Power-law asymptotic mass distributions for systems of accreting or fragmenting bodies

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Summary. Mass distributions of interplanetary particles are often well represented empirically by a power law of the form \( n(m) = A m^{-s} \). This paper considers analytically the possibility of such power-law distributions as asymptotic solutions to the rate equations governing both accretion and fragmentation of particles that interact only on collision. Particle relative velocities are assumed to be random with mean speeds proportional to \( m^{-h} \).

A general two-body collision cross-section of the form \( (m^a + M^a)^b (m^{-2h} + M^{-2h})^{1/2} \) leads to steady-state \( (dA/dt = 0) \) solutions with exponent \( s_1 = (3 + ab - h)/2 \) provided \( ab < 1 - |h| \). This applies equally to accretion and fragmentation, but in the case of accretion a continuous supply of small particles from an external source is required. Solutions with \( s \) constant, but \( dA/dt \neq 0 \) have also been found, subject to some restrictions on the cross-section parameters \( a, b \) and \( h \), in particular \( h < 0 \). These time-dependent solutions are not the same for accretion and fragmentation.

1 Introduction

The observed distributions of condensed matter in the Solar System and beyond have, for substellar masses, evolved through processes of accretion, fragmentation and sorting by gravitational, thermal and electromagnetic mechanisms. Experimentally, one of the more accessible characteristics of such groups of particles is their mass distribution \( n(m) \). Data exist (Hughes 1978) over an impressive range of masses to some 48 orders of magnitude above micromass particles of some \( 10^{-20} \) kg. In the absence of better theoretical guidelines, the data are usually plotted as \( \log n(m) \) versus \( \log m \) to yield an approximately straight line of downward slope \( s \). Not surprisingly quite a wide range of mass distribution indices \( s \) is observed, but, at least for particles with masses in the range \( 10^{-10} \) kg < \( m < 10^{18} \) kg, \( s \approx 1.8 \pm 0.2 \) for most sets of data. It is therefore of interest to enquire whether any of the forming processes mentioned above could lead to such ‘power-law’ distributions. In particular one might hope that the observed mass index for any group might reveal its history and serve to differentiate between distributions that have evolved for example through accretion or through fragmentation. For instance, Napier & Dodd (1974), in their discussion of the origin of the asteroids, conclude that they arose by fragmentation and not by accretion because their observed distribution index \( s \approx 1.8 \) accords with values obtained
theoretically for fragmentation by Dohnanyi (1969, 1970, 1978) and Hellyer (1970, 1971) whereas they estimate $s \approx 1.5$ for accretion. Hughes (1978) also concludes that 'high $s$ values result from fragmentation processes while low $s$ values are produced by accretion'.

Fragmentation has been quite extensively modelled by a number of authors – see for example the work of Dohnanyi and Hellyer cited above. Accretion in solar orbit has been discussed by Alfven & Arrhenius (1970).

Much work has also been done on the accretion of aerosols: see for example Hidy & Brock (1970), Wadden (1975), Klett (1975) and Drake (1972, 1976). Accretion and fragmentation of interstellar clouds have been discussed by Taff & Savedoff (1972, 1973), Field & Saslaw (1965), Field & Hutchins (1968), Kwan (1979) and by Norman & Silk (1980). Zvyagina & Safronov (1972), Zvyagina, Pechenikova & Safronov (1974), Pechenikova (1975), Pechenikova, Safronov & Zvyagina (1976) have treated accretion and fragmentation of protoplanetary bodies. Monte Carlo simulations of accretion have been reported by Napier & Dodd (1974), Handbury, Simons & Williams (1977, 1979) (referred to as HSW hereinafter) and very recently by Daniels & Hughes (1981) (DH hereinafter).

Most of this earlier work has been concerned with obtaining power-law solutions $n(m) = A m^{-s}$ for the mass-distribution function, and in a number of cases conflicting exponents $s$ have been reported. It is often not made particularly clear whether $A$ is being regarded as time-dependent or time-independent, a distinction we find to be crucial. In most cases only specific cross-sections have been treated so that restrictions on the range of cross-section parameters and solution exponents $s$ have not received much attention. In particular, little attention has been paid to the important question of the mass dependence of particle mean speed and its influence on the cross-section for collisions; only Kwan (1979) has attempted to treat this aspect analytically, and he makes use of an approximation which would significantly alter our findings in the time-dependent case.

In this article we discuss, analytically, power-law solutions to the asymptotic mass distributions attained by both accreting and fragmenting systems. Note that we do not set out to treat the much more general problem posed by the temporal evolution of distributions from arbitrary initial conditions; that is indeed a formidable task. We restrict our discussion to the possibility of asymptotic power-law solutions of the type found to describe, at least approximately, much of the observed data available.

An important feature of our treatment is that we make some attempt to take into account the effect of the mass dependence of particle mean speed, $\bar{v}(m)$ (strictly the rms velocity), on the collision cross-section. Strictly the determination of the speed distribution $\bar{v}(m)$ is inseparably linked with the problem of determining the mass distribution $n(m)$. The self-consistent calculation of $\bar{v}(m)$ and $n(m)$ together represents a most formidable task. In this paper, just as we restrict our attention to asymptotic power-law solutions $n(m) \propto m^{-s}$, so also we consider only power-law speed distributions $\bar{v}(m) \propto m^{-h}$. This is necessary for self-consistency. Only a power-law distribution can be free from any characteristic mass. If $\bar{v}(m)$ were to involve a characteristic mass, this would necessarily be reflected also in $n(m)$ which could not then be a power law. We seek to determine a relationship between $s, h$ and the other physical parameters assumed to govern the collision cross-section. We treat accretion and fragmentation separately, not the combined phenomenon. Nevertheless we compare and contrast our findings for these two processes. For simplicity we assume all bodies are spherical and of uniform density. Only two-body collisions will be considered.

For accretion we assume that bodies combine completely on collision. For fragmentation we consider only catastrophic collisions in which the masses are disrupted into fragments whose mass distribution is itself a power law. In this respect our work follows that of Hellyer (1970) and Dohnanyi (1969). The exponent of the power-law fragment distribution proves
to have little effect on the overall mass distribution index $s$. Both for fragmentation and accretion we ignore any effects arising from the angular momentum of the colliding bodies.

In our analysis we distinguish carefully between what we will describe as 'time-independent' mass distributions that have the general form

$$n(m) = A \ m^{-s}$$  \hspace{1cm} (1a)

where $A$ is strictly a constant and $dn/dt = 0$ so that a true steady-state exists, and what we term 'time-dependent' power-law distributions

$$n(m) = A(t) m^{-s}$$  \hspace{1cm} (1b)

in which $A(t)$ varies with time but is, of course, independent of mass $m$. In each case the exponent $s$ is a constant, independent both of $m$ and $t$. We shall assume always that $A$ is so large that we may safely ignore any statistical fluctuations associated with limited numbers of masses. We adopt a quite general collision cross-section of the form

$$\Sigma(m_1, m_2, a, b, h) \equiv (m_1^{a} + m_2^{b} + (m_1^{a-2h} + m_2^{b-2h})^{1/2}).$$  \hspace{1cm} (2)

Our primary interest is in the geometric collision cross-section for spherical bodies. This has the area of the circle whose radius is the sum of their radii. For bodies of uniform density it is proportional to the first factor in equation (2) with $a = 1/3$ and $b = 2$. However, for the sake of generality, and in order to make comparisons where possible with work by other authors who have discussed other cross-sections, we retain the more general parameters $a$ and $b$, but we assume throughout that $a > 0$ and $b > 0$.

Inclusion of the velocity factor $(m_1^{a-2h} + m_2^{b-2h})^{1/2}$ in the collision cross-section $\Sigma$ enables us to take account of this physically important aspect, largely ignored in earlier work, without incurring an excessive increase in complexity. This velocity factor is based on the following three assumptions. First, that the probability of collision between any two particles of unknown location is proportional to their relative velocity; secondly that the velocity distribution of all particles of given mass is Gaussian; and thirdly that the mean speed $\bar{v}(m) \propto m^{-h}$ where $h$ is some constant. Kennard (1938) has shown that the average relative velocity of two particles belonging to two distinct Gaussian velocity distributions with mean speeds $\bar{v}_1$ and $\bar{v}_2$ is $(\bar{v}_1^2 + \bar{v}_2^2)^{1/2}$.

Different choices of the velocity distribution index $h$ give rise to a variety of models. Most of the authors we cite have ignored any systematic dependence of velocity on mass. Comparison with their work may be made by setting $h = 0$ in $\Sigma$. Another quite common assumption is kinetic energy equipartition (KEE) in which the expectation value of the kinetic energy of a particle is independent of its mass. The assumption of KEE is expressed by setting $h = \frac{1}{2}$ in the collision cross-section $\Sigma$. KEE has been discussed critically by HSW who point out that it is a consequence of the assumption that in collisions all angles between the velocities of the colliding bodies occur with equal probability. In fact, because the collision rate is proportional to the relative velocity, head-on collisions are favoured. Head-on collisions involve a greater fractional loss in mechanical energy so that particles of large mass which have accreted as a result of many collisions ought to have less kinetic energy on average than those of lower mass whose formation will usually have involved fewer collisions. A Monte Carlo calculation by HSW suggests that $h$ should be about 0.6 rather than 0.5 for a purely accreting system.

In this context we should mention another Monte Carlo calculation reported by HSW in which the collisions involved a mixture of accretion and fragmentation that conserved particle numbers. The smaller mass $m$ is deemed to remove an equal mass $m$ from the larger
mass $M$ involved in the collision. The two masses $m$ coagulate to form a new mass $2m$ with conservation of momentum but, of course, destruction of mechanical energy. The remnant mass $(M-m)$ retains the original velocity of $M$. All masses are integral multiples of the unit mass $m_0$.

