutions. It is shown that if the one additional \textit{(ad hoc)} assumption is made that the mass be spherically symmetrically distributed then the gas pressure and the boundary conditions are sufficiently constraining that the general topological structure of the solution is effectively determined. The further assumption of isothermal conditions for this case demands that all solutions possess force-free axisymmetric magnetic fields. We also outline how the construction of aspherical (but axisymmetric) configurations can be achieved in some special cases and we show that the detailed form of the possible equilibrium shapes depends upon the arbitrary choice of the functional form of the variation of the gas pressure along the field lines. We have done these calculations to demonstrate that it is the additional \textit{ad hoc} assumptions (such as the assumed equation of state or the assumed geometrical form of the equilibrium shape) which produce constraints on the topological nature of the possible equilibrium structures (over and above those provided by Maxwell’s equations, the equations of magnetostatic equilibrium and Poisson’s equation). We suggest that these arbitrary constraints are too confining and that, accordingly, a wealth of physical and mathematical information is still to be gleaned concerning the number and spatial behaviour of the equilibrium structures available to self-gravitating gas clouds containing axisymmetric magnetic fields.

I Introduction

The formidable mathematical complexities introduced by the presence of magnetic fields into the problem of constructing the equilibria of a self-gravitating mass of hot gas are well known (see, e.g. Mestel 1965a, b). Usually these difficulties are reduced by introducing
additional *ad hoc* assumptions: that the field is uniform, dipolar, or whatever, inside the gas; that the equilibrium shape is spherical, spheroidal, ellipsoidal, etc.; that the equation of state is adiabatic, isothermal, incompressible, etc. But it has long been recognized that these additional assumptions are essentially palliative in character and, while defence of any particular such assumption can readily be made on physical grounds, it nevertheless is the case that such assumptions are indeed made in order to circumvent as much as possible of the mathematical difficulties introduced by the presence of prevailing magnetic fields inside the cloud of gas.

It seems to us to be in order to investigate the equilibrium states available to a magnetized cloud of self-gravitating gas when the conventional additional assumptions are systematically relaxed, or abandoned. Only by so doing can we see to what extent an equilibrium structure is crucially dependent on a particular assumption and to what extent a particular assumption is merely an illustrative device yielding firm results but not basically altering the fundamental structure of the class of equilibria available.

The general problem, involving non-uniformly rotating gas clouds, and arbitrary magnetic field structures, has never been fully investigated although attempts have been made over the years to investigate parts of the problem (Davies 1968; Wright 1969, 1973; Moss 1973, 1974, 1975, 1977a, b, c; Mestel 1976; Mestel & Moss 1977; Monaghan 1973; Markey & Tayler 1973; Mouschovias, 1976a, b).

We shall restrict our considerations in this paper to non-rotating gas clouds containing axisymmetric magnetic fields. As we shall see, even under these simplifying assumptions a general understanding is difficult to achieve of all the possible equilibrium configurations available to a self-gravitating gas cloud. The reason seems to be that the distribution of gas pressure along the magnetic field lines must be known before the equations describing the balance of the Lorentz force against gravity and gas pressure can be solved for the field structure, but the field structure must itself be known before the distribution of gas pressure along the field lines can be determined. Thus it appears that one is forced into making *a priori* ansätze concerning the distribution of pressure along field lines — with *a posteriori* consistency and justification to be sought. Presumably the different forms of equation of state that have been used in the past (incompressible, isothermal, adiabatic) reflect, in some sense, equivalent assumptions.

In any event, we will demonstrate that even within the axisymmetric framework there are two types of basic additional assumptions which force the equations of magnetostatic balance, and Poisson's equation for the self-gravity, into particular moulds. The first assumption is on the *geometry* of the configuration, the second is on the equation of state. The two are not completely divorced from each other since the equations are non-linear and hence a particular type of assumption influences what is, and is not, available for, say, the field configuration.

