A gas model for multicomponent star clusters: method and comparison with Fokker–Planck theory

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Summary. A multi-mass stellar system is simulated by integrating numerically a set of four gas-model equations for each component. The components interact by mutual gravitational interaction, by transfer of kinetic energy and by dynamical friction effects. Pre-collapse calculations of two-component systems are presented and compared with the results obtained by direct integration of the fundamental Fokker–Planck equation. The gas model shows a higher degree of equipartition of the kinetic energies. Later-phase evolution is somewhat slower as compared with results of the Fokker–Planck theory. The qualitative behaviour of evolution is, however, in fair agreement.

1 Introduction

Gas models of stellar dynamical systems are invoked, since the basic physics and results are easily comprehensible in terms of gravothermal or thermodynamical concepts (cf. for example, Hachisu & Sugimoto 1978; Heggie 1984; Bettwieser & Sugimoto 1984). Furthermore the calculation is cheap and, concerning the technique, well established in the theory of stellar evolution and hydrodynamics (cf. for example, Tscharnuter & Winkler 1979). In the present paper we extend gas-model equations from one to many populations of different star masses. Each population is described by its own set of fluid-dynamical equations. They are coupled by mutual gravitational attraction, by exchange of energy due to the tendency to equipartition and by dynamical friction effects. The main points of the theory of the gas model are recalled and the coefficient of heat conductivity is generalized so that it can simulate stellar encounter effects in a multi-mass system. Appropriate subscripts distinguish the different populations. In the present paper the method is applied to a two-component cluster in the pre-collapse phase of their evolution. In Section 2 we discuss the model equations. Section 3 deals with an application to a two-component system. The results are compared with those obtained by direct integration of Fokker–Planck equation (Inagaki & Wiyanto 1984) in Section 4. Post-collapse evolution and
further results concerning two- and a ten-component cluster will be presented in a forthcoming paper.

2 Model equations

We consider a stellar system of \( N \) stars with total mass \( M \). A star population \( j \) is defined by all stars having the same mass \( m_j \). Each population is assumed to be spherically symmetric in coordinate space and isotropic in velocity space, i.e. velocity dispersion in radial and tangential direction are assumed to be equal. For each population the following quantities are calculated:

mass density, \( \rho_j \),
velocity of mean mass motion, \( u_j \),
energy density, \( E_j \),
energy flux density, \( F_j \).

Subsidiary quantities to those of the last equations are the following

pressure, \( P_j = \frac{2}{3} E_j \),

one-dimensional velocity dispersion, \( \sigma_j^2 \).

mass of component \( j \), \( M_j = \int_0^r 4\pi r^2 \, dx \, \rho_j(x) \),

total mass density, \( \rho = \sum_j \rho_j \),

total number of stars, \( N = \sum_j M_j m_j^{-1} \),

mean star mass, \( \bar{m} = M N^{-1} \).

The gravitational acceleration corresponding to the density distribution is indicated by means of

\[ g(r) = \frac{G}{r^2} \int_0^r \sum_j \rho_j(x) 4\pi x^2 \, dx. \]  \tag{2}

The material derivative for component \( j \) is introduced, as defined by

\[ D_j = \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial r}. \]  \tag{3}

Gas-model equations can now be written in the following form:

\[ \frac{\partial}{\partial t} \rho_j + \text{div} (\rho_j u_j) = 0, \]  \tag{4}

\[ D_j u_j + \frac{1}{\rho_j} \frac{\partial}{\partial r} P_j + g(r) = \frac{\partial}{\partial t} u_j, \]  \tag{5}

\[ \frac{\partial}{\partial t} E_j + \text{div} (E_j u_j) + P_j \text{div} u_j = \rho_j \varepsilon_j - \text{div} F_j + \frac{\partial}{\partial t} E_j, \]  \tag{6}

\( \varepsilon_j \) is the specific internal energy.
\[ F_j + \Lambda_j \frac{\partial}{\partial r} \sigma_h^2 = 0. \] (7)

