The continuous spectrum in differentially rotating perfect fluids – II. The effect of gravitational radiation reaction

Eugene Balbinski

Department of Applied Mathematics and Astronomy,
University College, PO Box 78, Cardiff CF1 1XL

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Summary. The effect of gravitational radiation reaction on the continuous spectrum of normal modes in a differentially rotating perfect fluid is considered for the first time. This is done by evaluating the criterion for secular stability for a specific case, for which the continuum is found not to give rise to a secular instability. The calculation highlights certain differences between discrete and continuum modes, as well as between the Eulerian and Lagrangian representations of the continuum modes.

1 Introduction

In a previous paper (Balbinski 1984, Paper I), we discussed the continuous spectrum of normal modes which arises in differentially rotating perfect fluids in Newtonian gravity with particular reference to a model for which we were able to obtain an analytic solution for a certain class of modes. In this paper we shall consider the effects of gravitational radiation reaction on the continuous spectrum by evaluating the criterion for secular stability, which is an integral of the canonical energy density (Friedman & Schutz, 1978a, b FS). Notwithstanding the result of the calculation, the main aim is to show how this is done for the singular modes of the continuous spectrum. In order to do this in the clearest possible way we shall restrict ourselves to the simple model and analytic solutions of Paper I, though the method is generally applicable.

For bounded systems the criterion for secular stability to the emission of gravitational radiation is based on the canonical energy, $E_C$, a quadratic functional of the perturbation. If this is negative for some perturbation, then the configuration is secularly unstable in the linearized theory, otherwise it is secularly stable. For unbounded systems, however, such as the infinite cylinder we are considering here, this is an inappropriate criterion, as the integral over the canonical energy density diverges due to the infinite volume. Instead we shall evaluate the canonical energy per unit length of the cylinder, and assume that it plays an analogous role to the canonical energy for our unbounded model.

The modes that we shall consider are a continuum whose frequencies lie between 0 and –1 in our dimensionless units, and two discrete modes which lie at each end of the continuum (see Paper I). This is a further complication in that in such a case the canonical energy
must be evaluated for the discrete and continuum modes together (FS, 1978, b), as the total canonical energy is not just simply the sum of contributions from each. However, this situation is not an accident, but is the norm, as has been shown by the numerical calculations of Schutz & Verdaguer (1983) who found discrete modes within the continuum for a whole class of differentially rotating discs. We shall show that the total canonical energy per unit length of the continuum and discrete modes is positive, and also that the contribution to this from the continuum alone is itself positive. In doing this we shall encounter some special features of continuum modes which carry over to the general case, and we shall point these out where they occur. An important case in point is described below.

The canonical energy density seems to be most easily expressed in terms of the Lagrangian displacement vector, $\xi$, rather than the Eulerian variables of (in our case) velocity and pressure. This is the form in which it is given by FS, however, these authors describe a certain ambiguity in $\xi$, which, without remedy, leads to $E_c$ being arbitrary. It is due to the existence of solutions for $\xi$, known as ‘trivial’ solutions, as they correspond to zero solutions for the Eulerian variables. The ‘trivials’ amount to a re-labelling of the fluid elements in the Lagrangian picture, and so should not have physically detectable effects, but the addition of a suitable ‘trivial’ $\xi$ to the displacement vector can cause $E_c$ to change its sign. All this has been described by FS (1978a,b) in greater detail than here, as well as the following remedy. This is to restrict admissible perturbations to a subset termed ‘canonical’, but, at least in the isentropic case, this does not represent a real restriction on the physical perturbations. For isentropic systems a ‘canonical’ perturbation is one for which the Lagrangian change in the vorticity is zero. In the case of discrete normal modes, with frequencies outside the range of the continuum, this condition is automatically satisfied, but for discrete modes within the continuum, and for the continuum itself, it is not. We shall therefore construct an explicitly ‘canonical’ perturbation in Part (1). In Part (2) we shall derive a simplified expression for the canonical energy per unit length appropriate to our model, and use the ‘canonical’ perturbation derived in Part (1) to evaluate it.

