Cowling's theorem in non-flat space-time

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Summary. The paper deals with Cowling's theorem in non-flat space-time. Space-time and the electromagnetic field are stationary and axially symmetric. If the charge density is negligibly small the electromagnetic field as a whole vanishes everywhere.

Cowling's theorem states that a stationary axially symmetric magnetic field cannot be self-maintained (Cowling 1934; Alfvén & Falthammer 1963). We attempt to prove this theorem in the background of non-flat space-time. Such a demonstration seems important in view of the recent discovery of compact objects which are believed to be seats of intense gravitational field and consequently of a geometry deviating significantly from the Euclidean.

In the proof of the Cowling theorem, the following assumptions are introduced.

(1) The space-time and electromagnetic field are stationary and axisymmetric. These allow us to introduce two killing vectors one space-like with closed orbits and the other time-like with open orbits. Without loss of generality we choose time coordinate $t$ and angular coordinate $\phi$ so that these killing vectors are coordinate vectors. Hence we can write the metric of the form (cf. Carter 1969)

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} d\phi dt + g_{\phi\phi} d\phi^2 + g_{ab} dx^a dx^b,$$

where $a, b = r, z$ and $g_{\alpha\beta}$ are functions of $r, z$ only and all variables are independent of $\phi$ and $t$.

(2) The fluid is moving along the trajectories of a time-like killing vector $\xi^r$ which is constructed out of the two killing vectors already introduced. Hence $\xi^t = \xi^z = 0$.

(3) The current is a pure conduction current obeying Ohm's law, the charge density being negligibly small (cf. Cowling 1934; Alfvén & Falthammer 1963). Hence current density $J^\alpha$ and electric field are related by

$$J^\alpha = \sigma E^\alpha$$

where $\sigma$ = electrical conductivity.

Maxwell's equations are

$$F^\alpha{}_{\beta} = -J^\alpha$$,  \hspace{1cm} (3a)

$$*F^\alpha{}_{\beta} = 0$$,  \hspace{1cm} (3b)
where $F_{\alpha\beta}$ is Maxwell's field tensor and $*F_{\alpha\beta}$ is its dual. With the help of the killing vector $\xi^\alpha$ we define the electromagnetic field and $F_{\alpha\beta}$ as (Carter 1972)

\begin{align}
E_\alpha &= -F_{\alpha\beta} \xi^\beta, \\
B_\alpha &= -*F_{\alpha\beta} \xi^\beta = -\frac{1}{2} \eta_{\alpha\beta\mu\nu} F^\lambda_{\mu\nu} \xi^\beta, \\
\Phi F_{\alpha\beta} &= \xi_\alpha E_\beta - \xi_\beta E_\alpha - \eta_{\alpha\beta\mu\nu} \xi^\lambda B^\mu,
\end{align}

with

\begin{align}
\Phi = \xi^\alpha \xi_\alpha, \quad E_\alpha \xi^\alpha = 0, \quad B_\alpha \xi^\alpha = 0.
\end{align}

In view of equation (3b) we define a potential vector $A_\alpha$ such that

\begin{align}
F_{\alpha\beta} &= A_{\alpha,\beta} \equiv A_{\beta,\alpha},
\end{align}

so that due to assumption (1)

\begin{align}
F_{\phi t} = 0, \quad F_{\phi a} = A_{\phi, a}, \quad F_{ta} = A_{t, a}.
\end{align}

From $F_{\phi t} = 0$ and $E_\alpha \xi^\alpha = 0$ we have

\begin{align}
E_\phi = E_t = 0.
\end{align}

Hence Maxwell's equations then admit a solution $A_\phi = A_t = 0$, and if we impose a continuity condition and vanishing of field at the boundary of a surface at infinity, then this solution is the unique solution. Hence from (4a) and (4b)

\begin{align}
E_a = B_a = 0,
\end{align}

the only remaining components are $B_\phi$ and $B_t$. Now from equations (7), (8) and (2) $J^\alpha = 0$ altogether. Then from (4a) we can also define another potential vector $C_\alpha$ such that

\begin{align}
*F_{\alpha\beta} &= C_{\alpha,\beta} - C_{\beta,\alpha},
\end{align}

such that

\begin{align}
*F_{\phi t} = 0,
\end{align}

and using equations (4c) and (4d)

\begin{align}
B_\phi = B_t = 0.
\end{align}

Hence there is no magnetic field, which is Cowling's theorem. More accurately there do not exist any electromagnetic fields as a whole in such a case.

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**References**