HSW report the remarkably high value $h = 1.65$ for the asymptotic mass—velocity distribution arising from their calculation. This compares with their value $h \approx 0.6$ for a pure accretion simulation. They attribute this high value, $h = 1.65$, essentially to the fragmentation occurring in the mixed model and go on to suggest that 'it is reasonable to expect that any trends appearing from this model would appear more strongly in a model with stronger fragmentation'. We fear this may be misleading, and draw attention to the following features of the HSW mixed model.

(a) Because all masses are integral multiples of $m_0$, the fragmentation process can only hinder the accretion of masses from $m_0$ units; it cannot fragment the unit masses themselves.

(b) Masses greater than $100m_0$ were arbitrarily removed and replaced by an equivalent number of unit masses with the initial velocity distribution.

(c) The loss of mechanical energy is in fact associated with the mass-doubling accretion process $m + m = (2m)$, rather than the fragmentation because the residue $(M-m)$ is not impeded in the collision and retains its velocity unaltered.

Observe that because of (a) and (b) above the model is forced to be accretive overall and the direction of net mass flow is upwards from $m_0$ to higher masses. Fresh masses $m_0$ are continually being introduced with the initial velocity distribution. These masses cannot fragment, but by collision larger masses are formed from them, so they give rise to an upward mass flux. Energy is lost in these collisions so that on average larger masses move more slowly. What makes $h$ positive, therefore, is essentially the energy loss on accretion and the net upward mass flux. Certainly the fragmentation occurring leads to an amplification of $h$, relative to the pure accretion model, because it requires more collisions, and consequently greater loss of energy, on average, for a body to grow by a given amount.

For fragmentation, in the absence of accretion (not discussed by HSW) two opposing factors compete to determine the sign of $h$. The first is essentially just that described above for accretion, but in reverse. Energy is lost in collisions, and the mass flow is downwards; this gives lighter masses, resulting from many collisions, a tendency to move more slowly than heavier masses which have usually been involved in fewer collisions. This process works towards negative $h$. A competing tendency, towards positive $h$, arises as a consequence of momentum conservation. When a mass breaks into two fragments, the ratio of their speeds, measured relative to their centre of mass, is the inverse of their mass ratio. This tendency may well persist when more than two fragments are produced. Any bias of the fragment speed distribution towards the least massive fragments works towards a positive value for $h$. Without a more detailed knowledge of the (presumably mainly empirical) law of the speed distribution with respect to fragment mass, it is difficult to ascertain which of these opposing tendencies is likely to dominate and decide the sign of $h$. Certainly negative values of $h$ for fragmentation would not seem to be excluded a priori.

Our findings, both for accretion and for fragmentation, are summarized in Table 1. An interesting feature is that we find time-dependent solutions to be excluded both for accretion and for fragmentation unless the velocity index $h < 0$, $h < 0$, which corresponds to mean speed increasing with mass, is physically most implausible for an accreting system of bodies. The borderline case $h = 0$, corresponding to speed independent of mass, though unlikely, has, however, been treated (presumably on grounds of expediency) by many authors some of whose results we confirm.
Table 1. Summary of general results.

<table>
<thead>
<tr>
<th>Equation solved</th>
<th>Time-independent accretion and fragmentation</th>
<th>Time-dependent accretion</th>
<th>Time-dependent fragmentation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dn}{dt} = 0$</td>
<td>$1 \frac{dn}{n , dt} = \text{constant, independent of } m^{-s}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Power-law solution sought</th>
<th>$n = A m^{-s}$</th>
<th>$n = A(t) m^{-s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution exponents found</td>
<td>$s_1 = \frac{3 + k}{2}$</td>
<td>$s_f$. See Table 2.</td>
</tr>
<tr>
<td></td>
<td>$s_1 = \frac{3 + k}{2}$</td>
<td></td>
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<tr>
<td></td>
<td>$s_f \to 1 + k, , \epsilon \to 0$</td>
<td></td>
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</tbody>
</table>

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<tr>
<th>Behaviour of coefficient $A$</th>
<th>$\frac{dA}{dt} = 0$</th>
<th>$\frac{dA}{dt} &lt; 0$</th>
</tr>
</thead>
</table>

| Requirements necessary for existence or stability of solutions | $s > 1 + k + \frac{(h + |h|)}{2}$ | $h < 0$ |
|----------------------------------------------------------------|----------------------------------|--------|
|                                                                 | $s < 2 - \frac{(h + |h|)}{2}$         |        |
|                                                                 | $k < 1 - (h + |h|)$                  |        |
|                                                                 | $s_f < \frac{3 + k}{2}$              |        |
|                                                                 | $2 < s_f < 2 + l$                   |        |
|                                                                 | $s_f < 2$                           |        |
|                                                                 | $k < 1 + l$                         |        |
|                                                                 | $s_f > 2$                           |        |
| The lower of $s_f$ and $s_1$ cannot be stable.                  |                                    |        |

General collision cross-section: $(m^a + M^b)(m^{-2/b} + M^{-2/b})^{1/2}, \, a > 0, \, b > 0$.

Meaning of symbols used: $k = ab - h$.

For $b \neq 0$, $l = a$ or $2 |h|$, whichever is lower but non-zero.

For $b = 0$, $l = 2 |h|$.

$\epsilon$ is defined in Section 3.

Another feature of this paper is our finding that the same time-independent distribution exponent $s_1 = (3 + k)/2$, where $k = ab - h$, holds both for fragmentation and for accretion. This implies that these processes cannot be distinguished by the power-law distribution index $s_1$ alone, at least in this time-independent case. Furthermore the range of possible solutions $s_1$ and the constraints on cross-section parameters $a$, $b$, $h$ leading to such solutions are also found to be the same for both processes. In particular, for the geometric cross-section ($a = 1/3, \, b = 2$) discussed by DH, Hellyer, Dohnanyi and others, we find even time-independent power-law solutions to be excluded for $|h| > 1/3$. This excludes KEW with its exponent $h = 0.5$ and also the higher, more realistic, value 0.6 suggested by HSW for accreting systems.

2 Accretion

We assume a continuous distribution of particle masses with $n(m) \, dm$ bodies whose masses lie in the range $m$ to $m + dm$. Merging of these bodies of mass $m$ with others whose masses lie in the range $M$ to $M + dM$ is assumed to occur at the rate $K \Xi(m, M, a, b, h) \, n(m) \, n(M) \, dm \, dM$ where $K$ is a positive constant and $\Xi$ is the collision cross-section defined by equation (2). Merging is assumed to be complete, i.e. the masses $m$ and $M$ combine to form
a single mass \((m + M)\) and fragmentation is excluded. It follows that bodies with masses in the range \(m\) to \(m + dm\) are being formed by merging of smaller masses \(\mu\) and \((m - \mu)\) at the rate

\[
C\ dm = \frac{1}{2} K dm \int_{0}^{m} \Xi[\mu, (m - \mu), a, b, h] n(\mu) n(m - \mu) d\mu. \tag{3a}
\]

At the same time, however, bodies with masses in this range, \(m\) to \(m + dm\), are also being removed by merging with other masses \(M\) of all magnitudes at the rate

\[
D\ dm = K n(m) dm \int_{0}^{\infty} \Xi(m, M, a, b, h) n(M) dM. \tag{3b}
\]

The nett population increases at the difference between these rates:

\[
dn(m)/dt = C - D. \tag{3c}
\]

Note that equations (3) exhibit no ‘characteristic mass’ at which the nature of the accretion process changes. This implies that the asymptotic solution distribution must likewise be free of any such characteristic mass, and in particular suggests that power-law functions of type (1a) or (1b) are prime candidates for asymptotic distribution functions. In Section 2.1 below we seek a time-independent solution of type (1a) to the strict steady-state equation \(dn/dt = 0\) for all \(m\). In Section 2.2 we discuss time-dependent solutions of type (1b). In either case the power-law distribution can in reality extend only over a finite mass range \(m_{0} < m < m_{\infty}\). Outside this range we impose a crude cut-off so that \(n(m < m_{0}) = n(m > m_{\infty}) = 0\). However, at such boundaries, \(m_{0}\) and \(m_{\infty}\), no asymptotic steady state can be achieved. As our solutions are essentially steady-state solutions, we expect them to be valid only for masses \(m\) sufficiently far removed from the ends of a very wide distribution so that \(m_{0} < m < m_{\infty}\). Thus we seek only solutions for which \(m_{0}\) and \(m_{\infty}\) may be replaced by 0 and \(\infty\) respectively without serious error.

2.1 TIME INDEPENDENT ACCRETION

In this section we discuss the conditions for a distribution of type (1a), \(n(m) = A m^{-s}\), with \(A\) independent of time and \(m_{0} < m < m_{\infty}\), to constitute an asymptotic solution to the strict steady-state equation \(dn/dt = 0\) obtained by setting equation (3c) to zero.

Adopting distribution (1a) over the range \(m_{0} < m < m_{\infty}\), and substituting \((m + x)/2\) for \(\mu\) in equation (3a) yields

\[
C = \frac{KA^{2} m^{-(p+1)}}{4} \int_{1-\alpha'}^{1-\alpha} \Xi(u, v, a, b, h) (uv)^{-s} dx \tag{4}
\]

where \(u \equiv (1 + x)/2, v \equiv (1 - x)/2, p \equiv 2s - 2 - k, k \equiv ab - h\) and \(\alpha' = 2m_{0}/m\). (The upper limit of the integral, \(1 - \alpha'\), corresponds to \(m - m_{0}\) which is the greatest mass that can combine with another to form \(m\).)

Similarly, the substitution \(M = m(1 + x)/(1 - x)\) enables equation (3b) to be rewritten

\[
D = \frac{KA^{2} m^{-(p+1)}}{4} \int_{1-\beta}^{-\beta} 2v^{p} \Xi(u, v, a, b, h) (uv)^{-s} dx \tag{5}
\]

where \(\alpha \equiv 2m_{0}/(m + m_{0})\) and \(\beta \equiv 2m/(m + m_{\infty})\).