2 Part (1)

First we shall summarize various details of the model described in Paper 1. It is an incompressible uniform density cylinder, extending to infinity along the $z$-axis in cylindrical polar coordinates $(r, \phi, z)$, where $r$ is a dimensionless radial coordinate defined as the fraction of the radial distance to the surface. We also define a dimensionless unit of time, $t$, by

$$t = \frac{\text{time} \times \sqrt{2\pi G \rho_0}}{\text{time}}$$  \hspace{1cm} (1.1)$$

where $G$ is Newton's gravitational constant, and $\rho_0$ is the cylinder's uniform density. Note that $t, r$ were referred to as $t_0, r_0$, in Paper I. The cylinder rotates differentially with an angular velocity, $\Omega$, given by,

$$\Omega = r.$$  \hspace{1cm} (1.2)$$

Our first task is to evaluate the Lagrangian change in the vorticity, $\Delta V$. The reader may wonder how it is that $\Delta V$ is necessarily zero for modes outside the continuum, but not for modes within it. This happens because the perturbation equations only actually imply

$$(\omega + m \Omega) \Delta V = 0$$  \hspace{1cm} (1.3)$$

for a mode with frequency $\omega$, and angular dependence $\exp (im \phi)$, where $m$ is an integer. The solution of equation (1.3) for $\Delta V$ is

$$\Delta V = f(r, \omega) \delta(\omega + m \Omega)$$  \hspace{1cm} (1.4)$$

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where \( f \) is a smooth function, but otherwise arbitrary and \( \delta \) is a Dirac delta function. For frequencies outside the range of \( -m\Omega(r) \), (i.e. outside the continuum) the right hand side of equation (1.4) is equivalent to zero, so \( \Delta V \) is zero.

We shall consider perturbations which have been Fourier analysed in both \( \phi \), and \( z \), so that their dependence on these variables is \( \exp \left[ i(m\phi + \beta z) \right] \) where \( m, \beta \) are integers. In the following we set \( \beta = 0 \) and later, shall further restrict ourselves to \( m = 1 \). The Lagrangian change in the vorticity, \( \Delta V \), is

\[
\Delta V = \nabla_r (\Delta u_r) - \nabla_{\phi} (\Delta u_{\phi}) = \frac{\partial}{\partial r} (\Delta u_r) - \frac{\partial}{\partial \phi} (\Delta u_{\phi})
\]

(1.5)

where \( \Delta u^i \) is the Lagrangian change in the velocity \( u^i \), and \( \nabla \) represents the covariant derivative. In the following we denote an Eulerian change in \( x \) by \( \delta x \), and use a natural coordinate basis to evaluate components of vectors. \( \Delta u_i \) is given by

\[
\Delta u_i = \delta u_i + \xi^i \nabla_i \xi + u_j \nabla_j \xi^i
\]

(1.6)

so that \( \Delta V \) is

\[
\Delta V = \frac{\partial}{\partial r} \left( r^2 \delta u^\phi + r \omega_0 \xi^r \right) + im \omega_0 \xi^\phi - im \delta v^r
\]

(1.7)

where \( \xi^i \) are the components of the Lagrangian displacement vector, and \( \omega_0 \) is the equilibrium vorticity defined by

\[
\omega_0 = 2 \Omega + r \omega, r.
\]

(1.8)

For our incompressible model \( \nabla_i \xi^i \) must vanish, and so defining

\[
\eta = im \xi^\phi, \quad \xi = \xi^r
\]

(1.9)

we have

\[
\eta = -\frac{1}{r} \frac{\partial}{\partial r} (\xi r).
\]

(1.10)

With the definitions \( u = i \delta v^\phi, v = \delta v^\phi \), the linearized continuity equation is

\[
mv = u_r + \frac{u}{r}
\]