The encounter terms of a component with all other components appear on the right-hand side of equation (5) and (6). They are derived and discussed in Section 2.1. It is assumed that heat within one component is redistributed due to the local gradient in the velocity dispersion (cf. equation 7). The coefficient of heat transport is derived in Section 2.2. The quantity \( \varepsilon_j \) is introduced to stimulate core-heating effects in order to describe post-collapse evolution. Throughout this paper \( \varepsilon_j \) is set to zero, i.e. we deal with pre-collapse phases of evolution under the assumption of a vanishing rate for the hardening processes of tight binaries. For \( j \) mass components we have to solve \( 4j + 1 \) partial differential equations; up to now the method presented has been used for one, two, three and a ten-component system.

An Eulerian grid was adopted, the grid points chosen at equal intervals in \( \log(r) \). The equations are coded with backward time differences in implicit form and a Henyey solver was applied. Since we distinguish between inflowing and outflowing matter, conservation of mass is achieved within the limits of the calculation accuracy. The non-conservation of energy is secular, but only in extreme high-density stages noticeable (100 mesh points). In typical pre-collapse runs total energy was conserved with a precision of the order of a few per cent, depending on the total increase of the central density.

### 2.1 Encounter Terms

In calculating the effects caused by distant encounters we adopted the hypotheses of Larson (1970) and followed his procedure, but made the assumption that the radial and tangential components of the velocity dispersion are equal. Starting from Fokker–Planck equation we obtained results similar to those of Angeletti & Giannone (1976), provided the isotropic limit of their collisional terms is taken, i.e. the radial and tangential velocity dispersions are taken to be equal.

For convenience we denote by \( t_{ij} \) a relaxation time of component \( i \) in interaction with component \( j \).

\[ t_{ij} = \frac{3[8(2\pi)^{1/2}]^{-1} \cdot (\sigma_i^2 + \sigma_j^2)^{3/2}}{[Cm_i \varrho_i]} \] (8)

\[ C = G^2 \ln \{3 \sigma_i^2 R_{\text{max}}/[G(m_i + m_j)]\}. \] (9)

For the dynamical friction terms we have

\[ \frac{\partial}{\partial t} \bigg|_{c} u_i = \frac{F_i}{4Q_i} \sum_{j=1}^{N} \left( 1 + \frac{m_i}{m_j} \right) [(\sigma_i^2 + \sigma_j^2) t_{ij}]^{-1}. \] (10)

The heat transferred to component \( i \) is described by

\[ \frac{\partial}{\partial t} \bigg|_{c} \sigma_h^2 = - \sum_{j=1}^{N} (\sigma_i^2 m_i/m_j \cdot G_{ij} - \sigma_i^2 H_{ij})/t_{ij}. \] (11a)

Here the quantities \( G_{ij} \) and \( H_{ij} \) depend on the functional form of the velocity distribution. Both \( G_{ij} \) and \( H_{ij} \) become unity when the distribution function is Maxwellian (Spitzer 1956). In the present gas model the highest-order moment treated is the temperature. Therefore as a first approximation of the gas model, we put

\[ G_{ij} = H_{ij} = 1. \] (11b)
The resulting structure for the expression of the coefficient of heat transfer can also be derived directly from the Fokker–Planck equation, if one assumes from the beginning for all components a Maxwellian (Spitzer 1956). Note that in the derivation of Angeletti & Giannone further approximations are used during the calculation of the collision terms. Their procedure invoked to derive them introduces a perturbation expansion beyond the expansion into moments of the velocity distribution. This is the reason that their results are more complicated.

2.2 COEFFICIENT OF THERMAL CONDUCTIVITY

In this section the coefficient for heat transport is derived from the following requirements. (i) A trivial multicomponent system, i.e. $m_j = \text{constant}$ for all $j$, evolves in the same way as a one-component system. (ii) The coefficient of thermal conductivity is derived from scaling arguments. In other words, it agrees with that of the one-component system, if the number of components is one. (iii) The numerical value is adjusted so that a one-component system collapses completely in about 15 half-mass relaxation times, a value obtained by Cohn (1980).