All three quantities \(\alpha, \alpha'\) and \(\beta\), are to be regarded as vanishingly small because we are concerned with values of \(m\) far removed from the limits \(m_{0}\) and \(m_{\infty}\). Exploiting this together
with the fact that the integral in equation (4) is an even function of \( x \), enables equation (3c) to be expressed for \( m_0 \ll m_\infty \) as

\[
\frac{dn}{dt} = KA^2 m^{-(p+1)} \left[ \frac{1}{2} \int_0^{1-\alpha} \Xi(u, v, a, b, h) H(x, s, h) \, dx + E_1(m, s, h) + E_2(m, s, h) \right]
\]

(6)

where

\[
H(x, s, h) \equiv (uv)^x \left[ 1 - u^p - v^p \right].
\]

(7)

\( E_1 \) and \( E_2 \) are end corrections given in the limit \( \alpha, \beta, \alpha' \rightarrow 0 \) by

\[
E_1(m, s, h) = C_0 \int_{\alpha/2}^{\beta/2} v^q \, dv,
\]

(8)

\[
E_2(m, s, h) = C_0 \int_{\alpha/2}^{\alpha/2} v^r \, dv
\]

(9)

where \( q = s - k - 2 - (h + |h|)/2 \), \( k \equiv ab - h \), \( r \equiv -s - (h + |h|)/2 \) and \( C_0 = 1 \) for \( h \neq 0 \), \( C_0 = \sqrt{2} \) for \( h = 0 \).

For values of \( m \) well inside the range of the distribution \((m_0 \ll m \ll m_\infty)\) end correction \( E_1 \) may be neglected provided \( q > -1 \) which requires

\[
s > 1 + k + (h + |h|)/2.
\]

(10)

Because the limits \( \alpha/2, \alpha'/2 \) of the integral in equation (9) converge as \((m_0/m) \rightarrow 0\), the condition for neglect of \( E_2 \) is \( r > -2 \), in which case

\[
s < 2 - (h + |h|)/2.
\]

(11)

One observes that when \( s = s_1 \equiv (3 + k)/2 \), making \( p = 1 \), the factor \( [1 - u^p - v^p] \) in expression (7) for \( H \) becomes identically zero for all \( x \). For \( s = s_1 \), both conditions (10) and (11) impose the same constraint

\[
k < 1 - (h + |h|).
\]

(12)

Provided the collision cross-section parameters are such as to satisfy condition (12), both end corrections \( E_1 \) and \( E_2 \) may be neglected for \( m_0 \ll m \ll m_\infty \) and, to this approximation, the right-hand side of equation (6) will be zero. Thus \( n = A m^{-s_1} \) constitutes a solution to the strict steady-state equation \( dn/dt = 0 \).

For the important special case of the geometric collision cross-section (with \( a = 1/3 \) and \( b = 2 \)) \( k = 2/3 - h \) and \( s_1 = (3 + k)/2 = 11/6 - h/2 \). Condition (12), with \( k = 2/3 - h \), implies that no time-independent solution of type (1a) can exist unless \( |h| < 1/3 \). This excludes KEV for which \( h = 1/2 \) and also the rather higher value \( h \approx 0.6 \) suggested by Handbury, Simons & Williams (1977, 1979). For a geometric collision cross-section with velocity independent of mass (the model treated by DH) \( s_1 = 11/6 \).

The accretion process continually transfers mass upwards through the mass distribution. When \( s = s_1 \), \( dn/dt \) is essentially zero except near the distribution limits \( m_0 \) and \( m_\infty \). Mass is being transferred from the lower to the upper limit without any nett change at intermediate mass levels. For this to be a steady state, an inexhaustible reservoir of mass is required below any arbitrary level \( m_1 > m_0 \). The mass in the distribution itself below level \( m_1 \) is (for \( s \neq 2 \)):

\[
M(m < m_1) = A \int_{m_0}^{m_1} m^{1-s} \, dm = A \left[ m_1^{2-s} - m_0^{2-s} \right] / (2 - s).
\]

(13)
If $s < 2$, as it must be to satisfy constraint (11), this mass of small particles, $M(m < m_1)$, is necessarily finite even in the limit $m_0 \to 0$. An external source of low mass particles is therefore required to sustain the steady-state distribution.

The introduction of such a source of small particles corresponds to the Monte Carlo simulation in which DH maintained their number of unit mass particles constant by 'creating' new particles of unit mass to replace those removed by collision. For their geometric collision cross-section and neglect of any dependence of velocity upon mass, we obtain $s_1 = 11/6$ where DH found $s = 1.65 \pm 0.11$. The discrepancy, which is not much greater than their estimated error, probably reflects the absence of a sufficiently long tail of very massive objects in the necessarily restricted conditions of the DH Monte Carlo calculation.

It is an interesting feature that both the solution distribution index $s_1$ and constraint (10) can be readily derived by direct consideration of the flux of matter $\phi(m)$ through level $m$ in the distribution from lower to higher masses. This flux is given by

\[
\phi(m) = K \int_0^m \mu n(\mu) d\mu \int_m^\infty \Xi(\mu, M, a, b, h) n(M) dM \\
+ K \int_0^{m/2} n(\mu) d\mu \int_{m-\mu}^m \Xi(\mu, M, a, b, h) (M+\mu) n(M) dM.
\]  

(14)

Here the first contribution is from bodies of mass $\mu < m$ combining with others of mass $M > m$, while the second is from bodies of mass $\mu < m/2$ uniting with masses in the range $m/2 < M < m$. The first group transport mass $\mu$ through level $m$, the second transport $M + \mu$.

Adopting a power-law distribution $n(m) = A m^{-s}$, it is clear that equation (10) expresses the condition that flux $\phi(m)$ be finite. Furthermore, it is evident that for a scaling factor $\theta$

\[
\phi(\theta m) = \theta^{(3 + k - 2s)} \phi(m)
\]  

(15)

provided end effects may be neglected for both $m$ and $\theta m$. Now in a steady state with $dn/dt = 0$ for all $m$, the flux must be the same at all $m$, i.e. $\phi(\theta m) = \phi(m)$. This requires $s = s_1 = (3 + k)/2$.

Many authors (for example, Wadden 1975; Taff & Savedoff 1972; Field & Hutchins 1968; Field & Saslaw 1965) have considered what is perhaps the simplest special case, namely accretion with constant cross-section ($ab = 0, h = 0$). In agreement with them we find $s_1 = 3/2$ in this case. Three other papers deal with more general cross-sections. Kwan (1979), in an interesting appendix to his discussion of the mass spectrum of interstellar clouds, briefly examines a model in which the largest clouds ($m_\infty$) disrupt into a gas of the smallest constituents ($m_0$) of the system. By matching this production of unit ($m_0$) particles to their consumption by accretion in this closed cyclic system, he shows that a power-law distribution requires an exponent $s = (3 + ab - h)/2$ identical to our $s_1$. Kwan's analysis is a remarkable complement to ours in that by matching only the fluxes into and out of the distribution at its ends, and essentially ignoring the range between, he nevertheless obtains the same distribution index $s_1$ we have derived by a discussion of the equilibrium of the distribution at intermediate masses well removed from both limits $m_0$ and $m_\infty$. Although he does not expressly discuss this aspect, Kwan's treatment also requires that constraint (10) be satisfied; lower values of $s$ lead to infinite collision rates from $m_0$ particles when the range ratio $m_\infty/m_0$ of the distribution is increased without limit.

Zvyagina et al. (1974) have considered a system in which accretion and fragmentation proceed simultaneously, and looked for power-law solutions which depend on the
dimensionality $k$ of the cross-section. They do not explicitly discuss velocity dependent cross-sections, though negative values of $h$ can be readily accommodated within their $k$. Positive values of $h$ (i.e. negative powers of mass in the cross-section) require more careful treatment, particularly when establishing a limit to $k$ like that of equation (12). Because they are considering a gravitating system, their description of fragmentation is rather more complicated than that used in Section 3 of the present work. That is, they allow their equivalent of our (constant) fragmentation parameter $e$ to be mass dependent, except when their parameter $\lambda$ is zero. We show in Section 3 that fragmentation with constant $e$ leads to the same value of (and constraints upon) $s_1$ as with accretion, so long as the distributions are time-independent. We therefore expect exponent $s_1 = (3 + k)/2$ to apply to a system undergoing both accretion and fragmentation if the distribution is time-independent. Zvyagina et al. do not state if their solutions are time-dependent or independent, but they are in fact the latter because they have exponent $s_1$ (for $\lambda = 0$). They also find the inequality $k < 1$, which is the same as our equation (12) when positive values of $h$ are excluded.

