Extragalactic jets — II. Shape and stability

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Summary. A general formulation is given for determining the internal structure and stability of extragalactic jets confined by external pressure. Solutions are found for the jet cross-sectional shape and internal pressure distribution for high Mach number flows. With rapidly decreasing external pressures, a circular jet is found to be stable. A circular jet is found to become unstable and its cross-section to flatten with increasing jet-path length when the rate of decrease of external pressure is sufficiently slow. This global instability will lead to jet splitting and breakup.

1 Introduction

Detailed, high resolution maps of jets now allow us to infer not only jet trajectories, but also some aspects of internal jet structure and stability. We have therefore, undertaken a detailed extension of the trajectory calculations given in Paper I in order to model internal flow properties of jets. Once again, comparison of our model with the recent observations should provide further understanding of both jets and their environments.

First, we determine the cross-sectional shape to zeroth order in the slenderess parameter $e$ and the velocity potential and pressure up to order $e^2$ for large distances from the jet’s origin (Sections 2 and 3). Secondly, the equations for higher order terms are found; and, by generalizing the work of Geer (1977), we discuss the global instability of the jet cross-section to deformation (Section 4). Using these results, we then discuss the interpretation of observed properties of jets (Section 5).

The mathematical details are rather complex; and can be found in Smith (1979).

2 Cross-sectional shape of jets

2.1 Detailed Analysis

In Paper I, differential equations were obtained and solved for $\phi^0$ and $\phi^1$. In so doing, the jet curvature and cross-sectional area to zeroth order was obtained. Now higher-order coefficients are considered and solutions for $\phi^2$ and $\delta^0$ are found.
The solution for \( K = 0 \) with \( S^0 \) independent of \( \theta \) (a straight circular jet) is

\[
\frac{\partial \phi^2}{\partial r} = \frac{gr}{2},
\]

\[
\frac{\partial \phi^2}{\partial s} = \frac{g}{4} \left( \gamma^2 - (S^0)^2 \right) - \frac{g^2}{8 \phi^0} (S^0)^2,
\]

and

\[
(S^0)^2 = c_0^3 (\phi^0)^{-1} \left( 1 - (\phi^0)^2 \right)^{\nu(1-\gamma)},
\]

where \( c_0 \) is a normalization constant so that \( (S^0)^2 = A/A_0 \),

\[
g = \frac{2}{\gamma - 1} (\phi^0)^2 \phi^0 \left[ 1 - (\phi^0)^2 \right]^{-1} - \phi^0
\]

and the dot represents differentiation with respect to \( s \).

For \( K \neq 0 \), we look for separable solutions. The equations are greatly simplified for high Mach number flows and the cross-section is an ellipse with semi-axes \( a \) and \( b \) given by

\[
a = \left[ \frac{-\dot{\beta}}{\dot{g}/4 + \dot{\alpha} + \frac{1}{2} (g/2 + 2 \alpha)^2} \right]^{1/2},
\]

\[
b = \left[ \frac{-\dot{\beta}}{\dot{g}/4 - \dot{\alpha} + \frac{1}{2} (g/2 - 2 \alpha)^2} \right]^{1/2}.
\]

The semi-axis in the \( y-z \) plane (\( \theta = 0^\circ \)) is \( a \), and the semi-axis in the \( x \)-direction (\( \theta = 90^\circ \)) is \( b \).

The separable solution leads to a particular solution for \( \phi^2 \) and \( S^0 \) that results in a jet with an elliptical cross-section.