In a trivial multi-component system, $\sigma_j^2$ should not depend on $j$. Since $F_j \propto \varrho_j \sigma_j^2$, $\Lambda_j$ should be proportional to $\varrho_j$ from the first requirement and equation (7). The second requirement determines the functional form of $\Lambda_j$. We find an expression

$$\Lambda_j = \lambda \ln (0.4 N) G \varrho \varrho_j \sigma_j^{-1}$$

(12)

similar to that for a one-component system (Lynden-Bell & Eggleton 1980; Bettwieser 1983), where

$$\sigma^2 = \sum_{j=1}^{N} \varrho_j / \varrho \sigma_j^2.$$  

(13)

The third requirement determines the value of $\lambda$.

2.3 INITIAL AND BOUNDARY CONDITIONS

As the initial model we assume a multicomponent Plummer’s model. The velocity dispersion of all components are obtained from hydrostatic equilibrium. Hence the quantities of equation (1) obtain the following initial values:

$$\varrho(r) = M / \left( \frac{4\pi}{3} r_0^3 \right) \left[ 1 + (r/r_0)^2 \right]^{-5/2}$$

$$\varrho_j = \varrho M_j M^{-1}$$

$$\sigma_j^2 = GM/(2 r_0) [1 + (r/r_0)^2]^{-1/2}$$

$$u_j = 0; \quad F_j = 0.$$  

(14)

Since all velocity dispersions are equal we assume here a state before mass segregation. Such a state might be the result of collective relaxation processeses, which do not lead to thermal equilibrium but in ideal cases to a ‘Boltzman’ distribution for the kinetic energy per mass (Lynden-Bell 1967). The boundary conditions applied during time evolution are the symmetry condition at the centre ($r=0$),

$$u_j(r, t) = 0; \quad F_j(r, t) = 0$$

(15)

and a similar set of equations at the last Eulerian mesh $r=R$. The system is enclosed within a
perfectly reflecting wall. This choice makes results more easily comprehensible in terms of gravothermal effects.

3 Two-component clusters

The quantities which are equal in all calculations reported below are the following:

\[ R = 100 \text{ pc} \]

\[ r_0 = 3 \text{ pc} \]  

\[ M = 10^6 M_\odot \]

\[ m_1 = 1 M_\odot \].

100 mesh points were used in most runs and for the innermost radius $10^{-5}$ pc. Such a one-component system with $m = 1 M_\odot$ has its half-mass within 3.83 pc and a half-mass relaxation time of $1.29 \times 10^9$ yr. It collapses within 15.4 of such reference time-scales to infinite density leaving behind a power-law halo of an index of about $-2.2$ (cf. Lynden-Bell & Eggleton 1980).

3.1 Models and results

Time-scales are given in terms of initial half-mass relaxation time. In its definition we consider the system to be composed of $N$ stars each of mass $\dot{m}$ (compare equation 1b).

\[ t_{\text{rel}} = 0.06 (MR^3 G^{-1})^{1/2} [\dot{m} \log (0.4N)]^{-1}. \]  

(17)

Since we start from a state which is far from thermal equilibrium, stellar encounters are very efficient from the onset of evolution. The more massive stars possess larger kinetic energy than the less massive ones. The latter receive heat and thereby allow the heavier component to sediment rapidly towards central parts. Redistribution of heat, as caused by gradients in velocity dispersion, has a marginal influence on the initial evolution, especially if $m_2/m_1$ is large. The influence of dynamical friction was checked by a special tailored run, choosing $m_2/m_1 = 10$. In such cases time evolution merely responds in slight, quantitative details to effects caused by dynamical friction.

3.2 Models

In a sequence of simulations $m_2/m_1$ was set equal to two. Models with different fractions of the total mass of the heavier component are calculated. If the fraction $M_2/M$ is small, collapse of the heavier component proceeds in the external field of lighter stars. The core-collapse time decreases rapidly with increasing mass fraction of the heavier component. On the other hand, if this fraction is very large, the amount of thermal energy contained in the core of the more massive component is so large that the core of the less massive component cannot receive all the redundant energy. Hence the tendency to collapse is postponed, since the heat gained has to be dumped away into the halo and this process is slower than the tendency to equipartition. This is the reason that the core-collapse time in units of the half-mass relaxation time attains a minimum value at a certain fraction of the heavier stars (Fig. 1). Finally it approaches its single-mass value for negligible contribution of the less massive component. This effect is further illustrated by the evolution of the central density of the less massive stars in Fig. 2. For the case $M_2 = 0.1 M$ the density profiles at a late stage of evolution are plotted in Fig. 3.