Taff & Savedoff (1973) report numerical calculations of $s$ for a number of cross-sections which they label $C_1$ to $C_4$. For all but the first they find exponents $s_{TS} \approx 1.32 + k/2$ where we (and Kwan) obtain $s_1 = 1.50 + k/2$. All these five values of $s_{TS}$ fail to satisfy constraint (10). However, for $C_1$ (mass independent cross-section $k = 0$) they obtain $s_{TS} = 1.50$ in agreement with us (and several other authors). Numerical calculations are hard to check, but we note that these results $s_{TS}$ were obtained using a distribution restricted to $n = m_{\infty}/m_0 = 40$. In an earlier paper, Taff & Savedoff (1972) report the sequence of exponents

$$2.83, 2.45, 2.11, 1.82 \text{ for } n = 5, 10, 20 \text{ and } 40 \text{ respectively in a similar calculation for } a = 1/3, b = 2, h = 0.$$ 

There is little evidence that this sequence converges to 5/3 (at $n = \infty$) as the authors tentatively suggest. It is strange that their result for $C_1$ lies so far from the line $s_{TS} = 1.32 + k/2$ of the other five points so that although $C_1$ and $C_2$ have the same dimensionality, $k = 0$, their exponents $s_{TS}$ differ by 0.18. For $C_2$, the three-dimensional geometric cross-section with velocity independent of mass ($a = 1/3, b = 2, h = 0$) Taff & Savedoff (1973) find $s_{TS} = 1.65$ in agreement with the result of Daniels & Hughes (1981) Monte Carlo calculation. The value 5/3 is claimed for this exponent by Field & Saslaw (1965) but without supporting argument. In Taff & Savedoff (1972) it is also attributed to an unpublished thesis by Taff. In the following Section (2.2) we show how this exponent $s_0 = 5/3$ ($s_0 = 1 + k$ in general) arises naturally from the dimensionality of the cross-section when one requires that $n^{-1} dn/dt$ be independent of $m$. Because end effects are not negligible, this solution $s_0$ is excluded by condition (10). This exclusion is, however, marginal, so it is perhaps not altogether surprising that approximations to $s_0$ should sometimes arise in numerical calculations of restricted range like those of Daniels & Hughes (1981) and Taff & Savedoff (1973).

We now discuss the uniqueness of the exponent $s_1 = (3 + k)/2$ for a power-law solution to the strict steady-state condition $dn/dt = 0$. Differentiating equation (7) with respect to $s$, while maintaining $x$ and $h$ constant, gives

$$\left( \frac{\partial H}{\partial s} \right)_{x, h} = (uv)^{s} \left[ - \ln(uv) - (u^p - v^p) \ln(u/v) \right].$$

(16)

Now for all $x$ in the relevant range, $0 < x < 1$, $- \ln(uv) > \ln(u/v) > 0$ and $u^p - v^p < 1$.

Accordingly, for all $x$ in the range, $(\partial H/\partial s) > 0$ and so $H$ is positive for $s > s_1$ and negative for $s < s_1$. As all factors other than $H$ in the integral term of equation (6) are essentially positive, it follows that $dn/dt$ must have the same sign as $H$, assuming of course that condi-
tions (10) and (11) for negligible end corrections $E_1$ and $E_2$ are met. The zero at $s_1$ is therefore unique.

We have already seen that this solution $n_1 = A^{-s}$ requires an additional steady source of mass in the form of particles at the lower mass limit. We now discuss its stability subject to such support. There is no problem with local stability in the sense that time will smooth out local spikes or pits in any continuous distribution. As equation (3b) shows, bodies of mass $m$ are removed at a rate proportional to their numbers $n(m)$. Consequently any local roughness should decay exponentially.

A rigorous assessment of the global stability of the distribution would seem to require a study of the temporal evolution of distributions of quite general form and is accordingly outside the scope of a limited study of power-law distributions such as this. The following argument, however, provides some degree of support for the hypothesis that the distribution $n_1 (m > m_0) = A m^{-s_1}$ will be stable when supported at the $m_0$ level by a steady source of small particles. Suppose that we have a power-law distribution with $s < s_1$ so that $n > n_1$ for $m > m_0$. As shown, $dn/dt < 0$ for $s < s_1$ so that $n$ will at least initially decay towards the solution distribution $n_1$ for all $m > m_0$, as it must for stability. Similarly if $s > s_1$ so that $n < n_1$ for $m > m_0$, $n$ will increase towards $n_1$. However, the mass dependence of $dn/dt$ is such that $n$ will approach $n_1$ at a rate that depends on $m$ in such a way that $n$ will pass through non-power-law states outside the scope of our present analysis.

Some further support for the stability of power-law distributions with exponent $s_1$ when provided with a steady source of small particles is given by the work of Field & Saslaw (1965). They have shown that a distribution of accreting particles, whose collision cross-section is independent of mass, asymptotically approaches the power-law distribution with exponent $s_1 = 1.5$, independent of initial conditions, when supported by a constant supply of unit mass particles. Wadden (1975) in a treatment of the same mass-independent collision cross-section also found an asymptotic approach to $s_1 = 1.5$ regardless of the shape of the distribution of particles constantly fed to the accreting system.

### 2.2 Time-Dependent Accretion

In Section 2.1 we discussed the requirements for a strictly time-independent power-law solution, $n = A m^{-s}$ (equation 1a), to the accretion equation. As one such requirement is an external source of low mass particles, no strictly time-independent power-law solution exists for a distribution isolated from any such external source. In this section we relax the constraint of strict time-independence to permit the coefficient $A$ to vary with time, as in equation (1b), so that the distribution as a whole can rise or fall without changing the index $s$ of its power-law mass dependence.

Assuming a solution of type (1b) we rewrite the accretion equation (6) in the form

$$\frac{1}{KA^2} \frac{dA}{dt} + \frac{1}{KAn} \frac{\partial n}{\partial t} = G + m^\gamma [I(m, a, b, h, s) + E'_1(m, s, h) + E_2(m, s, h)] \tag{17}$$

where $\gamma = k - s + 1$, $G$ is independent of $m$, $I$ is an integral whose integrand is independent of $m$ and $E'_1$ and $E_2$ (equation 9) are end corrections. $I$, $G$ and $E'_1$ are defined below.

Apart from the factor $(KAn)^{-1}$, the essential difference between equations (6) and (17) is the explicit separation of the mass independent term $G$ in the latter. For generality we define

$$G = QC_0 \int_{m_o}^{m_\infty} M^{k-s} dM \tag{18}$$
where $Q$ is an arbitrary numerical constant. A mass-independent term is always present because equation (3b), after the substitution $n(m) = A m^{-s}$, includes the term

$$K n(m) \int_{m_0}^{m} M^{k-s} dM.$$ 

$A(t)$ will be essentially independent of mass $m$ if the right-hand side of equation (17) is likewise. The special case that this right-hand side be zero corresponds to strict time-independent accretion which was discussed in Section 2.1. Less restrictive solutions of type (1b) may exist if the end corrections $E_1'$ and $E_2$ are both negligible and either $\gamma = 0$ or $I = 0$ for all $m$. In the following analysis we show that the $\gamma = 0$ solutions are excluded by divergence of the integrals $I$ and $E_1'$. By making an appropriate choice of the arbitrary constant $Q$ in equation (18) that defines $G$, namely $Q = -1$, it is possible when $h < 0$ to eliminate the leading term in the end correction $E_1'$. This is a generalization of a technique explicitly used by Zvyagina & Safronov (1972) in their treatment of the special case of the cross-section with $h = 0$ and $a = b = 1$. It may also have been used by them to obtain their numerical solutions for a number of other special cross-sections. A necessary condition for the stability of power-law distributions whose exponent $s_I$ renders $I = 0$ is that the product $\gamma \partial I/\partial s$ be positive in the neighbourhood of $s_I$. (Suppose for example that $\gamma \partial I/\partial s < 0$. Then for a power-law distribution with exponent $s_I + ds$, slightly steeper than the solution, $n^{-1} dn/dt$ would be greater for smaller $m$ and the downward slope would become still steeper.)

The explicit form of the integral $I$ in equation (17) is

$$I(m, a, b, h, s) = \frac{1}{2} \int_0^{1-\alpha} (uv)^{q-s} (T_1 - QT_2) \, dx$$

(19)

where

$$T_1 \equiv \mathcal{Z}(u, v, a, b, h) (1 - u^p - v^p)$$

(20)

and

$$T_2 \equiv C_0 (u^k v^p + u^p v^k).$$

(21)

The end corrections are $E_2$ (defined by equation 9 in Section 2.1) and

$$E'_1(m, a, b, h, s) \equiv \frac{1}{2} \int_{1-\beta}^{1-\alpha} (uv)^{q-s} [\mathcal{Z}(u, v, a, b, h) + QC_0 u^k] \, v^p \, dx$$

(22)

$$\approx C_1(h) \int_{\alpha/2}^{\beta/2} v^q \, dv$$

(23)

in the limits $\alpha \to 0$, $\beta \to 0$, where $q = s - 2 - ab + (h - |h|)/2$, as in Section 2.1, and

$$C_1(h) \equiv \begin{cases} C_0(1 + Q) & \text{if } h < 0 \\ C_0 & \text{if } h > 0. \end{cases}$$

(24)

When $h > 0$, $C_1 = C_0 \neq 0$, so that $E'_1$, like $E_1$, may be neglected only if $q > -1$. This again imposes the condition

$$s > 1 + ab, \quad h > 0 \quad \text{or} \quad C_1 \neq 0$$

(25)

which corresponds to condition (10) of Section 2.1.
When \( h < 0 \), the choice \( Q = -1 \) renders \( C_1 = 0 \), nullifying the leading term (lowest power of \( v \)) in the integrand of expression (22) for \( E'_1 \).

Approximation (23) must then be replaced by

\[
E'_1 \approx C_2 \int_{a/2}^{\alpha/2} v^{q'/v} dv
\]

(26)

where \( C_2 \) is a numerical constant and \( q' \equiv s - 2 - k + l \) is the lowest remaining exponent of \( v \) in the integrand of equation (22). If \( b \neq 0 \), \( l \) represents \( a \) or \( 2|h| \), whichever has the lower but non-zero value (we consider only \( a > 0 \)). If \( b = 0 \) and \( h \neq 0 \), then \( l = 2|h| \). If \( b = 0 \) and \( h = 0 \) also, then the integrand of \( E'_1 \) in equation (22) is identically zero. We discuss this last special case later. There is no need to consider the case that \( a = 0 \) because it is in effect indistinguishable from setting \( b = 0 \). Accordingly we have excluded \( a = 0 \) throughout.