The general solution for high Mach number flows yields

\[
\dot{\eta} = (\dot{g} + \frac{1}{2} g^2) \frac{1 - \eta^2}{1 + \eta^2} \eta - \dot{\eta} g + \frac{1}{2} \frac{\dot{\eta}^2 + 3 + \eta^2}{\eta} - \frac{1}{1 + \eta^2},
\]

where \( \eta = a/b \) is the aspect ratio. By the change of variables \( s \equiv \exp(t) \), this can be written as

\[
\frac{\partial \eta'}{\partial \eta} = 1 + \frac{n}{\gamma} + \frac{n}{2 \gamma} \left( \frac{n}{\gamma} + 2 \right) \frac{1 - \eta^2}{\eta'} \frac{\eta' - \eta^2}{1 + \eta^2} + \frac{1}{1 + \eta^2}.
\]

Here, the dashes denote differentiation with respect to \( t \). If \( \eta(t) \) is a solution to equation (5), then so is \( 1/\eta(t) \). The problem is independent of whether we take the aspect ratio as \( a/b \) or \( b/a \). The aspect ratio can be studied by consideration of trajectories in the \( \eta, \eta' \) phase plane.

A further solution is a jet with a circular cross-section given by \( \alpha = 0 \).

### 2.2 Illustrations of Jet Shapes

We graph the solution of equation (5) in Figs 1 to 4. The jet shape in the limit as \( s \to \infty \) is strongly dependent on \( n/\gamma \), as we discuss below. In all graphs, the arrows denote the direction of increasing jet arc length. An important point to note is that if \( \eta' \) is replaced by \( -\eta' \) and \( 1 - n/\gamma \) by \( 1 + n/\gamma \), then the value of \( \partial \eta'/\partial \eta \) keeps the same magnitude but changes its sign. Hence the diagram for \( 1 - n/\gamma \) is simply the mirror-image of the diagram.
for \((1 + n/\gamma)\), with the reflection about the \(\eta' = 0\) axis and the directions of increasing \(s\) reversed.

Fig. 1, specifically for \(n/\gamma = -3\), is representative of the change in jet shape with arc length for \(n/\gamma < -2\). Given any initial conditions \(\eta\) and \(\eta'\) where the high Mach number approximation is valid, the final shape of the jet can be immediately determined from the figure. In this case, all the trajectories spiral in towards the stable point \(\eta = 1, \eta' = 0\), that is, the circular jet. In the limit as \(s \to \infty\) and \(\eta \to 1\), the ellipticity oscillates between negative and positive values with a rapidly decreasing amplitude of oscillation. The decrease of the amplitude of the spiral is a factor \(\exp(2\pi) \approx 500\) per revolution around \(\eta = 1\) as \(\eta \to 1\) and with \(-n/\gamma > 2\), and a factor \(\exp(2\pi\sqrt{2}) \approx 7000\) per revolution for Fig. 1. Therefore, very few oscillations can be seen in the figure.

Fig. 2 is a special case \(n/\gamma = -2\), in which the ellipticity remains bounded and \(\eta' \to 0\) in the limit as \(s \to \infty\). Consequently, if \(n/\gamma < -2\) and initial conditions of the form \(\eta \sim 1, \eta' \ll 1\) are given, then the aspect ratio does not vary significantly from unity. Furthermore, the ellipticity remains bounded for all initial conditions.

Fig. 3, specifically for \(n/\gamma = -\frac{3}{2}\), is representative of the jet shape for \(-2 < n/\gamma < -1\). An initially circular jet with \(\eta' = 0\) is now highly unstable. Initial conditions in the neighbourhood of this point result in a jet in which \(\eta \to 0\) or \(\eta \to \infty\). Hence, in all cases, \(\eta \to 0\) or \(\eta \to \infty\), in the limit as \(s \to \infty\).

Fig. 4 is the special case \(n/\gamma = -1\) in which there is reflection symmetry about the \(\eta' = 0\) axis. The form of the trajectories in the range \(-1 < n/\gamma < 0\) can be obtained by reflecting Fig. 3 about the \(\eta' = 0\) axis. Therefore, in the range \(-2 < n/\gamma < 0\), \(\eta \to 0\) or \(\infty\) in the limit as \(s \to \infty\).

Figure 1. Phase plane trajectories for \(n/\gamma = -3\) (see text). The arrows denote the direction of increasing jet arc length.
Figure 2. Phase plane trajectories for $n/\gamma = -2$ (see text). The crosses denote the stable point on each trajectory.