This physical situation is in accord with the criterion for the possibility of approximate equipartition between stars of different masses (Spitzer 1969). From the latter paper we take the
Figure 1. Core collapse time $t_{cc}$ as a function of the total mass $M_2$ of the more massive component for a sequence of models with $m_2/m_1=2$. In units of the half-mass relaxation time $t_{rel}$, $t_{cc}$ has a minimum value at about $M_2=0.2 M$ of 0.5 of its single mass value ($t_{cc}=15.4 t_{rel}$).

result that the typical time to establish equipartition in terms of the relaxation time of the lighter stars is equal to $m_1/m_2$, provided the velocity dispersions are equal. This explains the reduction of the evolution time by a factor of two with $M_2$ increasing from zero, and also indicates that the tendency to equipartition drives the initial evolution. The latter determines the core-collapse time since evolution accelerates. For large values of $M_2$, however, self-gravity of the heavier component requires larger and larger velocity dispersions of the heavier masses. As a consequence equipartition with the lighter stars is harder and harder to achieve and the evolution proceeds finally only due to redistribution of heat within the two, almost separated, components.

Figure 2. For the less massive stars the time evolution of the central density for different value of $M_2/M$ is plotted. The value for $M_2$ indicated in percentage of the total mass. Units of mass, length, velocity are equal to $1 M_\odot$, 1 pc and $10$ km s$^{-1}$, respectively.
Figure 3. Density profiles of the components at a late stage of evolution for a model with $m_2/m_1 = 2$ and $M_2 = 0.1 M$. Time = $12.9 t_{ref}$. Units as in Fig. 2.

4 Comparison with Fokker–Planck calculations

In this section we compare the results of the gas model with those of the Fokker–Planck model (henceforth FP for short) shown in Inagaki & Wiyanto (1984). We compare two models $M_2/M = 0.01$ and 0.33. In both models $m_2/m_1$ is set as 2. Some basic quantities are compared in Table 1. Note that the quantities $K$ and $\xi_2$, which are given in the table, are defined

$$K = \ln \{ m_2 \sigma^2_2 / (m_1 \sigma_i^2) - 1 \}$$

$$\xi_2 = t_{\text{rx}}^{(2)} \frac{\partial}{\partial t} \ln \varrho_2.$$  (18b)

Here $\sigma^2_i$ is the velocity dispersion of the component $i$, $\varrho_2$ is the central density of the more massive component and $t_{\text{rx}}^{(2)}$ is the central relaxation time defined by

$$t_{\text{rx}}^{(2)} = (2 \sigma^2_2)^{3/2} / [2 \pi G^2 m_2 \varrho_2 \ln (0.4N)].$$  (19)

In equations (18a) and (18b) all quantities are evaluated at the cluster centre in the latest stage of evolution. The differences of $\xi_2$ between the gas model and the Fokker–Planck model is seen in the table. The rate of evolution is smaller in the gas than in the Fokker–Planck model. Since the rate of evolution is smaller, there is more time to achieve the equipartition of kinetic energies. This explains why $K$ in the gas model is much smaller than that in the Fokker–Planck model. Therefore more heat is transported from the more massive component to the less massive component in the gas model. In Fig. 4(a) and (b) the profiles of the kinetic energies are plotted for the gas (FP) model, respectively. Note the discrepancy in the centre for this model with $M_2 = M_1/2$. The corresponding density profiles are plotted in Fig. 5(a) and (b). Note the indication