(1.11)

and so substituting \( \delta u^\phi, \xi^\phi \) from equations (1.9), (1.10), (1.11) in (1.7) we have

\[
\Delta V = r \omega_0, r \xi + \frac{1}{m} \left( r^2 u_{rr} + 3 ru_r + (1 - m^2) u \right).
\]

(1.12)

In Paper I it was shown that the general solution for perturbations with \( m = 1, \beta = 0, u(r; t) \), consists of contributions from two discrete modes, \( u_d(r; t) \), and the continuous spectrum, \( u_c(r; t) \). The discrete mode with zero frequency does not contribute to the canonical energy as we shall see in Part (2), so we only need to consider the contribution to \( u_d \) from the mode with frequency \( \omega = -1 \). This is

\[
u_d = \frac{3(r - 1) g(1) \exp(-it)}{\sqrt{2\pi G\rho_0}}
\]

(1.13)
where
\[
g(r) = \int_0^r ds \, s^2 \xi_0(s).
\]
Setting \( \Delta V \) to zero in equation (1.12) and substituting (1.13) for \( u_d \) we have a ‘canonical’ \( \xi_d \)
\[
\xi_d = \frac{-3g(1) \exp(-it)}{\sqrt{2\pi G\rho_0}}.
\]
(1.14)

A generalization of the solution for \( u_c(r; t) \) given in Paper I to arbitrary initial data is
\[
u_c = \frac{(-3)}{\sqrt{2\pi G\rho_0}} \int_{-r}^{-1} d\omega \exp(i\omega t) (r + \omega) \left( \frac{\xi_0(-\omega)}{\omega} + \frac{3g(-\omega)}{\omega^4} \right).
\]
(1.15)

In obtaining equation (1.15) we have reversed the order of the contour integration over frequency and the spatial integration which comes from the Green’s function solution for \( u_\omega \) (see Paper I), and have further assumed that \( g(0) \) is zero. These are both correct if the initial data are suitably smooth. A canonical \( \xi_c \) is now obtained as before by setting \( \Delta V \) to zero in equation (1.12), substituting equation (1.15) for \( u \), and solving for \( \xi_c \). The result is
\[
\xi_c = \frac{1}{\sqrt{2\pi G\rho_0}} \left\{ rh(-r) \exp(-it) + 3 \int_{-r}^{-1} d\omega \exp(i\omega t) h(\omega) \right\}
\]
(1.16)

where
\[
h(x) = \frac{\xi_0(-x)/x + 3g(-x)/x^4}{x^3}.
\]
(1.17)

We can write \( \xi \) as an integral over the continuous spectrum by introducing Dirac delta functions, so that
\[
\xi_c(r; t) = \frac{1}{\sqrt{2\pi G\rho_0}} \int_{0}^{-1} d\omega \exp(i\omega t) \left\{ rh(-r) \delta(\omega + r) + 3h(\omega)H[-(\omega + r)] \right\}
\]
(1.18)

Here \( H(x) \) is the Heaviside step function. From equation (1.10) we can see that \( \eta \) will not only contain delta functions, but also their derivatives. Because of these delta functions the Lagrangian displacement vector for a particular continuum mode is not square-integrable, and so does not belong to the usual Hilbert space defined for \( \xi \). This contrasts with the Eulerian perturbations discussed in Paper I which were all square-integrable. Unfortunately, it is not clear how to define a Hilbert space for the Eulerian variables. A point related to the presence of the delta functions in \( \xi \) is that the continuum displacement vector does not tend to vanish at large times, as do the continuum Eulerian perturbations, as can be seen from equation (1.16). At large times \( \xi \) tends to a constant in time, whereas \( \eta \) grows linearly to leading order. Both these differences between Eulerian and Lagrangian perturbations for the continuum carry over to more general configurations, see, for example, Balbinski (1982).