In the form in which we shall use them, the general equations describing magnetostatic balance of a self-gravitating gas cloud supporting an axially symmetric magnetic field have been given in spherical coordinates by Uchida & Low (1980). Their interest was centred on the problem of the equilibrium states available to a non-self-gravitating cloud of gas supported by a magnetic field against the gravitational attraction of a point mass. But the equations are readily generalized to the self-gravitating mass case and this is done in Section 2. We then show, in Section 3, that the apparently simple additional assumption that the mass is distributed spherically puts some severe constraints on the behaviour of the magnetic field, and the gas pressure, in order that an equilibrium state can exist. In Section 4 we present the broad outline of an argument for constructing shapes for non-spherically symmetric equilibria. Finally, in Section 5 we present our discussion.
2 The equations of equilibrium

It is well known (Chandrasekhar 1961) that in spherical coordinates \((r, \theta, \phi)\) any axially symmetric magnetic field can be written in terms of two independent components of a vector potential:

\[
\mathbf{B} = B_0 \left[ \hat{r}(r \sin \theta)^{-1} \frac{\partial}{\partial \theta} (A \sin \theta), -\hat{\theta} r^{-1} \frac{\partial}{\partial r} (r A), \hat{\phi} B \right]
\]  

(1)

where the scalars, \(A\) and \(B\), are functions only of \(r\) and \(\theta\), \(B_0\) is a constant, and careted quantities denote unit vectors. If we define,

\[
A = \Phi(r \sin \theta)^{-1}, \quad B = \Psi(r \sin \theta)^{-1},
\]  

(2)

we can write

\[
\mathbf{B} = B_0 (r \sin \theta)^{-1} \left[ \hat{r} r^{-1} \frac{\partial \Phi}{\partial \theta}, -\hat{\theta} \frac{\partial \Phi}{\partial r}, \hat{\phi} \Psi \right].
\]  

(3)

The equations describing magnetostatic and gravitational equilibrium are

\[
(4 \pi)^{-1} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla p + \rho \nabla \varphi = 0,
\]  

(4)

and Poisson's equation

\[
\nabla^2 \varphi = -4 \pi G \rho,
\]  

(5)

where \(\varphi\) is the gravitational potential, \(p\) is the gas pressure and \(\rho\) is the mass density.

Using equation (3), the \(r\) and \(\theta\) components of equation (4) can be written

\[
B_0^2 (4 \pi)^{-1} \left[ -\frac{\partial \Psi}{\partial r} - \frac{\partial \Phi}{\partial r} \mathcal{L} \Phi \right] (r \sin \theta)^{-2} - \frac{\partial p}{\partial r} + \rho \frac{\partial \varphi}{\partial r} = 0,
\]  

(6)

\[
B_0^2 (4 \pi)^{-1} \left[ -\Psi r^{-1} \frac{\partial \Psi}{\partial r} - \frac{\partial \Phi}{\partial r} \mathcal{L} \Phi \right] (r \sin \theta)^{-2} - r^{-1} \frac{\partial p}{\partial r} + \rho r^{-1} \frac{\partial \varphi}{\partial r} = 0
\]  

(7)

where we introduce the operator

\[
\mathcal{L} = \frac{\partial^2}{\partial r^2} + r^{-2} \sin \theta \frac{\partial}{\partial \theta} \left( \csc \theta \frac{\partial}{\partial \theta} \right),
\]

while the azimuthal component of equation (4) yields

\[
\frac{\partial (\Phi, \Psi)}{\partial (r, \theta)} = 0,
\]  

(8)

which implies that

\[
\Psi = \Psi(\Phi).
\]  

(9)

Write \(h_0 = 4 \pi m G \rho_0 / (k T_0)\), \(p = p_0 P\), \(\beta = 4 \pi p_0 B_0^{-2}\), \(T = T_0 \Theta\), \(\rho = \rho_0 R\), \(\varphi = 4 \pi G \rho_0 \phi\). If we set \(\rho_0 = p_0^\beta / (k T_0)\), the ideal gas law takes the form,

\[
P = R \Theta.
\]  