The condition \( q' > -1 \) for neglect of \( E'_1 \) now takes the form

\[
s > 1 + k - l, \quad h < 0, \quad (b, h \text{ not both zero})
\]

(27)

which is less restrictive than its counterpart for time-independent accretion, condition (10). Condition (11) for the neglect of \( E_2 \) remains unaltered. Combining conditions (11) and (27) gives, in place of (12),

\[
k < 1 + l, \quad h < 0.
\]

(28)

The \( \gamma = 0 \) solutions, \( s_0 = 1 + k = 1 + ab - h \), are clearly excluded by condition (25) for \( h > 0 \). They are also excluded for \( h < 0 \) because the integral \( I \) diverges at its upper limit when \( v = a/2 \rightarrow 0 \). As this upper limit is approached, the following approximation (valid only to its lowest power in \( v \)) applies to the integrand of \( I \) in equation (19) with \( Q = -1 \) and \( h < 0 \) (but excluding \( b = 0 = h \)):

\[
(uv)^{-s} (T_1 + T_2) \approx C_0 (pv + v^k - C_3 v^{p+l}) v^{-s}, \quad v \rightarrow 0
\]

(29)

where \( C_3 = b \) if \( l \neq 2|h| \) or \( C_3 = 1/2 \) if \( l \neq a \), or \( C_3 = b + 1/2 \) if \( a = l = 2|h| \).

Clearly for \( s = s_0 = 1 + k \) the lowest exponent of \( v \) in equation (29) cannot exceed \(-1\) and so the integral for \( I \) must diverge at its upper limit. Thus solutions \( s_0 \) are excluded for all values of \( h \).

Consideration of the convergence of \( I \) at its upper limit also reveals restrictions on the range of solutions \( s_I \) obtainable by the alternative route of setting \( l = 0 \). If \( h > 0 \) and \( s \) satisfies condition (25) for negligible \( E'_1 \) one finds

\[
p - k > 2h > 0.
\]

(30)

Consequently terms in the factor \( (T_1 - QT_2) \), in the integrand of \( I \), that involve \( v^p \) may be neglected by comparison with those that involve \( v^k \) as \( v \rightarrow 0 \). The lowest exponent of \( v \) in the integrand of \( I \) is therefore either \( 1 - s - h \) or \( k - s \). In either case a necessary condition that this lowest exponent exceed \(-1\) is

\[
s < s_0 \equiv 1 + k \equiv 1 + ab - h.
\]

(31)

This condition (31) conflicts with condition (25). Thus solutions \( s_I \) are excluded for \( h > 0 \).

If \( h < 0 \), convergence of integral \( I \) at its upper limit, requires, as we have seen, that the lowest exponent of \( v \) in equation (29) should exceed \(-1\). This imposes the following simultaneous constraints on the exponent \( s \):

\[
s < 2
\]

(32)

\[
s < 1 + k
\]

(33)

\[
s > 1 + k - l.
\]

(34)
Constraint (32) arises also from the requirement that end effect \( E_2 \) be negligible. Note that in order to satisfy both constraints (32) and (34) one requires

\[ k < 1 + l. \tag{35} \]

As condition (34) is equivalent to condition (27) for neglect of \( E'_1 \), it is clear that solutions \( s_f \) are not excluded provided \( h < 0 \). Although it is difficult to envisage circumstances in which \( h < 0 \) could have physical relevance, the limiting case \( h = 0 \) (just not excluded) has been discussed by a number of authors, and for that reason we shall give it particular attention. For generality, however, we shall continue to discuss solutions \( s_f \) over the broad range \( h < 0 \). We will now show that provided \( h < 0 \) and \( k < 1 + l \), a solution \( s_f \) that satisfies conditions (32)–(34) and the stability criterion \( \gamma \partial I/\partial s > 0 \) always exists, and that \( s_f < s_1 \) where \( s_1 = (3 + k)/2 \) is the time-independent solution index. We shall also prove that if \( p > 0 \) (i.e., \( s > 1 + k/2 \)) this solution \( s_f \) is unique. We suspect (but have not been able to prove) that it is unique regardless of the sign of \( p \).

Consideration of equation (29) shows that if \( s \) satisfies equations (32)–(34) the term \( -C_0C_3u^{p+1} \) dominates the integrand of \( I \) as \( v \to 0 \) towards the upper limit of the integral. Consequently as \( s \) approaches its lower limit \( 1 + k - l \) set by equation (34), \( I \to -\infty \). Similarly, as \( s \) increases towards its upper limit, \( 2 \) or \( 1 + k \), whichever is lower, \( I \to \infty \). Because \( I \) is continuous between these limits, at least one root \( s_f \) at which \( \partial I/\partial s > 0 \) must lie between them. Now, because of condition (33), \( \gamma = 1 + k - s \) must also be positive and consequently the stability criterion \( \gamma \partial I/\partial s \) will be fulfilled at any such root.

To show that \( s_f \) is unique, at least for \( p > 0 \), we now demonstrate that \( \partial I/\partial s > 0 \) for \( p > 0 \). Differentiating the integrand of \( I \) (with \( Q = -1 \)) with respect to \( s \) gives

\[ \partial[(T_1 + T_2)(uv)^s]/\partial s = (uv)^s g \tag{36} \]

where

\[ g = -\Xi \ln(uv) - \Xi(u^p - v^p) \ln(u/v) + C_0(u^p - v^p) \ln(u/v) \tag{37} \]

and

\[ \Xi = \Xi(u, v, a, b, h). \]

Now \( \Xi > C_0u^k \), and therefore

\[ g > -\Xi \ln(uv) - \Xi u^p \ln(u/v) + C_0u^p v^k \ln(u/v) \tag{38} \]

because \( 0 < v < u < 1 \), \( u^p < 1 \) for \( p > 0 \) and \( -\ln(u/v) > \ln(u/v) > 0 \). Therefore \( g > 0 \) and \( \partial I/\partial s > 0 \) for \( p > 0 \).

We now show that for \( s > s_1 \), \( I > 0 \) and consequently \( s_f < s_1 \). We saw in Section 2.1 that \( H \equiv (uv)^s(1 - u^p - v^p) \) has a unique zero at \( s = s_1 \) (making \( p = 1 \)) because \( \partial H/\partial s > 0 \). It follows that \( T_1(uv)^s = H\Xi > 0 \) for \( s > s_1 \). As it is clear from equation (21) that \( T_2(uv)^s \) is always positive, it follows that \( (T_1 + T_2)(uv)^s \), the integrand of \( I \), is positive for \( s > s_1 \) and consequently \( s_f < s_1 \).

In the remainder of this section we discuss briefly the special case \( h = 0 \) which is of considerable interest; examples with specific \( a, b \) have been discussed by a number of authors. We are not aware of any general solution to \( I = 0 \) even when \( h = 0 \). However, for \( b = 1 \) a solution to \( I = 0 \) is readily obtained for any value of \( a \) in the range \( 0 < a < 2 \). In this case the factor \( (T_1 + T_2) \) in the integrand of \( I \) is identically zero for \( p = 0 \) corresponding to the solution \( s_f = 1 + a/2 \). This solution includes and confirms the stable approximate numerical results \( s = 1.31, 1.50 \) and \( 1.67 \) found respectively for \( a = 2/3, 1 \) and \( 4/3 \) with \( b = 1 \) and \( h = 0 \), by Zvyagina & Safronov (1972).
The special case of the mass-independent cross-section, obtained by setting \( b = 0 = h \), deserves particular mention, and provides a link with the work of other authors. The usual choice \( Q = -1 \) renders \((T_1 - Q T_2) = C_0\) because \( k = 0 \) in equation (21) for \( T_2 \) and \( \Sigma(u, v, a, 0, 0) = C_0 = \sqrt{2} \) in equation (20) for \( T_1 \). Consequently for \( Q = -1, I > 0 \) for all \( s \). For any other choice of \( Q \) one requires \( s > 1 \) to avoid divergence of \( E_1' \) — see equation (25). This excludes the curious solution \( s = 1, Q = -1/2 \) for which the integrand of \( I \) is identically zero; for all other \( Q \neq 0 \) one requires \( s < 1 \) to avoid divergence of \( I \) — equation (33). Thus no solution \( s_f \) can exist for this limiting case although the neighbouring cross-section with \( b = 1, a \to 0, \alpha \neq 0 \) has solution \( s_f = 1 + a/2 \approx 1 \). This finding of ours is in agreement with the work of Schumann (1940) who showed that a mass-independent cross-section leads to an exponential distribution of the form \( n(m, \tau) = A(\tau) \exp[-m B(\tau)] \) for isolated accreting system. Other examples of time-dependent, non-power law, solutions of the accretion equation have been given by Hidy & Brock (1970) and Drake (1972). It is possible that solutions of these more general (non-power law) type may exist for cross-sections whose parameters are excluded by our constraint (35).

In a related context, it is interesting to observe that the choice \( Q = 2 \) leads in the specific case \( a = b = 1, \ h = 0 \), to an identically zero integrand also for \( p = 2 \) which corresponds to a solution \( s = 5/2 \). However, in this case \( \gamma \partial I/\partial s < 0 \) so that this solution is unstable as reported by Zvyagina & Safronov; \( s = 5/2 \) is in any case excluded by constraint (32) because \( E_2 \to \infty \) when \( m_0 \to 0 \).

Further evidence of stability in the case of the \( s = 3/2 \) solution for the cross-section with \( a = b = 1, \ h = 0 \) is provided by the work of Pechernikova (1975) who has studied, numerically, the evolution of the distribution towards this power-law solution starting from a number of different initial distributions. This case has also been studied analytically by Safronov (1963) and by Golovin (1963) who have shown that an initially exponential distribution evolves via a Bessel function form into an asymptotic power law with \( s = 3/2 \). Golovin finds that an initially \( \delta \) function distribution also reaches the same asymptotic form.