Figure 3. Phase plane trajectories for $n/\gamma = -\frac{3}{2}$ (see text).
In the determination of $S^0$ it has been assumed that the slenderness approximations are still valid at large distance from the origin of the jet, despite the possible degeneration of the jet into a sheet. In equation (20) of Paper I, the term $(\phi^0)^2 \phi^0$, which is of the order of $\epsilon^4$, was ignored when deriving $\phi^0$ and $\phi^1$. Since, in the limit as $s \to \infty$, $z \propto s$, it follows that $(\phi^0)^2 \phi^0 = O(\mu^{-1})$; and so, with $\mu < 0$, the approximation made in equation (20) is valid. Similarly, in Paper I, the terms of order $\epsilon^2$ and $\epsilon^4$ in equations (21) and (22), respectively, were ignored in deriving $\phi^0$ and $\phi^1$. We now calculate that these are $\xi^0 \phi^0 = O(1)$ and $(\phi^0)^2 - 1 + \xi z^\mu = O(1)$, in the limit as $s \to \infty$; and hence, our results are consistent with the original approximations.

We have determined the boundary surface of a jet with an elliptical cross-sectional shape in the case of a high Mach number flow. Examples of boundary surfaces are shown in Fig. 5, where an initial jet angle of injection of $\Omega = 90^\circ$ has been chosen. The width is displayed in the $y$-direction, and the breadth in the $x$-direction.

If $0 > n/\gamma > -2$, the jet degenerates into a sheet of fluid whose width increases in proportion to the arc length. Consequently, the jet will appear to be conical (Fig. 5a,b,c). Furthermore, if $n/\gamma > -1$, the thickness of the sheet decreases to zero in the limit as $s \to \infty$ (Fig. 5a). If $n/\gamma < -1$, then both the width and thickness approaches infinity in the limit as $s \to \infty$ (Fig. 5c). If $n/\gamma < -2$, the jet cross-section will approach a stable circular shape via oscillations (in the ellipticity) of decreasing amplitude in the limit as $s \to \infty$. The axes become of the order of $s^{-n/2\gamma}$, and so the boundary streamlines diverge away from the locus of centroids (Fig. 5d).

The confinement condition derived in Paper I will not hold in the case of $n/\gamma > -2$, since it has been shown that the circular cross-section assumption is invalid. Therefore, we now
write the confinement condition as \( e (da/ds) < 1/M \) and \( e (db/ds) < 1/M \). Applying the asymptotic formulae for the jet shape as given here by equation (16) and by equation (50) of Paper I, this confinement condition becomes

\[
e^2 < \xi z^{\mu}.
\]

(6)

Stipulating the transverse velocity to be less than one-tenth of the sound-speed of the jet, the confinement condition can be written as

\[
z^{-n} < 10^{\gamma/(\gamma-1)} \left( \frac{\xi}{0.1} \right)^{-\gamma/(\gamma-1)} \left( \frac{e}{0.01} \right)^{2\gamma/(\gamma-1)}.
\]

(7)

Thus, confinement can be maintained up to \( z \geq 10 \) for all \( n > -2 \) and \( \gamma \leq 2 \), for \( e \sim 0.01 \) and \( \xi^2 \sim 0.01 \).

The introduction of expansions \( \phi^1 \) and \( S^1 \) in powers of \( \xi \) also gives a relationship between \( \xi \) and \( e \). In obtaining the zero- and first-order results, \( \phi^0 = 1 \) and \( \phi^1 = 0 \), the terms ignored were of the order \( e^2 \) and \( \xi z^{\mu} \). Since \( \phi^2 \) is of the order \( e^2 \), for correct ordering we require \( e^2 > \xi z^{\mu} \), in contradiction to the confinement condition (19). Obviously, in \( \phi^0 \) the expansion in \( \xi \) is not valid. Nevertheless, it is still possible to expand \( \phi^2 \) and \( S^0 \) in powers of \( \xi \), i.e. \( \phi^0 = 1 - \xi z^{\mu} \), \( \phi^2 = \phi_0^0 + \phi_1^2 \xi + \phi_2^2 \xi^2 + \ldots \); and it follows that \( \phi_0^2 \) has been correctly calculated here.
We have demonstrated that the width of the jet expands uniformly for \( n/\gamma > -2 \), large \( s \) and under external pressure confinement. Observations have shown that resolved jets often do have such a constant opening angle, with estimates of \( n > -2 \) (and hence, \( n/\gamma > -2 \) if \( \gamma > 1 \)). Since an unconfined expansion can result in a conical jet, a free-jet model has been proposed (Readhead, Cohen & Blandford 1978; Perley, Willis & Scott 1979). We point out that, in the direction that gives greatest resolvability of the width, a confined beam will also be conical in shape.