(10)

Here, \(p_0\) and \(T_0\) are scaling constants, \(k\) is the Boltzmann constant and \(m\) is the mass of a single fluid particle. With these normalizations, Poisson's equation takes the form,

\[
\nabla^2 \phi = -R.
\]  

(11)
Now change the variable dependence of $\Theta$, $P$ and $\varphi$ by writing $\Theta = \Theta(\Phi, r)$, $P = P(\Phi, r)$, $\varphi = \varphi(\Phi, r)$ which, once $\Phi(r, \theta)$ is known, express $P$, $\Theta$ and $\varphi$ in terms of $r, \theta$. Equations (6) and (7), together with equation (10), then reduce to the single equation,

$$(r \sin \theta)^{-2} \left[ \mathcal{L} \Phi + \Psi(\Phi) \psi(\Phi) \right] + \beta \left[ \frac{\partial P}{\partial \Phi} - h_0 P \Theta^{-1} \frac{\partial \varphi}{\partial \Phi} \right] = 0,$$

with $P$ constrained to be of the form,

$$P(\Phi, r) = P_0(\Phi) \exp \left\{ h_0 \int_{r_0}^r \Theta(\Phi, r')^{-1} \frac{\partial \varphi}{\partial r'} (\Phi, r') \, dr' \right\},$$

where $r_0$ is a constant and $P_0(\Phi)$ is, at the moment, an arbitrary function of its argument. Uchida & Low (1980) have recently analysed the behaviour of magneto-active gas clouds in the presence of a given point mass using a similar method of analysis — i.e. they replace $\nabla \varphi$ in equation (4) by a term proportional to $r^{-2}$, representing a point mass at $r = 0$, and they then remove Poisson’s equations so that there is no self-gravitational field. Our calculations, involving the non-linear, self-consistent, equilibrium of self-gravitating, magneto-active, gas clouds, are of considerably greater complexity and deal with very different equilibria than the magnetospheres of stars.

To proceed further, we note that $\Theta(\Phi, r)$ in equation (13) is a free function so far. Some specific statements have to be made on the structure of the mass of gas. It has been customary in the past to make simplifying assumptions concerning the equation of state of the gas. Thus Prendergast (1956, 1958) assumed an incompressible fluid. Others (e.g. Simon 1958; Wolter 1958) have considered polytropic equations of state ($P \propto \rho^n$). Any attempt to specify a realistic equation of state so complicates the situation that is is only possible to treat the problem by the use of expansions truncated at the lowest order (Davis 1968; Wright 1969; Monaghan 1973; Moss 1973, 1974, 1975, 1977a, b, c) as pointed out to us by the referee. There are two types of difficulty encountered — namely, the non-linearity of the equilibrium equation and the complication arising from the equation of state of the gas. Our formulation of the magnetostatic equation has the merit of being able to generate consistent forms for the arbitrary functions defining the equilibrium configuration. Using this way of handling the non-linear magnetostatic relationship, it then seems appropriate to investigate what self-consistent equilibrium structures are available when we impose further constraints on the system. We demonstrate in the next section that, if one assumes that the configuration has a prescribed geometrical behaviour, then such an assumption imposes constraints on the equation of state and vice versa, of course.

### 3 Spherically symmetric density

If $R = R(r)$ only, it then follows from Poisson’s equation that $\varphi$ is also a function solely of $r$ with

$$r^{-1} \frac{d^2}{dr^2} (r \varphi) = - R(r).$$

Equations (10) and (13) imply that the pressure must then be expressible in the form

$$P(\Phi, r) = \Pi(\Phi) + h_0 \int_{r_0}^r R(r') \frac{d \varphi(r')}{dr'} \, dr' \equiv \Pi(\Phi) + p_0(r),$$

where $\Pi(\Phi)$ is an arbitrary function of $\Phi$. Equation (12) then becomes

$$(r \sin \theta)^{-2} \left[ \mathcal{L} \Phi + \Psi(\Phi) \psi(\Phi) \right] + \beta \frac{d \Pi(\Phi)}{d \Phi} = 0.$$
Equation 15 allows us to express $\tilde{\varphi}$ in terms of $p_0(r)$ as follows,

$$\left( \frac{d\tilde{\varphi}}{dr} \right)^2 = Cr^{-4} - 2h_0^2r^{-4} \int_0^r r^4 \frac{dp_0(r)}{dr} dr,$$

where $C$ is a constant to be determined by invocation of the appropriate boundary conditions.