Having sought in vain for variable transformations that might lead, with an appropriate choice of \( s \), to identically zero integrands for more general values of \( b \), we have instead obtained solutions to \( I = 0 \) by numerical methods for three collision cross-sections of particular interest, namely those with \( a = 1/3, 1/2 \), and \( 2/3 \), all with \( b = 2 \) and \( h = 0 \). Our results for \( s_f \), presented in Table 2, are respectively 1.5518, 1.7743 and 1.9129, all accurate to at least three and probably four decimal places. All satisfy the stability criterion \( \gamma \partial I/\partial s > 0 \) and conditions (32)–(34). The result for \( a = 1/3 \) is in reasonable agreement with Zvyagina & Safronov's (1972) \( s = 1.54 \) (for \( a = 1/3, \ b = 2 \)), but their value 1.80 for \( a = 2/3, \ b = 2 \) is markedly lower than ours. Without access to the details of their calculation, we cannot explain this discrepancy. Our numerical integrations were performed using several low fractional powers of \( v \) as the integration variable to avoid large variations in the integrand. The integrand was evaluated at 4000 equispaced values of the integration variable.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( s_f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 &lt; a &lt; 2 )</td>
<td>1</td>
<td>( 1 + a/2 )</td>
</tr>
<tr>
<td>( 1/3 )</td>
<td>2</td>
<td>1.5518</td>
</tr>
<tr>
<td>( 1/2 )</td>
<td>2</td>
<td>1.7743</td>
</tr>
<tr>
<td>( 2/3 )</td>
<td>2</td>
<td>1.9129</td>
</tr>
</tbody>
</table>

Table 2. Some results for time-dependent accretion. General velocity independent cross-section function \( \Sigma = C_o (m^a + M^a)^b \).

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It should be remarked that these few examples demonstrate that $s_f$ is not uniquely determined by the dimensionality $k \equiv ab - h$ of the collision cross-section. This contrasts with $s_1 = (3 + k)/2$, the solution for time-independent (supported) accretion discussed in Section 2.1, and with (divergence excluded) $\gamma = 0$ solutions $s_0 = 1 + k$.

It is also of some interest to observe that condition (32) requires $s_f < 2$ so that the mass of the distribution is concentrated in its most massive bodies and accretion, unsupported by any source of small particles as in Section 2.1, must lead to a distinct fall in the distribution, i.e. $dA/dt < 0$. If a solution $s_f > 2$ were possible, then the concentration of mass in the smallest particles would provide a practically inexhaustible supply and the distribution would be effectively time-independent with $dA/dt \sim 0$.

3 Fragmentation

In this section we discuss the question of wide power-law mass distributions arising asymptotically as a result of bodies fragmenting on collision in the absence of accretion. Our assumptions are essentially those of Hellyer (1970) whose treatment is broadly similar to that of Dohnanyi (1969). However, as in our treatment of accretion, we employ the more general factor $(m_1^a + m_2^a)^b$ in the collision cross-section $\Xi$ (Hellyer and Dohnanyi discuss the geometric cross-section for which $a = 1/3$ and $b = 2$) and we again also include a velocity factor $(m_1^{-2h} + m_2^{-2h})^{1/2}$.

Dohnanyi (1969) has shown that erosive collisions, i.e. those that do not fragment the target mass, may be neglected. We consider only catastrophic collisions in which the target mass $\mu$ is struck by a projectile of mass $M > e\mu$ where $e\mu < \mu$ represents the mass of the smallest projectile capable of fragmenting a target mass $\mu$. (e will of course depend on the velocity index $h$.) Following Hellyer (1970) and Hartman (1965) we assume the resultant fragment masses $m$ to have a distribution

$$\xi(\mu, m < \mu) = (1 - \omega)\mu^\omega m^{-(\omega + 1)} \tag{39}$$

where $\omega$ is an empirical constant whose value lies between 0.5 and 1.0. Dohnanyi (1969) adopts a fragment distribution of similar dimensionality.

We assume that bodies whose mass lies in the range $\mu$ to $\mu + d\mu$, suffer catastrophic collision at the rate

$$\psi(\mu) d\mu = K d\mu \int_{e\mu}^{m_\infty} n(M)\Xi(\mu, M, a, b, h) dM \tag{40}$$

where $m_\infty$ is the largest mass in the distribution, and the cross-section function $\Xi(\mu, M, a, b, h)$ is as defined by equation (2).

The rate of change of the mass distribution is formally

$$\frac{dn(m)}{dt} = \int_m^{m_\infty} n(\mu) \xi(\mu, m) \psi(\mu) d\mu - n(m) \psi(m) \tag{41}$$

where the first term represents the generation of new masses $m$ by fragmentation of larger masses $\mu$, and the second represents the destruction of masses $m$ by catastrophic collisions.

As in our discussion of accretion, we consider both time-independent and time-dependent power-law distributions (equations 1a and b respectively) as asymptotic solutions to the fragmentation equations (39)–(41). In both cases the analysis which follows yields the same exponent $s_1 = (3 + k)/2$, with $k = ab - h$, found in Section 2.1 for time-independent accretion, as one solution. For time-independent fragmentation $s_1$ is the only solution, and it exists subject to the same condition (12) $[k < 1 - (h + |h|)]$ required for time-independent accretion. This condition implies that the distribution index $s_1 < 2$ (condition 11). It follows
that the mass of the distribution is concentrated near $m_\infty$ where it constitutes a reservoir capable of maintaining the flow of mass down the distribution without appreciable attenuation. This renders the distribution practically time-independent.

For time-dependent fragmentation, a second solution $s_F$, in addition to $s_1$, is found to satisfy the criteria for convergence. The stability criteria for these two solutions are mutually exclusive: the lower exponent $s$ is always unstable. The convergence criteria for time-dependent fragmentation require $s_1 > 2$. Consequently $s_F$, when stable, also exceeds 2. For stable time-dependent solutions, therefore, the mass of the distribution is concentrated in its smallest particles so that fragmentation leads to a fall in the distribution as a whole, i.e. $dA/dt < 0$.

### 3.1 Time-Independent Fragmentation

We seek in this section a time-independent power-law distribution $n(m) = A m^{-s}$ that will satisfy the fragmentation equation (41) with $dn/dt = 0$. Substituting $A M^{-s}$ for $n(M)$, equation (40) may be rewritten

$$\psi(\mu) = K \mu^{\gamma} J(a, b, h, e, s, m_\infty/\mu)$$

where

$$J \equiv \int_e^{m_\infty/\mu} y^{-s} (1 + y^{a})^b (1 + y^{-2h})^{1/2} dy$$

and $\gamma \equiv k - s + 1$, $k \equiv ab - h$, as in Section 2.2.

$J$ can be approximated by $J_\infty(a, b, h, e, s) \equiv J(a, b, h, e, s, \infty)$, for values of $\mu < m_\infty$ provided

$$s > 1 + k + (|h| + h)/2.$$  

This is precisely the same condition (10) that required to be satisfied for negligible end effects in the case of time-independent accretion treated in Section 2.1. Combining equations (39)–(43) and substituting $A m^{-s}$ for $n(m)$, equation (41) may be rewritten

$$\frac{dn}{dt} (m < m_\infty) \approx AKJ_\infty \left[ (1 - \omega)m^{-(\omega + 1)} \int_{m_\infty}^{m_\infty} \mu^{(\gamma - s + \omega)} d\mu - m^{\gamma - s} \right]$$

$$= AKJ_\infty m^{\gamma - s} [ (1 - \omega) (\gamma - s + \omega + 1)^{-1} - 1 ].$$

The solution

$$s = s_1 = (3 + k)/2$$

nullifies the factor in square brackets in equation (46). It also satisfies the constraint $\gamma - s + \omega + 1 < 0$ required for convergence of the integral in equation (45), given that $\omega < 1$. It is clear that $s_1$ is the unique solution to $dn/dt = 0$ because the integrand of $J$ in equation (43) is positive definite. If $s_1$ is to satisfy convergence condition (44), the collision cross-section parameters $a$, $b$ and $h$ must be such as to satisfy

$$k < 1 - (h + |h|).$$

Combining this condition (48) with condition (47) yields the constraint

$$s_1 < 2 - (h + |h|)/2.$$  

It is a remarkable feature that these conditions (44), (48) and (49) for time-independent fragmentation correspond precisely to conditions (10), (12) and (11) for time-independent
accretion. Because $s_1 < 2$ the mass is again concentrated in the major bodies of the distribution. This concentration provides a reservoir of mass near $m_\infty$ that is capable of sustaining the flux of fragmenting material down the distribution without suffering significant attenuation, given that the power-law distribution extends over a wide enough range. In this respect, because of the opposite signs of their mass flows, time-independent fragmenting and accreting power-law distributions of index $s_1 < 2$ are in sharp contrast, the accreting distribution requiring an external source to sustain it.

### 3.2 Time-Dependent Fragmentation

We turn now to the question of time-dependent power-law solutions of the type discussed for accretion in Section 2.2. Assuming such a solution, defined by equation (1b), we may rewrite equation (40) in the form

$$\psi(\mu) = KG + K\mu^y F(a, b, h, e, s)$$

where

$$F = \int_0^\infty f_1(y) \, dy - \int_0^e f_2(y) \, dy,$$

$$f_1 = [(1 + y^a)^b (1 + y^{-2h})^{1/2} - QC_0 y^k] \gamma^{-s}$$

$$f_2 = QC_0 y^{k-s}$$

and $G$, $\gamma$ and $C_0$ are as defined in Section 2.2. (The boundaries of the distribution, $m_0$ and $m_\infty$, have been extended to the limits 0 and $\infty$.)