3 Internal jet pressure distribution

The pressure distribution within the jet is found via the Bernoulli equation. For a circular jet, we obtain

\[
\left( \frac{p}{p_0} \right)^{1-1/\gamma} = z^\mu + \frac{e^2}{4\xi} \left( g^2 + 2\phi^0 \dot{g} \right) \left[ (S^0)^2 - r^2 \right].
\]

(8)

The \( z^\mu \) term signifies the pressure equilibrium with the surrounding medium at \( r = S^0 \). In the limit as \( s \to \infty \), we find that \( g^2 + 2\phi^0 \dot{g} \to \frac{1}{2} n/\gamma (n/\gamma + 2) \). Thus, if \( n/\gamma > -2 \), the pressure decreases as \( r \) decreases, i.e. the radial pressure force is inhibiting the jet from expanding and the opening angle of the jet decreases with increasing arc length. If \( n/\gamma < -2 \), then the radial pressure force is acting outwards, and the bounding streamlines curve away from the centre of the jet.

Taking \( \xi = 2(\gamma + 1) \), corresponding to \( M_0 = 1 \), and putting \( z = 1 \), then \( \phi^0 = (\gamma - 1)/ (\gamma + 1) \), \( \phi^0 = -n/\gamma [(\gamma - 1)/(\gamma + 1)]^{1/2} \), giving \( g = 0 \) and \( \dot{g} \phi^0 = (n/\gamma)^2 (\gamma - 1) \). When the flow is transonic, the radial pressure force is found to be always positive, tending to expand the jet. The outward-pressure force may inhibit the Kelvin–Helmholtz instability in the neighbourhood of the nozzle. The radial pressure gradient is reversed at large distances from the nozzle for the circular jet if \( n/\gamma > -2 \); and the entrainment of surrounding material could be significant.

The transverse expansion of a jet by pressure forces implies a lower velocity and a greater mass flux per unit area along the centroid of the jet than at the jet surface. This contrasts with models invoking higher central velocities resulting from frictional forces decelerating the fluid at the boundary surface.

In the more general non-axisymmetric case, the internal pressure distribution is given by:

\[
\left( \frac{p}{p_0} \right)^{1-1/\gamma} = \left( \frac{p_1}{p_0} \right)^{1-1/\gamma} - \frac{e^2}{2\phi^0 c^2} \left[ 1 - \frac{r^2}{(S^0)^2} \right],
\]

(9)

where \( p_1 \) is the pressure in the external medium. If \( \dot{\beta} < 0 \), the pressure within the jet is greater than the external pressure \( p_1 \); and conversely, if \( \dot{\beta} > 0 \), then \( p < p_1 \). The parameter \( \dot{\beta} \) can be calculated for large \( s \) in the case \( \xi = 0 \). For \( n/\gamma < -2 \), the circular cross-sectional jet solution is approached in the limit as \( s \to \infty \); and, \( \dot{\beta} < 0 \). If \( -1 > n/\gamma > -2 \), then \( \dot{\beta} \) is positive; if \( 0 > n/\gamma > -1 \), then \( \dot{\beta} \) is negative (Smith 1979). The direction of the pressure gradient at the boundary surface may have considerable influence on the stability of the jet.