We are dealing with a finite mass of gas of radius, $r = r_*$. As Prendergast (1956) and Roberts (1955) have pointed out: at the radius $r = r_*$ we must demand that the total pressure be continuous as well as demanding that the normal and tangential components of magnetic field be continuous. But, by equation (15), the gas pressure is a function of $r$ and $\theta$. Hence across $r = r_*$, the only way to have a continuous pressure is if

$$\Phi(r_*, \theta) = 0. \quad (18)$$

From the definition of magnetic field (equation 3) the continuity of the normal and tangential components then requires

$$\Psi(\Phi = 0)|_{r = r_* - 0} = \Psi(\Phi = 0)|_{r = r_* + 0}, \quad (19)$$

and

$$\frac{\partial \Phi(r, \theta)}{\partial r} \bigg|_{r = r_* - 0} = \frac{\partial \Phi(r, \theta)}{\partial r} \bigg|_{r = r_* + 0}. \quad (20)$$

Since the normal and tangential components of field are continuous, it follows that the gas pressure on its own must be continuous. But the gas pressure outside the mass is zero. Hence if follows that on $r = r_*$ we require

$$\Pi(\Phi = 0) + p_0(r_*) = 0. \quad (21)$$

We must also demand that the density $R(r)$, the pressure $P(\Phi, r)$, and the temperature $\Theta(\Phi, r)$, be positive quantities in the interior of the mass of gas.

To construct the equilibrium solutions, the steps are as follows. We first specify the free functions $\Pi(\Phi)$, $\Psi(\Phi)$ and $p_0(r)$. Equation (16) can then be solved for $\Phi$ subject to the boundary conditions (19) and (20) which match the magnetic field to an external magnetic field (e.g. $|B| \to 0$ as $r \to \infty$ or $|B|$ constant as $r \to \infty$). Equation (17) yields the gravitational potential $\tilde{\varphi}$. Equation (15) then gives both pressure and density in terms of the spatial coordinates. Finally, equation (10) gives the temperature distribution. The problem is formidable as equation (16) is in general non-linear. The particular case where equation (16) is linear is presented in the Appendix for the purposes of illustration.

The assumption of a spherically symmetric mass distribution is rather constraining, limiting the structural form of the pressure to that given by equation (15). If we were to further assume the gas to be isothermal, the situation reduces to the singular case of the magnetic field being force-free and there is no interaction between the field and the gas. To see this, set $\Theta = \Theta_0$, a constant. Since $\Phi$ is a function of $r, \theta$, and since $P \propto R(r)\Theta = $ function of $r$, then $\Pi(\Phi) = \Pi_0 = $ constant, for consistency. Absorbing this constant into the definition of $p_0(r) \propto R(r)$ we then see that equation (16) reduces to

$$\square \Phi + \frac{d}{d\Phi}\left[\frac{1}{2}\Psi(\Phi)^2\right] = 0,$$

which represents an axisymmetric force-free field configuration (see e.g. Chandrasekhar 1961).

† More precisely, that $\Phi(r_*, \theta) = $ constant, which can be chosen to be zero without loss of generality.
From equations (10) and (13) we then have, since \( p_0(r) \propto R(r) \),
\[
\rho(r) \propto \exp[\beta \varphi(r)],
\]
where \( \beta \) is constant. Poisson's equation then takes on the generic form
\[
r^{-2} \frac{d}{dr} \left( r^2 \frac{d\varphi}{dr} \right) + \exp(\varphi) = 0,
\]
whose solution is well-known (Walker 1915; Chandrasekhar 1939).