If the second integral is not to diverge at the lower limit, we require $\gamma > 0$, i.e.

$$s < 1 + k.$$  \hspace{1cm} (54)

For $h > 0$ this condition (54) is incompatible with convergence of the first integral at its upper limit. However, for $h < 0$, the choice $Q = 1$ (which applies hereinafter) relaxes the condition for convergence at this upper limit to

$$s > 1 + k - l, \quad h < 0, \quad (b, h \text{ not both zero})$$

where, as in Section 2.2, $l = a$ or $2|h|$ whichever is the lower non-zero value. Note that conditions (54) and (55) are precisely the same as the corresponding constraints (33) and (34) relating to time-dependent accretion in Section 2.2. (In the special case of constant cross-section, $b = 0 = h$, constraint (55) does not apply as $f_1 = 0$ for all $y$. However, power-law solutions to this case cannot meet both inequalities 54 and 57 below.)

With the aid of equations (39) and (50), equation (41) may be re-expressed in the form

$$\frac{1}{KA} \frac{dn}{dt} = \frac{(1-\omega)}{m^{\omega+1}} \int_m^\infty (G + F^\mu) \mu^{\omega-s} d\mu - Gm^{-s} - Fm^{\gamma-s}.$$

Subject to the conditions

$$s > 1 + \omega$$

and

$$p = 2s - k - 2 > \omega,$$

the integral in equation (56) may be evaluated yielding

$$\frac{1}{KA^2} \frac{dA}{dt} = \frac{1}{KAn} \frac{dn}{dt} = \frac{(2-s)G}{(s-\omega-1)} + \frac{(1-p)Fm^{\gamma}}{(p-\omega)}.$$

\[\text{(59)}\]
This equation (59) plays a role analogous to that of equation (17) in Section 2.2. It shows that $dA/dt$ will be independent of mass $m$ if any one of $\gamma, F$ or $(1-p)$ is zero. Solutions of type $\gamma = 0$, i.e., $s = s_0 = 1 + k$, are excluded by condition (54). Solutions corresponding to $p = 1$ occur at $s = s_1 = (3 + k)/2$, the same index found for both time-independent accretion and time-independent fragmentation in Sections 2.1 and 3.1. As $\omega < 1$, $p = 1$ always satisfies condition (58).

Solutions $s_1$ that conform to both conditions (54) and (55) are possible only for values of $a, b$ and $h < 0$ that satisfy

$$1 < k < 1 + 2l.$$  

(60)

(Contrast this with condition 48 for time-independent ($dn/dt = 0$) solutions $s_1$ in Section 3.1.) To satisfy conditions (54) and (55), $s_1$ must lie in the range

$$2 < s_1 < 2 + l$$

(61)

and so necessarily also satisfies condition (57).

Observe that the values of $\omega$ and $e$ (provided of course they are both less than unity) do not influence the value or existence of time-dependent solutions $s_1$, though they may determine their stability. As we shall see, the stability of $s_1$ solutions, when these exist, is bound up with the location and stability of solutions $s_F$ corresponding to $F = 0$ in the rate equation (59).

At the lower limit of the range of solutions that satisfy conditions (54) and (55), $F \to \infty$, while at the upper limit, $s_0, F \to -\infty$. Consequently there must exist at least one solution $s_F$ in this range for which $F = 0$.

In practice, one expects $e \ll 1$, which implies that $s_F$ must lie very close to $s_0$ in order to ensure that the integrand $f_2$ (in equations 51 and 53) is large enough to compensate for the short range of integration, $0 < y < e$, of the second integral of equation (51) which makes the negative contribution to $F$. Two specific examples may serve to illustrate this point. In the simple special case for which $a = 2, b = 1$ and $h = 0$ (these satisfy condition 42), $e$ and $s_F$ are readily found to be related by the equation

$$s_F = (3 + e^2)/(1 + e^2).$$

Thus, in this case, $s_F = s_1 = 5/2$ when $e = 1/\sqrt{3}$. However, for $e = 0.1, s_F = 2.98$, while for $e = 0.01, s_F = 2.9998$ very close to $s_0 = 3$. In another simple special case, also satisfying condition (42), namely $a = 1, b = 2, h = 0$, the relationship between $e$ and $s_F$ is

$$s_F = [5 + 3e + (1 + 6e + e^2)^{1/2}] / (2 + 2e).$$

(63)

In this case $s_F = 2.986$ for $e = 0.1$ and $s_F = 2.9998$ for $e = 0.01$, again very close to $s_0 = 3$. Even for $e = 1, s_F = 2.707 > s_1 = 2.5$.

We will now show that $s_F$ is the unique solution to $F = 0$ because $\partial F/\partial s$ is necessarily negative when $F = 0$. Writing $y = e x$, equation (51) may be re-expressed

$$F = e \int_1^\infty f_1(y) \, dx - e \int_0^1 f_2(y) \, dx.$$  

(64)

$$\frac{\partial F}{\partial s} = -e \int_1^\infty f_1(y) \ln y \, dx + e \int_0^1 f_2(y) \ln y \, dx$$

$$= -e \int_1^\infty f_1(y) \ln x \, dx + e \int_0^1 f_2(y) \ln x \, dx - F \ln e.$$  

(65)
Both $f_1$ and $f_2$ are positive for all $y$, while $\ln(x)$ is always positive in the first integral and always negative in the second. In equation (65), therefore, both integrals necessarily make negative contributions to $\partial F/\partial s$ while the final term is zero when $F = 0$. It follows that $\partial F/\partial s < 0$ when $F = 0$ and so $s_F$ must be unique. For any solution, $s_F$ or $s_1$, to be stable, consideration of equation (59) shows that $\gamma \partial (\phi F)/\partial s$ must necessarily be positive. (This criterion is analogous to the stability criterion, $\gamma \partial I/\partial s > 0$, introduced in Section 2.2.)

$$\phi \equiv (1-p)/(p-\omega) \equiv (3-2s+k)/(2s-2-k-\omega),$$  \hspace{1cm} (66)

so

$$\partial \phi/\partial s = 2(\omega - 1)/(p-\omega)^2 < 0.$$  \hspace{1cm} (67)

Since $\gamma > 0$ and $\partial F/\partial s < 0$ at $s = s_F$, stability of the solution $s_F$ requires $\phi < 0$ which is satisfied provided $s_F > s_1 = (3+k)/2$. Conversely, for $s_1$ to be stable, $\gamma F \partial \phi/\partial s$ must be positive, and that requires $s_1 > s_F$. This last condition, $s_1 > s_F$, is hard to satisfy because $s_1 < s_0 \equiv 1 + k$, in order to satisfy condition (54), while, as we have seen, $s_F$ is very close to $s_0$ for realistic values of $e < 1$. Thus it is $s_F$ rather than $s_1$ that is likely to be stable.

The following argument shows that sufficient conditions for $s_1$ to be unstable are $k > 1 + a$, $b > 1$. It is clear from equations (51)–(53) that $F(e_1) < F(e = 1)$ because both integrands $f_1$ and $f_2$ are positive for all positive $y$. Therefore if $F(s = s_1, e = 1) > 0$, $s_1$ must be unstable, given our initial assumption that $e < 1$. Writing $x$ for $1/y$ in $f_1$ and $x$ for $y$ in $f_2$ one obtains

$$F(s = s_1, e = 1) = \int_0^1 [(1 + x^a)^b (1 + x^{-2h})^{1/2} - C_0 (1 + x^{k-1})] x^{-(k+1)/2} dx.$$  \hspace{1cm} (68)

Now $(1 + x^{-2h})^{1/2} > C_0$ and, because $x < 1$, $(1 + x^a)^b > 1 + x^a > 1 + x^{k-1}$, for $k > 1 + a$, $b > 1$. It follows that the integrand in equation (68) is positive throughout the range of integration. Accordingly $F(s_1, e < 1) > 0$ and $s_1$ must give an unstable distribution.

When stable, $s_F > s_1$, and, as we have seen from our consideration of convergence criteria, $s_1 > 2$. It follows that any stable solution, $s_1$ or $s_F$, must exceed 2. This means that in contrast to the time-independent case treated in Section 3.1, the mass of the distribution is concentrated in its smallest particles and the fragmenting distribution of larger bodies has no virtually inexhaustible reservoir from which to draw support. The distribution density must fall throughout its range. This is confirmed by equation (59) in which the constant term $(2-s)G(s-\omega-1) < 0$ for $s > 2$ giving $dA/dt < 0$ when, at the solution value $s_1$ or $s_F$ of the index, the $m$ dependent term is zero.

4 Summary and conclusions

We have considered wide power-law mass distributions, $n(m) = A m^{-x}$, as candidates for asymptotic solutions to the differential equations describing both accretion and fragmentation (separately). These power-law solutions fall into two distinct categories which we have termed time-independent or time-dependent according to whether the coefficient $A$ is taken to be independent or dependent on time $t$. Although our principal interest was in the 'geometric' collision cross-section for two spherical masses $m_1$, $m_2$ of uniform density, which is proportional to $(m_1^{1/3} + m_2^{1/3})^2$, we have employed the algebraic cross-section factor $(m_1^2 + m_2^2)^{1/2}$ for greater generality and to facilitate comparison with published work concerned with other-than-geometric cross-sections. To take account of a dependence of particle mean speed $\bar{v}$ upon mass $m$, we have also included the factor $(m_1^{2h} + m_2^{2h})^{1/2}$.
appropriate to a power-law velocity distribution $\bar{v}(m) \propto m^{-h}$. Our general findings are summarized in Table 1, but we would draw attention to the following features.

For accretion and fragmentation alike, time-independent power-law solutions only exist if the cross-section parameters obey the constraint $k < 1 - (h + |h|)$ (where $k \equiv ab - h$ is the mass-dimensionality of the cross-section). Note that for the 'geometric' distribution with $a = 1/3$, $b = 2$, this restricts the velocity index to the range $-1/3 < h < 1/3$ and so excludes both kinetic energy equipartition ($h = 1/2$) and the more realistic $h = 0.6$ proposed for an accreting system by Handbury et al. (1979).