4 Jet stability

The internal structure of the jet has been determined by solving the non-linear boundary value problem for Poisson’s equation. The application of initial conditions leads to a unique solution that was specified for jets with an elliptical or circular cross-sectional shape.
However, the solutions for $\phi^k$ and $S^{k-2}$ with $k > 2$ are dependent on the particular solution found for $\phi^2$ and $S^0$; but, even after the initial conditions are given, solutions for $\phi^k$ and $S^{k-2}$ are not unique.

For the case of an incompressible fluid under gravity, Geer (1977) demonstrated that there exist non-trivial eigensolutions which have a physical interpretation in terms of the stability of the flow. For this reason, the problem will be limited to obtaining eigensolutions and determining the jet stability (Smith 1979).

For the circular jet in which the jet path is aligned with the external pressure gradient, we find that the cross-sectional shape is stable for $n/\gamma < -2$. As suggested in Figs 1–4, the circular jet is unstable in the rate $0 > n/\gamma > -2$. This can be interpreted physically by consideration of the centrifugal forces on the jet boundary. For $n/\gamma < -2$, the boundaries are convex (Fig. 5d) and the centrifugal forces are directed inwards, and act to stabilize the jet. For $0 > n/\gamma > -2$, the boundaries are concave, and a flattening of the beam will lead to greater outward centrifugal forces in the direction of flattening.

For an elliptical jet, the stable circular cross-section is approached as $s \to \infty$ for $n/\gamma < -2$. For $0 > n/\gamma > -2$, the jet is stable to disturbances that are symmetric about the plane into which the jet eventually flattens. However, the jet is spatially unstable to disturbances that are asymmetric about the plane into which the jet eventually flattens.

5 Observational implications and conclusions

The final jet cross-sectional shapes produced by this model can now be summarized.

(i) If $n/\gamma < -2$, a stable circular jet is attained, after possible oscillations in ellipticity about the circular state.

(ii) If $n/\gamma < -2$, a flattened jet results with an axial ratio proportional to $s^{n/\gamma + 2}$, where $s$ is the distance along the jet. The two axes are proportional to $s$ and $s^{-1-n/\gamma}$.

(iii) If $n/\gamma < -2$, the jets are unstable.

For example, the existence of the eigensolutions discussed above shows that flattening jets will tend to warp about their major axes, leading to a distortion and possibly a splitting of the jets as they continue to flatten. Although the final structure of an unstable jet is still somewhat uncertain owing to our ignorance of the non-linear regime of instability we can conclude in general that, if the confining external pressure distribution falls off sufficiently slowly ($n \leq 0$) along the jet, the jet will be deformed and eventually break up.

From statistical studies of radio sources, Pacholczyk (1977) and Valtonen (1979) estimate $n$ to be in the range of $-2 > n > -3.5$, for either a free or a confined jet model. If the jet is confined by the pressure of material in the radio lobe, consisting of previously ejected fluid, then $n \sim 0$ (Scheuer 1974).

The jet length over which the confinement condition (equation 7) holds is proportional to $10^{10} t^{1/(\gamma - 1)}$, and the maximum growth-rate of an unstable mode is proportional to $\delta^{2+n/\gamma}$. Thus, if $n$ is close to zero, the spatial instabilities discussed here should be significant enough to create observable jet distortion and fragmentation.

Finally, we emphasize that the flattening instability presented here for any confined jet with $n/\gamma > -2$, is quite different from other jet instabilities, such as the Kelvin–Helmholtz instability, (Turland & Scheuer 1976; Blandford & Pringle 1976; Hardee 1979), or the firehose instability, which are temporally growing modes. Here the flow remains time independent, but the amplitude of a disturbance grows with distance from the origin. All these instabilities evolve from centrifugal forces: the Kelvin–Helmholtz modes are local surface disturbances, the firehose mode arises from disturbances on scales larger than the jet.

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width, and these spatial instabilities arise from the intermediate scale of disturbance—the jet cross-section. However, the spatial instability will only result if there is some permanent slight distortion to the beam shape at some point along the beam path. While numerical techniques will be required for the more general investigation of jet shape and stability, particularly for low Mach number flows, the analytic bent-jet model given here seems to deal neatly with slender high Mach number jets.

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References