Finally in this section we consider the ‘trivial’ modes for our model. We have the following relation between \( \delta v' \) and \( \xi \).
\[
\delta v' = \xi + im \Omega \xi.
\]
(1.19)

For a ‘trivial’, \( \delta v' \) is zero so that
\[
\xi = \xi_0(r) \exp(-irt)
\]
(1.20)

\[
= \int_0^{-1} d\omega \exp(i\omega t) \xi_0(r) \delta(\omega + r).
\]
(1.21)
The Eulerian variable, $\delta v^\phi$, is given by

$$\delta v^\phi = -i\eta, t + m\Omega\eta - m\Omega, r\xi$$

(1.22)

If we set $\delta v^\phi$ to zero, we obtain a solution for $\eta$ which is consistent with equation (1.10). The addition of a trivial is therefore equivalent to adding a term proportional to $\xi_0$ to the exp ($-\text{i}rt$) term in equation (1.16), and this shows the arbitrariness inherent in attempting to use the simpler relation (equation 1.19) in converting to Lagrangian variables, rather than equation (1.12). A general solution for trivial perturbations was given by FS (1978a), but the solution for a single normal mode (equation 1.21) is due to Verdaguer (private communication 1980).

2 Part (2): The canonical energy per unit length

This is

$$E_{CL} = \frac{1}{2} [\xi, \xi] - (\xi, \xi) + (\xi, 2\Omega \nabla_\phi \xi)]$$

(2.1)

where $(\xi, \eta)$ is an inner product defined by

$$(\xi, \eta) = \rho_0 \int drd\phi r^2 \xi^* \eta,$$

(2.2)

The reason why the canonical energy is an inappropriate functional to calculate here is concerned with the definition of energy in general relativity. The difference in the canonical energy evaluated on two time-like hypersurfaces corresponds to the amount of energy radiated in gravitational radiation. However, the usual definition of energy in general relativity and stability criterion is based on the assumption that the space-time is asymptotically flat. This not the case for the infinite cylindrical system discussed here, and an alternative definition of energy must be used. Such an energy, the C-energy, has been defined by Thorne (1965) for infinite cylindrical systems such as ours, and it has the property that in Newtonian limit, the C-energy per unit length reduces to the Newtonian energy per unit length. This supports our considering energy per unit length instead of energy, and we believe a proof that this is a suitable stability criterion for infinite cylindrical systems can be constructed along similar lines to that for bounded systems.

Combining equations (2.1) and (2.2), and performing the $\phi$ integration we have

$$E_{CL} = \pi\rho_0 \int_{r_0}^{r_1} dr \left\{ \frac{r^3}{m^2} [\xi_r^*, \xi_r] - \xi_r^* \xi_r + 2im\Omega\xi_r^* \xi_r ] + \frac{r^2}{m^2} [\xi_{rr}^*, \xi_r] + \xi_{rr}^* \xi_r - \xi_{rr}^* \xi_r + 2im\Omega r\xi_r^* \xi_r ] \right\}$$

(2.3)

which after some integrations by parts becomes

$$E_{CL} = \pi\rho_0 \int_{r_0}^{r_1} dr \left\{ \frac{r^3}{m^2} [\xi_r^*, \xi_r] - \xi_r^* \xi_r + 2im\Omega\xi_r^* \xi_r ] + \left(1 - \frac{1}{m^2}\right)2im\Omega r\xi_r^* \xi_r ] \right\} + \pi\rho_0 [\xi_0^*, \xi_0] |_{r=1}. $$