We thus see that if the assumption of isothermality is added to the ansatz of a spherically distributed mass, then the only possible equilibrium structure for a self-gravitating gas cloud containing an axisymmetric magnetic field is one in which the magnetic field is force-free everywhere inside the cloud.*

The point here is that the complicated interplay of the Lorentz force, gas pressure and self-gravity is sufficiently self-locking that only one or two further, ad hoc, assumptions about, say, either the equation of state or the shape of the geometric configuration are enough to force the equilibrium properties of the system to be very tightly constrained, indeed. It is this fact that the present section of the paper is designed to emphasize.

In order to illustrate how non-spherical equilibrium shapes can be built, in the next section we consider, in outline only, a prescription for constructing such shapes.

4 Density a functional of the gravitational potential

Let us assume a priori that the density is a functional solely of the gravitational potential \( \varphi \) and hence is expressible in the form,
\[
R = \frac{dF(\varphi)}{d \varphi} > 0.
\]
Equations (10) and (13) then imply that
\[
P(\Phi, r) = \Pi(\Phi) + \alpha F(\varphi),
\]
where \( \Pi(\Phi) \) is an arbitrary function of \( \Phi \). Use of equation (23) in (21) then yields
\[
\mathcal{L} \Phi + \Psi(\Phi) \Psi'(\Phi) + \beta (r \sin \theta)^2 \frac{d \Pi(\Phi)}{d \Phi} = 0.
\]
We recognize that equation (24) has precisely the same structural form as equation (16) so that its solutions can be considered known to the same extent as those of equation (16). The fundamental difference arises from Poisson's equations which takes the form
\[
\nabla^2 \varphi + \frac{dF(\varphi)}{d \varphi} = 0
\]
and in which \( \varphi \) is no longer a function solely of \( r \). Further: since \( R \), the normalized density, is positive it follows that \( dF(\varphi)/d \varphi \) must be greater than, or equal to, zero everywhere. Again we must demand that the normalized pressure (equation 23) be positive inside the mass of gas and that the magnetic field (determined from equation 24) satisfy physically reasonable boundary conditions at large distances from the configuration. The 'surface' of the gaseous configuration is no longer so simple to determine as it was in the case where the gas density was a function solely of radius. The point here is that the solution to Poisson's equation

*The referee has pointed out that since configurations which are force-free everywhere do not exist, it follows that a solution which is force-free within the gas must exert forces somewhere outside.
determines the surface shape as \( \dot{\varphi} = \text{constant} \) (a surface of constant density), which yields a curve in \((r, \theta)\) space, say \( S(r, \theta) = 0 \). On this same curve we must demand that the gas pressure drop to zero and that the normal and tangential components of magnetic field be continuous. Equation (23) then tells us that \( \Phi \) must be constant on \( S(r, \theta) = 0 \) in order that the bounding surface of constant \( \dot{\varphi} \) be everywhere parallel to a surface of constant \( \Phi \) so that the pressure is constant (zero in fact) on the bounding surface of the configuration. In general, the curve \( S(r, \theta) = 0 \) is an unknown and is to be determined as a part of the solution — i.e. we have a free boundary value problem. The point is illustrated by the following example.

Suppose,

\[ \Pi(\Phi) = \Lambda \Phi, \quad (26) \]

\[ F(\dot{\varphi}) = F_0 + \alpha^2 \dot{\varphi}, \quad (27) \]

where \( \Lambda, F_0 \) and \( \alpha^2 \) are constants. Then \( P = \Lambda \Phi + F_0 + \alpha^2 \dot{\varphi} \) and the density is uniform with value \( R = \alpha^2 \). Poisson’s equation (31) becomes

\[ \nabla^2 \dot{\varphi} + \alpha^2 = 0, \quad (28) \]