Subject to this constraint, $k < 1 - (h + |h|)$, the same power-law distribution with exponent $s_1 = (3 + k)/2$ constitutes an unique time-independent asymptotic solution to both the accretion and fragmentation equations. Possible solutions $s_1$ are restricted to the range $1 + k + (h + |h|)/2 < s_1 < 2 - (h + |h|)/2$ both for accretion and for fragmentation. It is remarkable that the fragmentation parameters $\omega$ and $\epsilon$ (both defined in Section 3) exert no influence on the value of $s_1$ or on the range restrictions. The same index $s_1$ was obtained in a different manner by Kwan (1979) who treated an accreting distribution whose largest masses are disrupted into the smallest units so forming a closed cycle.

Time-independent accreting and fragmenting distributions $n(m) = A m^{-s_1}$ are of course distinguishable by the direction of mass flux. As $s_1 < 2$, the mass is strongly concentrated in the larger bodies. For the fragmenting distribution these large bodies constitute a virtually inexhaustible source for the downward mass flux, but for the accreting system the upward flux would rapidly erode the distribution at its low mass end and the steady state can only be maintained if a steady supply of small particles is provided by an external source.

Since both time-independent fragmentation and accretion lead to the same values $s_1$ and to the same restrictions on their range, it follows that a system which includes both accretion and fragmentation will also produce the same power-law distribution. If fragmentation dominates accretion in this mixed system, no external source of small particles is required to feed the accretion. Such a mixed system (with $h = 0$) was considered by Zvyagina et al. (1974), whose findings agree with our value of $s_1$ when their fragmentation law is equivalent to ours.

In Section 1 we argued that whilst $h$ for an accreting system is probably positive, it could well be negative for fragmentation. For the mixed system $h$ will in general be different from that found for either pure fragmentation or pure accretion. This emphasizes that $h$ is a result of the system dynamics rather than a parameter which determines them, a point we take up again later.

For the particular case of a 'geometric' collision cross-section with velocity independent of mass ($a = 1/3$, $b = 2$, $h = 0$) we obtain $s_1 = 11/6 \approx 1.83$ which is in broad agreement with treatments of fragmentation by Hellyer (1970, 1971) and Dohnanyi (1969, 1970, 1978) and with a limiting case of the treatment of mixed accretion and fragmentation by Zvyagina et al. (1974). Observed mass indices $s = 1.8$ are quite common. For example, such values are reported by Vedder (1966) and Hartman (1965) for meteorites and by Dohnanyi (1969) for asteroids. The compilation by Hughes (1978) from a variety of sources shows mean $s$ values between 1.6 and 2.0 for particles between mass limits of $10^{-10}$ kg and $10^{18}$ kg. It is remarkable that a model based on such simple assumptions — a 'geometric' collision cross-section with neglect of velocity and gravitational effects — should be in such good agreement with the data.

The remarkable similarity of time-independent asymptotic power-law distributions of accreting and fragmenting systems does not extend to the corresponding time-dependent distributions $n(t) = A(t)m^{-s}$, although some common features are present. The severe restriction, $h < 0$, on the mass-dependence of velocity required for such time-dependent
solutions is one such common feature. Another is the permitted range of time-dependent solutions when expressed in terms of the cross-section dimensionality \( k : 1 + k - l < s < 1 + k \).

This last similarity is, however, purely formal; the time-dependent exponent ranges are actually exclusive: \( s < 2 \) for accretion, \( s > 2 \) for fragmentation. This corresponds to the fact that the constraints on \( k \) for time-dependent solutions for the two mechanisms are partly exclusive, being \( k < 1 + l \) for accretion and \( k > 1 \) for fragmentation.

The distinction, \( s < 2 \) for accretion, \( s > 2 \) for fragmentation, in the time-dependent case, is physically associated with the mass-reservoir aspect discussed above. If the distribution coefficient \( A \) is to fall at a significant rate, it is necessary that mass flows towards, rather than away from, the end of the distribution at which the mass is concentrated. (Positive values of \( dA/dt \) are found only for 'solutions' that violate the constraints.)

Note that time-dependent and time-independent fragmentation are also distinguished by the sign of \((s - 2)\) for essentially the same physical reason. If \( k < 1, s_1 < 2 \), and we have a time-independent fragmenting distribution supported by the mass concentration in its largest bodies. If, however, \( 1 + 2l > k > 1 \), only time-dependent solutions are possible. \( s_1 = (3 + k)/2 \) is still a 'solution', but assuming \( e < 1 \), we have \( s_F \rightarrow 1 + k > s_1 \) and \( s_1 \) is not stable when less than \( s_F \). The favoured time-dependent distribution with index \( s_F > s_1 > 2 \) not only has its mass concentrated in its smallest particles, but has a dearth of massive bodies relative to the unstable distribution with index \( s_1 \).

However, in the case of accretion \( s < 2 \) for both time-dependent and time-independent asymptotic power-law distributions, the latter being possible only if the loss of small masses is constantly made good from an external source. Indeed, if \( k \) satisfies the condition \( k < 1 \) for time-independent accretion with \( h < 0 \), it necessarily also satisfies the condition, \( k < 1 + l \), for time-dependent accretion. In this case it is the presence or absence of an external source of small particles that physically determines which type of accreting distribution is established. Observe that the time-dependent accretion index \( s_t < s_1 \), reflecting the reduced proportion of small particles in the asymptotic distribution unsupported by any external small particle source.

Our analytic and numerical results for time-dependent accretion indices \( s_F \) corresponding to some specific cross-sections with \( h = 0 \) are set out in Table 2. These confirm, with greater precision, many of the results reported by Zyvagina & Safronov (1972). The values of \( s_F \) are not related to the cross-section dimensionality \( k \) by any such simple formula as \( s_1 = (3 + k)/2 \). While this is strictly true also for the time-dependent fragmentation index \( s_F \), in practice, with \( e < 1 \), \( s_F = 1 + k \) so that a simple dimensional formula provides at least a close approximation.

An important feature of the present work is the inclusion of the velocity factor \((m_1^{2h} + m_2^{2h})^{1/2}\) in the collision cross-section to take account of the fact that the probability of collision is proportional to the relative velocity of the bodies concerned. The question of the mass dependence of velocity has been largely ignored in most of the relevant literature to date. Our time-independent result, \( s_1 = (3 + k)/2 = 11/6 - h/2 \), for a 'geometric' cross-section with \( a = 1/3, b = 2 \), is valid only for \( |h| < 1/3 \). That is to say there are no time-independent power-law distributions arising as asymptotic solutions to the accretion or fragmentation equations when \( |h| > 1/3 \). For time-dependent solutions \( h > 0 \) is excluded for both accretion and fragmentation, while for the geometric cross-section in particular one requires \(-2/3 < h < 0\) for accretion and \( h < -1/3\) for fragmentation.

Both accretion and fragmentation are essentially irreversible processes in which a considerable fraction of the mechanical energy of the colliding bodies must inevitably be lost. Completely random accretion, in the sense of picking the combining particles at random, would lead to a mean speed \( \bar{v} \propto m^{-h} \) with \( h = 0.5 \) (KEE). However, as Handbury...
et al. (1977) point out, the accretion is not wholly random: collisions with greater relative velocity are favoured. This should lead to increased loss of mechanical energy and consequently to $h > 0.5$. For such values of $h$ neither time-dependent nor time-independent power-law solutions exist for accretion with the geometric cross-section.

Because of the loss of mechanical energy on fragmentation, the mean kinetic energy, per unit mass, of the fragments must necessarily be less than that of the colliding bodies. In the absence of any specific dependence of fragment velocity on fragment mass in a typical collision, this loss of mean specific kinetic energy on collision would lead to negative values of $h$. For fragmentation with the geometric cross-section, negative $h$ allows power-law solutions; these are time-independent for $h > - \frac{1}{3}$ and time dependent for $h < - \frac{1}{3}$.

Many real systems will involve both fragmentation and accretion, and will have time-independent power-law solutions with exponent $s_1$. For such systems the value of $h$ reached asymptotically by the evolving velocity distribution will depend on the balance between the two competing processes. This is an example of the general problem involved in predicting $h$. Throughout this paper we have tended to employ the particle velocity distribution $\bar{u}(m)$, taken to be a power law $\bar{u} \propto m^{-h}$ for the reasons set out in Section 1, as if it were a fundamental physical datum controlling the statistics of the collision process in accreting or fragmenting systems in rather similar fashion to the cross-section parameters $a$ and $b$. Of course a power-law velocity distribution would indeed influence the collision cross-section in the fashion we have described. However, this distribution of velocities, like the mass distribution itself, is strictly the outcome of the collision physics rather than a datum. One ought properly to determine $n(m)$ and $\bar{u}(m)$ together in a thoroughly self-consistent fashion. This appears to be an extremely difficult procedure, well beyond our present capabilities.

Instead we have restricted our analysis to the very limited objective of deciding in what circumstances power-law solutions may exist, and, when possible, finding a relationship between $a$, $b$, $s$ and $h$ for such power-law solutions to the mass and velocity distribution problem. Another relationship between $a$, $b$, $s$ and $h$ must of course be obtained before both $s$ and $h$ can be independently determined from the physics of the collision process. The derivation of such a second relationship must surely demand a much more detailed analysis of the collision process, both for accretion and for fragmentation, than appears possible today.

Though we are aware of the limitations of our approach to velocity effects, we have at least made an advance on the common practice of ignoring them altogether. Indeed this work would seem to be the first to discuss analytically the effects of the velocity distribution on both accretion and fragmentation. In addition to finding power-law solutions for both processes, we believe that this work has provided valuable insights into the problem, perhaps particularly by showing the distinction between time-dependent and time-independent solutions, and by finding limits to the range of existence of power-law solutions. We hope that these asymptotic solutions may also guide those interested in the behaviour of accreting and fragmenting systems at earlier stages of their evolution.

References


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