(2.4)
We can see immediately that there is no contribution to $E_{CL}$ from the zero frequency mode due to the time derivatives in equation (2.4). Here we encounter another difference in the calculation between discrete modes with frequencies outside the continuum, and modes which have frequencies within the continuum. The basis of the stability criterion is that $E_c$ decreases with time and this can be applied to individual normal modes with no co-rotation points, $\xi_n^d$, since

$$E_c \left( \sum_n \xi_n^d \right) = \sum_n E_c (\xi_n^d)$$

that there are no interaction terms, see FS(1978b). However, this is not necessarily the case for the continuum modes, or for a discrete mode within the continuum, since in these cases we cannot show that the combined canonical energy can be split into a contribution from each mode which decreases in time, and we have therefore to apply the stability criterion to the whole perturbation. Here we have a discrete mode at $\omega = -1$, one end of the continuum and shall therefore calculate the total canonical energy per unit length of the discrete mode, and the continuum, $E_{CL}^T$, where

$$E_{CL}^T = E_{CL}^d + E_{CL}^c + E_{CL}^l.$$  \hspace{1cm} (2.5)

Here $E_{CL}^d$ is the contribution due solely to the discrete mode, $E_{CL}^c$ is that from the continuum and $E_{CL}^l$ is an interaction term. We calculate $E_{CL}^d$ first; since $\xi_{CL,r} = 0$ from equation (1.14) and $m = 1$ this only contributes through the integrated terms giving

$$E_{CL}^d = 9 g^2 (1)$$ \hspace{1cm} (2.6)

which is positive definite for non-zero initial data. The contributions to $E_{CL}^c$ come from both the integrated part and the first term in brackets in equation (2.4). The latter term is, using equation (1.16) for $\xi_c$,

$$\dot{\xi}_c, r = \dot{\xi}_c, r - \xi_c, r \cdot \dot{\xi}_c, r + 2im \Omega \xi_c, r \dot{\xi}_c, r = 2rf, r + 7f^2 (1)$$ \hspace{1cm} (2.7)

where

$$f(r) = 3g(r)/r^3 - \xi_o (r).$$ \hspace{1cm} (2.8)

It should be noted that all the time-dependent terms in equation (2.7) have cancelled, as they should, but that both terms in equation (1.16) contribute to equation (2.7). Integrating by parts we have

$$E_{CL}^c = \frac{3}{2} \int_0^1 dr r^3 f^2 + \frac{3}{2} f^2 (1)$$ \hspace{1cm} (2.9)

which is again positive definite for non-zero initial data.

Finally, there is a contribution to $E_{CL}^c$ from $\xi_c$, $\xi_p$ taken together, $E_{CL}^l$, which comes from the integrated part in equation (2.4). This is

$$E_{CL}^l = 6 g (1) [\xi_o (1) - 3 g (1)]$$ \hspace{1cm} (2.10)

which is not positive definite. By expressing $g$ in terms of $\xi_o$, and integrating by parts we can re-write equation (2.10) as

$$E_{CL}^l = 6 \left( \int_0^1 ds s^3 \xi_o (s) s \right) \left( \int_0^1 ds s^2 \xi_o (s) \right).$$ \hspace{1cm} (2.11)
The sign of this interaction term is dependent therefore on the gradient of the initial data \( \xi_0 \).

If we add together the various contributions from equations (2.6), (2.9) and (2.10), we have the total canonical energy per unit length,

\[
E_{CL}^T = \frac{3}{2} \left\{ \int_0^1 dr \, r^3 f^2(r) + [\xi_0(1) - g(1)]^2 + 2g^2(1) \right\} 
\]  

(2.11)

which is again positive definite, so that the combined perturbation does not give rise to a secular instability.

3 Conclusion

We have shown by a specific example, for \( m = 1 \) modes, how the canonical energy density can be calculated for a continuous spectrum, and have obtained a result which suggests that for this example, the continuum does not give rise to a secular instability to gravitational radiation reaction. The calculation does however, show that the presence of a discrete mode within the continuum can be de-stabilizing in the sense that the total canonical energy per unit length can be less than is attributable to the discrete mode and continuum separately. It would be interesting to know whether the canonical energy associated with the continuum is necessarily positive in more general situations, and also whether the interaction energy can ever be sufficiently negative to make the total canonical energy negative.

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References