Let the surface \( S(r, \theta) = 0 \) be expressed in the form \( r = r_*(\theta) \). The requirement that \( S(r, \theta) = 0 \) be a gravitational equipotential is that,

\[ \dot{\varphi}_i(r_*, \theta) + \dot{\varphi}_e(r_*, \theta) = \dot{\varphi}_0 = \text{constant}, \quad (29) \]

where \( \dot{\varphi}_i \) and \( \dot{\varphi}_e \) are the interior and exterior gravitational potentials respectively. To keep the problem simple, let the surface be nearly spherical,

\[ r = r_*[1 + e \xi(\theta)] \equiv r_s(\theta) \quad (30) \]

where \( \xi(\theta) \) is some function of \( \theta \); \( e \) and \( r_* \) are constants. Continuity of the normal and tangential components of the magnetic field across \( r = r_s(\theta) \) then implies that at \( r = r_s(\theta) \):

\[ \frac{\partial \Phi_i}{\partial \theta} - \frac{\partial \Phi_e}{\partial \theta} = e r_* \frac{d \xi(\theta)}{d \theta} \left( \frac{\partial \Phi_i}{\partial r} - \frac{\partial \Phi_e}{\partial r} \right) + \mathcal{O}(e^2), \quad (31) \]

\[ \frac{\partial \Phi_i}{\partial r} - \frac{\partial \Phi_e}{\partial r} = -e r_*^2 \frac{d \xi(\theta)}{d \theta} \left( \frac{\partial \Phi_i}{\partial \theta} - \frac{\partial \Phi_e}{\partial \theta} \right) + \mathcal{O}(e^2). \]

Finally, the pressure must vanish at \( r = r_i(\theta) \),

\[ F_0 + \alpha^2 \dot{\varphi}_0' + \Lambda \Phi_i(r_*, \theta) = 0. \quad (32) \]

Since \( e \) is small, equations (29), (31) and (32) can be written out to first order in \( e \). Since the boundary is nearly spherical, we can expand \( \xi(\theta), \Phi_i(r_*, \theta), \Phi_e(r_*, \theta), \dot{\varphi}_i(r_*, \theta) \) and \( \dot{\varphi}_e(r_*, \theta) \) in terms of Legendre polynomials in \( \cos \theta \). Equations (29), (31) and (32) then pose a problem for determining the expansion coefficients. Notice that \( \xi(\theta) \) is left as an unknown. Clearly the result is that we have an eigenvalue problem for the expansion coefficients of \( \xi(\theta) \) in terms of Legendre polynomials.

5 Discussion and conclusion

We have given the general equations determining the equilibrium states of a self-gravitating mass containing an axisymmetric magnetic field. The equations are highly non-linear and involve arbitrary functionals, such as \( \Pi(\Phi) \), which have to be determined, or assumed, from
considerations other than those contained in the equations of hydromagnetic equilibrium or Maxwell's equations. *

In Section 3 we showed how the assumption of spherical symmetry of the mass density distribution placed rather severe restrictions on the functional form of the pressure distribution (with an illustrative example given in the Appendix). We showed how the assumption of isothermality for an assumed spherical mass distribution demanded that the axisymmetric magnetic field be force-free. We illustrated in Section 4 how the equations could be solved for aspherical shapes. †

It should also be noted that throughout this work we have restricted our consideration to non-rotating masses of gas and to axisymmetric magnetic fields. Relaxation of either (or both) of these assumptions raises even more formidable mathematical difficulties in determining the possible equilibrium states available than we have already encountered. In the final analysis such increases in complexity must be allowed for since it is unlikely that astrophysical masses of gas will (i) contain magnetic fields that are precisely axisymmetric, or (ii) have precisely zero angular momentum. These problems, too, must be addressed in an analysis more complete than the simple (but general) axisymmetric equations we have given here.