| Table 1. Comparison of the core collapse time, equipartition of the kinetic energies at the centre and of the central collapse rate resulting from gas model calculations and from Fokker–Planck theory. |
|-----------------|---------|-----------------|---------|-----------------|
| model with $M_2/M = 0.01$ | $t_{\text{cc}}/t_{\text{ref}}$ | $K$ | $\xi_2$ | model with $M_2/M = 0.33$ | $t_{\text{cc}}/t_{\text{ref}}$ | $K$ | $\xi_2$ |
| GAS | 12.8 | $2.3 \times 10^{-4}$ | $3.5 \times 10^{-4}$ | 13.8 | $1.4 \times 10^{-2}$ | $4.0 \times 10^{-3}$ |
| FP  | 13.8 | $1.4 \times 10^{-2}$ | $4.0 \times 10^{-3}$ | 8.1  | $8.7 \times 10^{-4}$ | $2.1 \times 10^{-3}$ | 8.5  | $1.7 \times 10^{-1}$ | $4.3 \times 10^{-3}$ |

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Figure 4. Radial profiles of the kinetic energies $m_i\sigma_i^2$ at a late stage of evolution for $m_2=2$ and $M_2=0.33 \, M$. (a) Ga model, time=$8.1 \, t_{\text{ref}}$. (b) Fokker–Planck, time=$8.52 \, t_{\text{ref}}$. Units as in Fig. 2.

of a shoulder near the core region, which is not present in the FP calculation (cf. Fig. 5a and b). There is also a significant greater spread between the central densities in the case of the gas mode calculation. Fig. 6 shows the comparison of evolution of the central densities of the model with $M_2/M=0.01$. Before $\rho_2$ dominates $\rho_1$, the evolution of $\rho_1$ and $\rho_2$ is nearly the same in both models. However, if $\rho_2$ dominates $\rho_1$ in the gas model is smaller than that in the Fokker–Planck calculation. This tendency is larger in the model with $M_2/M=0.33$. In the model with $M_2/M=0.33$ the central density of the less massive component is about 30 times smaller than that in the Fokker–Planck model. This is also explained by the efficiency of heat transport in the ga model. In the gas model, since the less massive component receives more heat from the more massive component, it expands more than that in the Fokker–Planck model. The discrepancy between the gas model and the Fokker–Planck model may become small, if we consider that $\lambda$ i
Figure 5. Radial profiles of the densities. Same model and (a), (b) as in Fig. 4.
not constant but a function of the distribution function. In the gas model $\lambda$ is determined so that $t_{\alpha}$ is about fifteen times the half-mass relaxation time. However, $\lambda$ thus determined gives smaller $\xi_2$ in the late phase. This suggests that $\lambda$ should be larger if the distribution function is closer to the Maxwellian.

5 Discussion

Gas model equations for multimass systems are formulated. They are numerically stable and the integration scheme is simple and cheap so that the dynamics can be followed for a relatively long time in pre- or post-collapse calculations. The results of the application of the method to a two-component system accord well qualitatively with the results of the Fokker–Planck theory and provide new evidence that heat transfer between different components does not always lead to complete equipartition of the kinetic energies. Both theories agree in so far as the halo of the system is concerned. Detailed quantitative comparison with the results of Fokker–Planck theory shows, however, that the gas-model prediction overestimates the degree of equipartition in the core. The speed of evolution in late phases is smaller in the gas model and allows for more time to transfer energy from the more massive to the less massive stars. This behaviour might stem from the fundamental difference in the formulation of the relaxation processes. The gas mode contains the assumption of a Maxwellian for the velocity distribution of the particles. Maybe this is the fundamental reason that the tendency to equipartition is more pronounced. One might speculate whether a different choice of the coefficient of heat conductivity exists, which leads to better agreement. On the other hand, our derivation given in Section 2.2 rests on scaling arguments which are considered to express general, but not very restrictive, relations. It seems that they have to be met for any functional form of the heat conductivity.
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In cases of astrophysical interest ($M_2/M$ small) such difference between the models seems to have no important physical consequences. Both models predict a high degree of equipartition in the core for such cases. In this sense the method presented can satisfactorily be applied to astrophysical problems, such as evolution of globular clusters or galactic nuclei.

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