The shape of a gas cloud and/or its equation of state are normally prescribed kinematically — i.e., the shape is assumed a priori or the relation between density and pressure is assumed (isothermal, adiabatic, incomprehensible) and the magnetostatic balance of the gas cloud is sought in the confining strait-jacket of such rigid assumptions. But the important point is that the requirement of magnetostatic equilibrium alone constrains the pressure distribution to be of a specific form, namely, that given by equation (13). The required form of pressure distribution still contains a large degree of freedom through the free functions \( p_0(\Phi) \) and \( \Theta(\Phi, r) \). The main difficulty with the physical problem is, of course, the non-linearity of the governing equations. Even if we do not impose a priori assumptions on the equation of state of the gas and merely ask what equilibria are available for the large class of admissible \( P(\Phi, r) \), the question cannot be readily answered except for particular cases — such as where the magnetostatic equation is reducible to a linear problem (as illustrated in the Appendix). It seems to us that the problem of the self-gravitating magnetic gas, as we have formulated it here, should be worked out on a kinematic basis as a first step — i.e. generation and investigation of the properties of magnetostatic equilibria for various forms of \( P(\Phi, r) \) should be undertaken. Such kinematic studies provide invaluable insight into the more general problem involving determining the pressure simultaneously with the magnetic field by solving the magnetostatic equation with a statement on the equation of state of the gas. In view of the importance customarily assigned to the general problem of star formation from self-gravitating gas clouds, it seems to us that detailed investigation of this particular aspect is long overdue.

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* The referee has noted that the incorporation of an equation of state and the equation of radiative equilibrium must surely remove a great deal of the arbitrariness.

† The referee has pointed out that the simple models we calculate for the purposes of illustration can be easily obtained by 'standard' methods of analysis.
References


Appendix

Let us choose

$$\Pi(\Phi) = \Lambda \Phi, \quad (A1)$$

where $\Lambda$ is an arbitrary constant which we can take to be positive without loss of generality.

Also, for simplicity we take $\Psi$ to be zero. Equation (16) then becomes

$$\frac{\partial^2 \Phi}{\partial r^2} + r^{-2}(1 - \mu^2) \frac{\partial^2 \Phi}{\partial \mu^2} + \Lambda \beta r^2 (1 - \mu^2) = 0, \quad (A2)$$

where $\mu = \cos \theta$.

Equation (A2) is linear and its solution can be readily written in terms of known functions. Direct calculation with the use of the appropriate boundary conditions at the stellar radius $r = r_* \ y$ields the solution

$$\Phi = \frac{1}{10} \Lambda \beta r^2 (1 - \mu^2) (r^2 - r^2) \quad \text{for} \quad r < r_*; \quad (A3)$$

$$= -\frac{1}{15} \Lambda \beta r_*^2 (1 - \mu^2) (r^2 - r_*^2 r^{-1}) \quad \text{for} \quad r > r_*.$$

It follows immediately that, since the signs of $\Phi_1$ and $\Phi_0$ are opposite, the field lines external to the self-gravitating sphere are oppositely directed to those interior to the sphere near $r = 0$.

The presence of the $r^2$ term in $\Phi_0$ also tells us that the field (as $r \to \infty$) is uniform. The pressure is given by equation (15):

$$P = p_0(r) + \Lambda \Phi. \quad (A4)$$
For purpose of illustration, set the free function $p_0(r)$ to be,

$$p_0(r) = 1 - r^2/r_*^2,$$

in the gas and to be zero outside. The pressure distribution in the gas is then,

$$P = 1 - r^2/r_*^2 + \frac{\Lambda^2}{10} \frac{r^2}{r_*^2} (r_*^2 - r^2) (1 - \mu^2),$$

while the density in the gas is

$$R(r) = r_*^{-1} (6/\hbar_0)^{1/2} = \text{constant},$$

giving rise to the gravitational field

$$\frac{\partial \phi}{\partial r} = - rr_*^{-1} \left[ 2/(3 \hbar_0) \right]^{1/2} \text{ for } r < r_*$$

$$= - r^{-2} (7/3 R r_*^3) \text{ for } r > r_*$$